Math 3210–1, Summer 2016
Solutions to Assignment 7

4.3. #1. We appeal to the mean-value theorem: First, there exists $a \in (-1\,,0)$ such that

$$f'(a) = \frac{f(0) - f(-1)}{0 - (-1)} = 0.$$

Next, there exists $b \in (-1\,,1)$ such that

$$f'(b) = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{1}{2}.$$

Finally, there exists $c \in (0\,,1)$ such that

$$f'(c) = \frac{f(1) - f(0)}{1 - 0} = 1.$$

4.3. #2. Let $f(z) = \sin z$. Since $f'(z) = \cos z$ for all $z \in \mathbb{R}$, it follows from the mean-value theorem that for all $x < y$ there exists $c \in (x\,,y)$ such that

$$\cos(c) = f'(c) = \frac{f(y) - f(x)}{y - x} = \frac{\sin(y) - \sin(x)}{y - x}.$$

Since $|\cos(c)| \le 1$, it follows that $|\sin(y) - \sin(x)| \le |y - x|$.

4.3. #3. If $y = x$ then the assertion holds trivially because indeed $0 \le 0$. Therefore, we may assume that $y > x > 0$.

Let $f(z) = \ln z$ for all $z > 0$, and recall that $f'(z) = 1/z$ for all $z > 0$. For every $0 < x < y$, there exists $c \in (x\,,y)$ such that

$$\frac{1}{c} = f'(c) = \frac{f(y) - f(x)}{y - x} = \frac{\ln y - \ln x}{y - x}.$$

Therefore,

$$\ln y - \ln x = \frac{y - x}{c} \le \frac{y - x}{r},$$

for every $0 < r < c$ including $0 < r \le x$.

4.3. #4. For every $0 \le y < x$ there exists $c \in (y\,, x)$ such that

$$M \ge |f'(c)| = \frac{|f(x) - f(y)|}{x - y} \quad \Rightarrow \quad |f(x) - f(y)| \le M(y - x).$$

Let $y \to 0^+$ to see from the continuity of f on $[0\,,\infty)$ — and from the continuity of $g(y) := x - y$ on $[0\,,\infty)$ — that $|f(x)| \le Mx$ for all $x > 0$. The inequality holds trivially when $x = 0$ as well.

4.3. #5. Let $A = \lim_{x\to\infty} f(x)$ and $B = \lim_{x\to\infty} f'(x)$. For every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that for every $a \ge N_\varepsilon$:

(a) $A - \varepsilon \le f(a) \le A + \varepsilon$; and
(b) $B - \varepsilon \le f'(a) \le B + \varepsilon$.

Choose and fix an arbitrary $\varepsilon > 0$. For all $y > x > N_\varepsilon$, there exists $c \in (x\,, y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x}.$$

Since y and x both exceed N_ε, it follows from the triangle inequality that

$$|f(y) - f(x)| \le |f(y) - A| + |f(x) - A| \le 2\varepsilon.$$

And since $c > N_\varepsilon$, $f'(c) \ge B - \varepsilon$. Combine to see that

$$B - \varepsilon \le \frac{2\varepsilon}{y - x} \qquad \text{for all } y > x > N_\varepsilon. \tag{1}$$

Similarly,

$$\frac{2\varepsilon}{y - x} \le B + \varepsilon \qquad \text{for all } y > x > N_\varepsilon. \tag{2}$$

On one hand, (2) implies that $B + \varepsilon > 0$, equivalently, $B > -\varepsilon$. Since ε were arbitrary, it follows that $B \ge 0$. On the other hand, we can let $y \to \infty$ in (1) to see that $B - \varepsilon \le 0$, equivalently $B \le \varepsilon$, whence $B \le 0$. Combine to conclude that $B = 0$.

4.3. #6. $f'(x) = 6x^2 + 6x - 12 = 6(x^2 + x - 2) = 6(x-1)(x+2)$. If $x < -2$, then $f'(x) > 0$; if $x \in (-2\,, 1)$, then $f'(x) < 0$; and if $x \in (1\,,\infty)$, then $f'(x) > 0$. Therefore, f is increasing on $(-\infty\,, -2)$ and on $(1\,,\infty)$, whereas f is decreasing on $(-2\,, 1)$.

4.3. #7. Let $f(x) = x - 1 - \ln x$ for $x > 0$, and observe that $f'(x) = 1 - (1/x)$ for all $x > 0$. If $x < 1$, then $f'(x) < 0$; if $x > 1$, then $f'(x) > 0$. Therefore, f is decreasing on $(0\,, 1)$ and increasing on $(1\,,\infty)$. In other words, $f(x) > f(1) = 0$ for all $x > 0$ that satisfy $x \ne 1$. This proves the stronger inequality that $\ln x < x - 1$ for all $x > 0$ that satisfy $x \ne 1$ [of course, $\ln x = x - 1$ for $x = 1$].

4.3. #8. Let $f(x) = \mathrm{e}^{-x} x^{\mathrm{e}}$ for all $x \ge 0$. Then, f is continuous on $[0\,,\infty)$ and differentiable on $(0\,,\infty)$ with derivative

$$f'(x) = -\mathrm{e}^{-x} x^{\mathrm{e}} + \mathrm{e}^{-x} \mathrm{e} x^{\mathrm{e}-1} = \mathrm{e}^{-x} x^{\mathrm{e}-1}(x - \mathrm{e}) \qquad \forall x > 0.$$

This shows that f is increasing on $(\mathrm{e}\,,\infty)$ and decreasing on $(0\,,\mathrm{e})$. In particular, $f(x) > f(\mathrm{e}) = 1$ for all $x > 0$ that satisfy $x \ne \mathrm{e}$. In particular, $f(\pi) > 1$, equivalently, $\pi^{\mathrm{e}} > \mathrm{e}^{\pi}$.

4.3. #13. Here is a variation of 4.3.4: *If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and its derivative is bounded uniformly by some finite constant $M \geq 0$ then $|f(x) - f(y)| \leq M|y - x|$ for all $x, y \in \mathbb{R}$.* The proof is just the same as the proof of 4.3.4: For every $y > x$ there exists $c \in (x\,, y)$ such that

$$M \geq |f'(c)| = \frac{|f(y) - f(x)|}{y - x} \qquad \Rightarrow \qquad |f(y) - f(x)| \leq M|y - x|. \tag{3}$$

Since the roles of x and y are interchangeable, then same inequality holds also when $x > y$; and the inequality holds trivially when $x = y$ [$0 \leq 0$].

Now, let us assume that $M := r < 1$, so that f is a contraction mapping. Let $x_0 := 0$ [say] and having defined $x_0, \ldots, x_n$ define $x_{n+1} := f(x_n)$, inductively. Then,

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| \leq r|x_n - x_{n-1}| \qquad \forall n \geq 1.$$

Thus, $|x_2 - x_1| \leq r|x_1 - x_0| := Cr$, $|x_3 - x_2| \leq r|x_2 - x_1| \leq Cr^2$, $|x_4 - x_3| \leq Cr^3, \ldots$. By induction,

$$|x_{n+1} - x_n| \leq Cr^n \qquad \forall n \geq 0. \tag{4}$$

Next we note that $x_1 = x_0 + (x_1 - x_0) = (x_1 - x_0) = \sum_{j=0}^{0}(x_{j+1} - x_j)$, $x_2 = x_0 + (x_1 - x_0) + (x_2 - x_1) = \sum_{j=0}^{1}(x_{j+1} - x_j)$, etc. By induction,

$$x_{n+1} = \sum_{j=0}^{n} (x_{j+1} - x_j) \qquad \forall n \geq 0. \tag{5}$$

Observe that (4) and (5) together imply that $|x_{n+1}| \leq \sum_{j=0}^{\infty} |x_{j+1} - x_j| \leq C \sum_{j=0}^{\infty} r^j = C/(1 - r)$ for all $n \geq 0$. Therefore, $\{x_n\}_{n=0}^{\infty}$ is a bounded sequence. By the Bolzano–Weierstrass theorem there exists a subsequence $x_{n_1}, x_{n_2}, \ldots$ and a number $x \in \mathbb{R}$ such that $\lim_{k\to\infty} x_{n_k} = x$. Because

$$x_{n_{k+1}} = f(x_{n_k}) \qquad \forall k \geq 0, \tag{6}$$

we can let $k \to \infty$ to see that the left-hand side converges to x and the right-hand side converges to $f(x)$ [by continuity]. We have thus shown that there exists a number x that satisfies $x = f(x)$; this x is a fixed point of f.

4.3. #16. Let $f(x) := \ln x$ for all $x > 0$ and recall that $f'(x) = 1/x$. Since $|f'(x)| \leq 1$ for all $x \geq 1$, the same argument that led to (3) shows us that

$$|f(x) - f(y)| \leq |x - y| \qquad \forall x, y \geq 1.$$

In particular, for every $\varepsilon > 0$, whenever $x, y \in [1\,, \infty)$ satisfy $|x - y| < \delta := \varepsilon/2$, we have $|f(x) - f(y)| < \varepsilon$. Therefore, f is uniformly continuous on $[1\,, \infty)$. However, since f is unbounded on $(0\,, 1)$, f cannot have a continuous extension to $[0\,, 1]$; this is because every continuous function on $[0\,, 1]$ is necessarily bounded. We have thus shown that $f(x) = \ln x$ is not uniformly continuous on $(0\,, 1)$.