Math 3210–1, Summer 2016
Solutions to Assignment 8

4.4. #1. Fix an $x > 1$, and consider the functions $f, g : [1\,, x] \to \mathbb{R}$, defined by

$$g(y) = y^r \quad \text{and} \quad f(y) = \ln y.$$

Then, f and g are continuous on $[1\,, x]$ and differentiable on $(1\,, x)$; $g(y) \neq 0$ for all $y \in (1\,, x)$ and $g'(y) = ry^{r-1} \neq 0$ for all $y \in (1\,, x)$. By the Cauchy's form of the mean-value theorem, there exists $c \in (1\,, x)$ such that

$$\frac{\ln x}{x^r - 1} = \frac{f(x) - f(1)}{g(x) - g(1)} = \frac{f'(c)}{g'(c)} = \frac{1/x}{rx^{r-1}} = \frac{1}{rx^r}.$$

Since $x > 1$ and $r > 0$, $rx^r > r$ and hence $1/(rx^r) < 1/r$. Therefore,

$$\frac{\ln x}{x^r - 1} < \frac{1}{r}.$$

4.4. #2. Because

$$\frac{|\sin(x) - x|}{|x|^3} = \frac{\sin(-x) - (-x)|}{|-x|^3} \qquad \forall x \in \mathbb{R},$$

it suffices to prove the assertion of the exercise for $x > 0$ [the case $x = 0$ holds trivially]. Choose and fix some $x > 0$. Define

$$f(y) = \sin(y) - y, \qquad g(y) = y^3 \qquad \forall y \in [0\,, x].$$

Because $g(y) > 0$ for all $y \in (0\,, x)$ and $g'(y) = 3y^2 > 0$ for all $y \in (0\,, x)$, the Cauchy mean-value theorem implies that there exists $c \in (0\,, x)$ such that

$$\frac{\sin(x) - x}{x^3} = \frac{f(x) - f(0)}{g(x) - g(0)} = \frac{f'(c)}{g'(c)} = \frac{\cos(c) - 1}{3c^2}.$$

We had proved in the lecture that $|1 - \cos\theta| \leq \frac{1}{2}\theta^2$ for all $\theta \in \mathbb{R}$. This implies that

$$\frac{|\sin x - x|}{|x|^3} \leq \frac{1}{6}.$$

Multiply boths sides by $|x|^3 > 0$ to complete the proof.

4.4. #10. If $x > 0$ then

$$x^x = \exp\{x \ln x\} = \exp\left\{\frac{\ln x}{1/x}\right\} = \exp\left\{-\frac{f(x)}{g(x)}\right\},$$

where $f(x) := -\ln x$ and $g(x) := 1/x$ for all $x > 0$; clearly, $\lim_{x\downarrow 0} f(x) = \infty$ and $\lim_{x\downarrow 0} g(x) = \infty$. Also, $g(x) \neq 0$ for all $x > 0$ and $g'(x) = -1/x^2 \neq 0$ for all $x > 0$. In particular, l'Hôpital's rule applies and tells us that

$$\lim_{x\downarrow 0} \frac{f(x)}{g(x)} = \lim_{x\downarrow 0} \frac{f'(x)}{g'(x)} = \lim_{x\downarrow 0} \frac{-1/x}{-1/x^2} = 0.$$

This and the continuity of the exponential function together yield $\lim_{x\downarrow 0} x^x = \mathrm{e}^0 = 1$.

4.4. #11. Similarly to above,

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} \exp\left\{\frac{\ln x}{x}\right\} = \exp\left\{\lim_{x\to\infty} \frac{\ln x}{x}\right\} = \exp\left\{\lim_{x\to\infty} \frac{1/x}{1}\right\} = \mathrm{e}^0 = 1.$$

4.4. #13. Let $f(x) := \ln x$ and $g(x) = \sqrt{x}$, both for $x > 0$. Then g and g' never vanish on $(0\,,\infty)$ and hence

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0.$$

In particular,

$$\lim_{n\to\infty} \frac{\ln(x_n)}{\sqrt{x_n}} = 0,$$

for any sequence $\{x_n\}_{n=1}^{\infty}$ that tends to infinity. Set $x_n := n$ to finish.

4.4. #14. We have to cheat a little and appeal to the fundamental theorem of calculus [not covered yet] in order to see that if $F(x) := \mathrm{e}^{f(x)}$ and $G(x) := \int_0^x F(t)\,\mathrm{d}t$ for all $x > 0$, then $G(x)$ and $G'(x) = \exp\{f(x)\}$ do not vanish for any $x > 0$. Therefore, l'Hôpital's rule implies that

$$\lim_{x\to\infty} \frac{\mathrm{e}^{f(x)}}{\int_0^x \mathrm{e}^{f(t)}\,\mathrm{d}t} = \lim_{x\to\infty} \frac{f'(x)\mathrm{e}^{f(x)}}{\mathrm{e}^{f(x)}} = L.$$