Math 3210–1, Summer 2016
Solutions to Assignment 6

3.4. #10. For all $n \in \mathbb{N}$ and $x \in (-1\,,1)$,

$$s_n(x) = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n.$$

According to Problem 9,

$$s_n(x) = \frac{1 - x^{n+1}}{1 - x}. \tag{1}$$

Here is the reason:

(a) $s_{n+1}(x) - s_n(x) = x^{n+1}$, so that $s_{n+1}(x) = s_n(x) + x^{n+1}$, for all $n \in \mathbb{N}$ and $x \in (-1\,,1)$;

(b) $xs_n(x) = x + \cdots + x^{n+1} = s_{n+1}(x) - 1$.

Plug (a) into (b) to see that $xs_n(x) = s_n(x) + x^{n+1} - 1$, which we can solve to obtain

$$(1 - x)s_n(x) = 1 - x^{n+1},$$

which is another way to state (1). Now, let us apply (1) to find that

$$s_n(x) - \frac{1}{1-x} = -\frac{x^{n+1}}{1-x}, \quad \text{therefore} \quad \left| s_n(x) - \frac{1}{1-x} \right| = \frac{|x|^{n+1}}{1-x} \qquad \forall n \in \mathbb{N},\ x \in (-1\,,1).$$

Optimize to find that if $0 < r < 1$, then

$$\sup_{x \in [-r,r]} \left| s_n(x) - \frac{1}{1-x} \right| = \sup_{x \in [-r,r]} \frac{|x|^{n+1}}{1-x} \le r^{n+1} \sup_{x \in [-r,r]} \frac{1}{1-x} = \frac{r^{n+1}}{1-r}.$$

[Challenge: Prove that the preceding inequality is in fact an identity.] In any case, it follows that

$$\lim_{r \to 0^+} \sup_{x \in [-r,r]} \left| s_n(x) - \frac{1}{1-x} \right| = \lim_{r \to 0^+} \frac{r^{n+1}}{1-r} = 0.$$

Therefore, $s_n(x) \to (1 - x)^{-1}$, uniformly for $x \in [-r\,, r]$, as $n \to \infty$. On the other hand, the same sort of computation shows that

$$\sup_{x \in (-1,1)} \left| s_n(x) - \frac{1}{1-x} \right| = \sup_{x \in (-1,1)} \frac{|x|^{n+1}}{1-x} \ge \frac{|y|^{n+1}}{1-y},$$

for any $y \in (-1\,,1)$. By picking $y \in (-1\,,1)$ to be arbitrarily close to one, we see that the preceding is $= \infty$, and hence cannot converge to 0 as $n \to \infty$.

3.4. #12. If $n, m \in \mathbb{N}$ satisfy $n \geq m$, then

$$s_n(x) - s_m(x) = \sum_{k=m+1}^{n} a_k x^k \quad \forall x \in (-1\,,1).$$

In particular, for every $r \in (0\,,1)$,

$$\sup_{x\in[-r,r]} |s_n(x) - s_m(x)| \leq \sup_{x\in[-r,r]} \sum_{k=m+1}^{n} |a_k||x|^k,$$

thanks to the triangle inequality. If $x \in [-r\,,r]$, then $|x|^k \leq r^k$ for all $k \in \mathbb{N}$. In particular,

$$\sup_{x\in[-r,r]} |s_n(x) - s_m(x)| \leq C \sum_{k=m+1}^{n} r^k \leq C \sum_{k=m+1}^{\infty} r^k,$$

where $C := \sup_{k\geq 1} |a_k|$ is a finite constant [because the a_ks are bounded]. Now we compute the geometric series in question to find that

$$\sup_{x\in[-r,r]} |s_n(x) - s_m(x)| \leq \frac{Cr^{m+1}}{1-r}.$$

Since $r^{m+1} \to 0$ as $m \to \infty$, it follows that for all $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $r^{m+1} \leq \varepsilon$ for all $m > N_\varepsilon$. Choose and fix an arbitrary $\varepsilon > 0$ to see that

$$\sup_{x\in[-r,r]} |s_n(x) - s_m(x)| \leq \frac{C\varepsilon}{1-r} \qquad \forall m, n \in \mathbb{N} \text{ s.t. } n \geq m > N_\varepsilon.$$

This implies that $\{s_n\}_{n=1}^{\infty}$ is uniformly Cauchy on $[-r\,,r]$. Since s_n is a polynomial, it is continuous on $(-1\,,1)$ for every $n \in \mathbb{N}$. Therefore, Theorems 3.4.4 and 3.4.10 together imply that $s(x) := \lim_{n\to\infty} s_n(x)$ exists for all $x \in [-r\,,r]$ and is a continuous function. Since this is true for every $r \in (0\,,1)$, s exists and is continuous for every $x \in (-1\,,1)$.

4.1. #8. Suppose to the contrary that $L_+ := \lim_{x\to 0^+} f(x)$ exists. It would then follow that $L = \lim_{n\to\infty} f(x_n)$ for every strictly decreasing positive sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \to 0$. But this is not so. For instance, $x_n := 2/(\pi n)$ defines a strictly decreasing sequence that converges to zero, yet $f(x_n) = \sin(\pi n/2)$ is infinitely often ± 1 and infinitely often zero, so there is no limit. By using $-x_n$ we see that $\lim_{x\to 0^-} f(x)$ also doesn't exist.

4.1. #9. Here,

$$f(x) = \begin{cases} -x & \text{if } x < 0, \\ \sin(x) & \text{if } x > 0. \end{cases}$$

Then $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(-x) = 0$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sin(x) = 0$. The two limits agree, therefore, $\lim_{x\to 0} f(x) = 0$ by Theorem 4.1.7.

4.1. #13. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of numbers in $(a\,,b)$ such that $x_n < x_{n+1}$ for all $n \geq 1$ and $\lim_{n\to\infty} x_n = a$. Then $\{f(x_n)\}_{n\in\mathbb{N}}$ is a bounded, strictly decreasing sequence. By Theorem 2.4.6,

$$L := \lim_{n\to\infty} f(x_n)$$

exists. Since f is bounded, L is a real number; that is $L \in \mathbb{R}$, equiv. $L \notin \{-\infty\,,\infty\}$. By the definition of limits, for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that

$$|f(x_n) - L| < \varepsilon \qquad \forall n > N_\varepsilon.$$

Choose and fix an arbitrary $\varepsilon > 0$. If $n > N_\varepsilon$ and $x < x_n$, there exists $m > n$ such that $x \in (x_{m+1}\,, x_m)$, which in turn implies that

$$f(x_{m+1}) \leq f(x) \leq f(x_m).$$

Because $m > n > N_\varepsilon$, $f(x_{m+1}) > L - \varepsilon$ and $f(x_m) < L + \varepsilon$. This proves that if $n > N_\varepsilon$ and $x \in (a\,, x_n)$, then $|f(x) - L| < \varepsilon$. This proves that $\lim_{x\to a^+} f(x) = L$. The same sort of argument can be used to prove that $\lim_{x\to a^-} f(x)$ exists and is a real number. I will leave the details to you.

4.2. #1.

$$\frac{f(x+h) - f(x)}{h} = \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right) = \frac{-1}{x(x+h)} \to -\frac{1}{x^2} \qquad \text{as } h \to 0.$$

4.2. #2. Set $f(x) := x^2 + 3x$. Then,

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 + 3(x+h) - x^2 - 3x}{h} = \frac{x^2 + 2xh + h^2 + 3x + 3h - x^2 - 3x}{h}$$
$$= 2x + 3 + h \to 2x + 3 \qquad \text{as } h \to 0.$$

Therefore, $f'(x) = 2x + 3$ for all x.

4.2. #3. Use the quotient rule [Theorem 4.2.6(d)]:

$$\left(\frac{\sin x}{\cos x}\right)' = \frac{(\cos x)^2 + (\sin x)^2}{(\cos x)^2} = \frac{1}{(\cos x)^2}.$$

4.2. #11. Define

$$f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}, \qquad g(x) = \begin{cases} x^2\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then, I claim that f is not differentiable at 0 and g is. Indeed,

$$\frac{f(x) - f(0)}{x - 0} = \sin(1/x)$$

does not have a limit as $x \to 0^+$ [see Exercise 4.1.9], whereas

$$\frac{g(x) - f(0)}{x - 0} = x\sin(1/x),$$

whence

$$\left|\frac{g(x) - f(0)}{x - 0}\right| = |x\sin(1/x)| \leq |x| \to 0 \qquad \text{as } x \to 0.$$