Math 3210-1, Midterm 1

Thursday June 2, 2016

This exams is comprised of 5 questions, on 5 pages, and for a total of 25 points.

1. (3 points each; 6 points total) Which of the following functions are one-to-one and which are onto? Justify your answers.

   (a) $f : [0\,,\pi/2] \to [0\,,1]$, defined as $f(x) = \sin(x)$ for $0 \le x \le \pi/2$.
   Solution. The function $f(x) = \sin x$ is strictly increasing on $(0\,,\pi/2)$ [one-to-one] and for every $y \in [0\,,1]$ there exists $x \in [0\,,\pi/2]$ such that $f(x) = y$. Therefore, f is both one-to-one and onto.

   (b) $f : [0\,,\pi] \to [0\,,1]$, defined as $f(x) = \sin(x)$ for $0 \le x \le \pi$.
   Solution. $f(0) = f(\pi) = 0$; therefore, f is not one-to-one. However, f is onto: For every $y \in [0\,,1]$ there exist at least one $x \in [0\,,\pi]$—in fact there exists one $x \in [0\,,\pi/2]$—such that $f(x) = y$.

2. (4 points) Use induction to prove that $\sum_{k=1}^n k = \frac{1}{2}n(n+1)$ for every integer $n \ge 1$.
   Solution. For every $n \in \mathbb{N}$, let $P(n)$ denote the assertion that $\sum_{k=1}^n k = \frac{1}{2}n(n+1)$. $P(1)$ is true since $\sum_{k=1}^1 k = 1 = \frac{1}{2} \times 1(1+1)$. Suppose next that $P(1), \ldots, P(n)$ are true for some integer $n \ge 1$. It remains to prove that $P(n+1)$ is true. But

$$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1) = \frac{n(n+1)}{2} + (n+1),$$

   by the induction hypothesis. Factor $n+1$ to see that

$$\sum_{k=1}^{n+1} k = (n+1)\left[\frac{n}{2} + 1\right] = (n+1)\left[\frac{n+2}{2}\right] = \frac{(n+1)(n+2)}{2}.$$

   That is, $P(n+1)$ is true, as desired.

3. (4 points) Prove that $\sqrt{5}$ is an irrational number.
   Solution. By Theorem 1.3.9, if $k \in \mathbb{Z}$, then all rational solutions to the equation $x^2 = k$ are integers. in particular, set $k = 5$ to see that if $\sqrt{5}$ were rational then it would have to be an integer. Now it is easy to see that $2 < \sqrt{5} < 3$, whence $\sqrt{5} \notin \mathbb{Z}$.

4. (5 points) Let $X = \{0\,, 1\}$ be a set with two elements. Define two binary operations "$\oplus$" and "$\otimes$" on X as follows:

$$0 \oplus 0 = 0, \quad 0 \oplus 1 = 1, \quad 1 \oplus 0 = 0, \quad 1 \oplus 1 = 0,$$
$$0 \otimes 0 = 0, \quad 0 \otimes 1 = 0, \quad 1 \otimes 0 = 0, \quad 1 \otimes 1 = 1.$$

Prove or disprove the following statement: X is a commutative ring.
Solution. $0 \oplus 1 \neq 1 \oplus 0$. Therefore, X is not a commutative ring.

5. (6 points) Find the supremum and infimum of the set

$$A = \left\{ \mathrm{e}^{-n} : \, n \in \mathbb{N} \right\}.$$

Solution. $\sup A = 1/\mathrm{e}$, $\inf A = 0$.