Math 3210–1, Summer 2016
Solutions to Assignment 5

3.1. #1. The question is really asking, "for which values of x is $x^2 - 1$ in $[0\,,1]$"? Equivalently, "for which values of x is x^2 in $[1\,,2]$"? Therefore, the domain is the union of $[-\sqrt{2}\,,-1]$ and $[1\,,\sqrt{2}]$.

3.1. #3. First of all, $f(x) := 1/(x^2+1)$ is well defined for all $x \in \mathbb{R}$. This is because $x^2+1 \geq 1 > 0$ for all $x \in \mathbb{R}$. To prove continuity we first observe that for all $x, y \in \mathbb{R}$,

$$f(x) - f(y) = \frac{1}{x^2+1} - \frac{1}{y^2+1} = \frac{y^2 - x^2}{(x^2+1)(y^2+1)}.$$

Therefore,

$$|f(x) - f(y)| = \frac{|x^2 - y^2|}{(x^2+1)(y^2+1)} = \frac{|x-y| \cdot |x+y|}{(x^2+1)(y^2+1)}.$$

For all real numbers x and y, we have: (a) $x^2 + 1 \geq 1$; (b) $y^2 + 1 \geq 1$; and (c) $|x+y| \leq |x| + |y|$. Therefore,

$$|f(x) - f(y)| \leq |x-y|(|x| + |y|). \tag{1}$$

Now we use the general inequality (1) in order to prove that $f(x) = 1/(x^2+1)$ is continuous at every point in $\mathbb{R}$. Choose and fix an arbitrary point $a \in \mathbb{R}$. It remains to prove that f is continuous at a regardless of the manner in which we chose a.

Choose and fix an arbitrary $\varepsilon > 0$, and define δ to be a positive number that is small enough to satisfy

$$\delta(2|a| + \delta) \leq \varepsilon.$$

There is no unique choice, of course. Just choose one.

For all $y \in \mathbb{R}$,

$$|y - a| \leq \delta \qquad \Rightarrow \qquad |f(a) - f(y)| \leq |y-a|(|a| + |y|) \leq \delta(2|a| + \delta) \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this proves that f is continuous at a. Since $a \in \mathbb{R}$ was arbitrary, this proves that f is continuous.

3.1. #9. f is *not* continuous if its domain were $\mathbb{R}$. Informally, this is because f has a jump at $x = 0$. Of course, the formal definition of continuity is more subtle. In order to *prove* that f is not continuous at 0, we show that there exists $\varepsilon > 0$ such that for every δ there exists $y \in \mathbb{R}$ such that $|y| \leq \delta$ and $|f(y) - f(0)| > \varepsilon$. Indeed, any $\varepsilon \in (0\,,2)$ works. For instance, set $\varepsilon = 1/2$. Then for every $\delta > 0$ and all $y \in (-\delta\,,0)$, $|f(0) - f(y)| = f(0) - f(y) = 1 - (-1) = 2 > \varepsilon$.

If the domain of f were $D := [0\,,\infty)$, then f would be continuous because f is a constant on D; that is, $|f(x) - f(y)| = 0 < \varepsilon$ for all $x, y \in D$ and $\varepsilon > 0$.

3.2. #1. Note that $x^2 - 2x + 1 = (x-1)^2 \geq 0$ for all $x \in \mathbb{R}$. Therefore, $f(x) \geq -1$ for all $x \in \mathbb{R}$. Because $f(1) = -1$ and $1 \in [0\,,3)$, the preceding shows that $f(x)$ is minimized at $x = 1$, and the minimum value of f over $[0\,,3)$ is $f(1) = -1$. For the maximum, we note that $f(x) = x(x-2)$. In particular, $f(x) \leq 0$ when $x \leq 2$ and $f(x) > 0$ when $x > 2$. This proves that if f had a maximum, then that maximum would have to occur somewhere in the interval $[2\,,3)$. Next I claim that

$$x^2 - 2x < y^2 - 2y \qquad \text{if } 2 \leq x < y. \tag{2}$$

In other words, I claim that f is a strictly increasing function on $[2\,,\infty)$. If (2) were true, then it would follows that $f(x) < f(3) = 3$ for all $x \geq 2$. Because the latter inequality is strict this shows that f is maximized at $x = 3$ and hence does not have a maximum in $[2\,,3)$.

It remains to prove (2). The assertion (2) is equivalent to the following: $y^2 - x^2 > 2(y - x)$ for all $y > x \geq 2$, which is in turn equivalent to the claim that $y + x > 2$ for all $y > x \geq 2$. Clearly, $y + x > 2$ for all $y > x \geq 2$; therefore, (2) holds.

3.2. #2. Because I is closed and bounded, Theorem 3.2.1 ensures that there exist two points $x, z \in I$ such that $f(x) \leq f(y) \leq f(z)$ for all $y \in I$. [In other words, the minimum of f is achieved at some point $x \in I$ and the maximum is achived at some point $z \in I$.] From now the argument is carried by considering three different cases.

Case 1. If $f(x) > 0$, then set $m := f(x)$ to see that $f(y) \geq m$ for all $y \in I$. This shows that such a number $m > 0$ exists when $f(x) > 0$ and proves the result in Case 1.

Case 2. If $f(z) < 0$, then set $m := -f(z)$ to see that $f(y) \leq -m$ for all $y \in I$. This shows that such a number $m > 0$ exists when $f(z) < 0$ and proves the asserted result in Case 1.

Case 3. If $f(x) < 0$ and $f(z) > 0$, then by the intermediate value theorem there would exist a point w, between x and z, such that $f(w) = 0$. This cannot be, therefore Case 1 and Case 2 are the only logical possible cases.

3.2. #4. Define $f(x) = 1/x$ for $x \in I := (0\,,1)$. Then f is continuous [this is proved as in Exercise 3.3.3] but clearly not bounded. The function $f(x) = x^2 - 2x$ is continuous and bounded on $[2\,,3)$ but does not have a maximum [see Exercise 3.2.1].

3.2. #5. Let $f(x) = x$ for all $x \in I := [1\,,\infty)$. Then f is easily seen to be continuous on I but not bounded. Neither does f have a maximum value on I.

3.2. #9. Define $g(x) = f(x) - x$ for all $x \in [0\,,1]$. Then, $g(0) = f(0) \geq 0$ and $g(1) = f(1) - 1 \leq 0$.

Case 1. If $g(0) = 0$ then this means that $f(0) = 0$ and hence there exists a fixed point at $x = 0$.

Case 2. If $g(1) = 1$ then a similar observation to the one in case 1 yields a fixed point at $x = 1$.

Case 3. In the remaning case, $g(0) > 0$ and $g(1) < 0$. Since g is a difference of two continuous functions it is continuous. Therefore the intermediate-value theorem yields some $y \in [0\,,1]$ such that $g(y) = 0$. By the definition of g, this y is a fixed point of f; i.e., $f(y) = y$.

3.3. #1. Yes, f is uniformly continuous on $(0\,,1)$. Here is why. Choose and fix $\varepsilon > 0$. Define $\delta = \varepsilon/2$. Then for all $x, y \in (0\,,1)$,

$$|y - x| < \delta \qquad \Rightarrow \qquad |f(x) - f(y)| = |x - y|(x + y) \leq 2|x - y| < 2\delta = \varepsilon.$$

3.3. #2. Since f is unbounded on $(0\,,1)$, it follows from the more general Exercise 3.3.7 below that f is not uniformly continuous [though it *is* continuous, from basic principles].

3.3. #7. Suppose to the contrary that f is uniformly continuous on its domain I. Then, Theorem 3.3.6 ensures f has a continuous extension to $\bar{I}$. Because f is unbounded on I, f would have to be also unbounded on the larger domain $\bar{I}$. But according to Theorem 3.2.1, a continuous function on a closed and bounded domain (here $\bar{I}$) attains its maximum and minimum and therefore cannot be unbounded.