Math 3210–1, Summer 2016
Solutions to Assignment 9

5.1. #1. Here, $x_0 = 1$, $x_1 = 5/4$, $x_2 = 3/2$, $x_3 = 7/4$, and $x_4 = 2$. Because f is decreasing, $m_1 = \inf_{x\in[1,5/4]}(1/x) = 4/5$, $m_2 = \inf_{x\in[5/4,3/2]}(1/x) = \frac{2}{3}$, $m_3 = \inf_{x\in[3/2,7/4]}(1/x) = 4/7$, and $m_4 = \inf_{x\in[7/4,2]}(1/x) = 1/2$. Because $x_k - x_{k-1} = 1/4$ for all k, this implies that

$$L(f\,,P) = \frac{m_1 + \cdots + m_4}{4} = \frac{1}{4}\left[\frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \frac{1}{2}\right].$$

Similarly, $M_1 = \sup_{x\in[1,5/4]}(1/x) = 1$, $M_2 = \sup_{x\in[5/4,3/2]}(1/x) = 4/5$, $M_3 = \sup_{x\in[3/2,7/4]}(1/x) = 2/3$, and $M_4 = \sup_{x\in[7/4,2]}(1/x) = 4/7$. Therefore,

$$U(f\,,P) = \frac{M_1 + \cdots + M_4}{4} = \frac{1}{4}\left[1 + \frac{4}{5} + \frac{2}{3} + \frac{4}{7}\right].$$

5.1. #5. Consider an arbitrary partition $P = \{x_0\,,\ldots,x_n\}$ of $[0\,,1]$, where as usual we write $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$. Because $[x_{k-1}\,,x_k]$ contains rationals and irrationals for all k, it follows that $M_k = 1$ and $m_k = 0$ for all k. This implies that

$$U(f\,,P) = \sum_{k=1}^{n}(x_k - x_{k-1}) = x_n - x_0 = 1,$$

because of properties of telescoping sums. Similarly,

$$L(f\,,P) = 0.$$

Since $U(f\,,P) - L(f\,,P) = 1$ for all partitions P, Theorem 5.1.7 implies that f is not integrable on $[0\,,1]$.

5.1. #9. Let P be an arbitrary partition of $[a\,,b]$. Since $M_i = m_i = k$ for all $i = 1,\ldots,n$,

$$U(f\,,P) = \sum_{i=1}^{n} k(x_i - x_{i-1}) = k(b-a),$$

using properties of telescoping sums. Similarly, $L(f\,,P) = k(b-a)$. Because $U(f\,,P) - L(f\,,P) = 0 < \varepsilon$ for all $\varepsilon > 0$, Theorem 5.1.7 implies that f is integrable and $\int_a^b f = k(b-a)$.

5.2. #9. Let $M := \sup_{x\in[a,b]}|f(x)|$. Since $f^2(x) - f^2(y) = (f(x) - f(y))(f(x) + f(y))$ for all $x, y \in [a\,, b]$, it follows that

$$\left|f^2(x) - f^2(y)\right| \le \{|f(x)| + |f(y)|\}\,|f(x) - f(y)| \le 2M|f(x) - f(y)|.$$

Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a\,, b]$ as before. For every $k = 1, \ldots, n$ we can find a sequence $x_1, x_2, \ldots,$—depending on k—such that $\lim_{j\to\infty} f(x_j) = \sup_{x\in[x_{k-1},x_k]} f(x) = M_k(f)$ and a sequence $y_1, y_2, \ldots,$—depending on k—such that $\lim_{j\to\infty} f(y_j) = m_k(f)$. Now,

$$\left|f^2(x_j) - f^2(y_j)\right| \le 2M|f(x_j) - f(y_j)| \qquad \forall j.$$

The right-hand side converges to $M_k(f) - m_k(f)$ as $j \to \infty$. The left-hand side converges to $M_k(f^2) - m_k(f^2)$ since f^2 is maximized and/or minimized where f is. [This is because $g(z) := z^2$ is increasing.] In this way we find that

$$M_k(f^2) - m_k(f^2) \le 2M\,[M_k(f) - m_k(f)] \qquad \forall k = 1, \ldots, n.$$

Therefore,

$$\begin{aligned} U(f^2, P) - L(f^2, P) &= \sum_{k=1}^{n} \left(M_k(f^2) - m_k(f^2)\right)(x_k - x_{k-1}) \\ &\le 2M \sum_{k=1}^{n} \left(M_k(f) - m_k(f)\right)(x_k - x_{k-1}) \\ &= 2M\,[U(f\,, P) - L(f\,, P)]\,. \end{aligned}$$

By Theorem 5.1.7 for every $\varepsilon > 0$ we can find a partition P of $[a\,, b]$ such that $U(f\,, P) - L(f\,, P) \le \varepsilon/(2M)$. The preceding implies that, for the same $(\varepsilon\,, P)$, $U(f^2\,, P) - L(f^2\,, P) \le \varepsilon$. A second appeal to Theorem 5.1.7 shows that f^2 is integrable.

5.2. #10. Because $(f + g)^2 = f^2 + 2fg + g^2$,

$$fg = \frac{1}{2}\left[(f + g)^2 - f^2 - g^2\right].$$

By the previous exercise, the right-hand side is integrable. Therefore, so is the left-hand side.