Math 3210–1, Summer 2016
Solutions to Assignment 3

2.1. #2. If $|x-1| \le 1/2$ and $|y-2| < 1/2$, then $1/2 < x < 3/2$ and $3/2 < y < 5/2$. In particular, $x \neq y$.

2.1. #5. We plan to prove that

$$\lim_{n\to\infty} \frac{2n-1}{3n+1} = \frac{2}{3}.$$

To this end, we first note that for every natural number n,

$$\left|\frac{2n-1}{3n+1} - \frac{2}{3}\right| = \left|\frac{3(2n-1)-2(3n+1)}{3(3n+1)}\right| = \left|\frac{-5}{3(3n+1)}\right| = \frac{5}{3(3n+1)} \le \frac{5}{9n}.$$

Therefore, for every $\varepsilon > 0$,

$$\left|\frac{2n-1}{3n+1} - \frac{2}{3}\right| < \varepsilon \qquad \text{for all } n > N_\varepsilon := \frac{5}{9\varepsilon}.$$

This does the job.

2.1. #8. First, we may observe that

$$0 \le \sqrt{n+1} - \sqrt{n} = \frac{\left(\sqrt{n+1}-\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n}\right)}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{\sqrt{n}}.$$

Therefore,

$$\left|\sqrt{n+1} - \sqrt{n} - 0\right| = \sqrt{n+1} < \frac{1}{\sqrt{n}} \qquad \forall n \in \mathbb{N}.$$

From this we can conclude that for every $\varepsilon > 0$,

$$\left|\sqrt{n+1} - \sqrt{n} - 0\right| < \varepsilon \qquad \text{for all } n > N_\varepsilon := \frac{1}{\varepsilon^2}.$$

Equivalently, $\sqrt{n+1} - \sqrt{n} \to 0$ as $n \to \infty$.

2.2. #5. Let us write, for all $n \in \mathbb{N}$,

$$\sqrt{n^2+n} - n = \frac{\left(\sqrt{n^2+n}-n\right)\left(\sqrt{n^2+n}+n\right)}{\sqrt{n^2+n}+n} = \frac{n}{\sqrt{n^2+n}+n}.$$

Simplify to find that

$$\sqrt{n^2+n} - n = \frac{1}{\sqrt{1+\frac{1}{n}}+1}.$$

In particular,

$$\left|\sqrt{n^2+n} - n - \frac{1}{2}\right| = \left|\frac{1}{\sqrt{1+\frac{1}{n}}+1} - \frac{1}{2}\right| = \left|\frac{1-\sqrt{1+\frac{1}{n}}}{2\sqrt{1+\frac{1}{n}}+2}\right|.$$

Simplify further, by examining signs, in order to see that for all $n \in \mathbb{N}$,

$$\left|\sqrt{n^2+n} - n - \frac{1}{2}\right| = \frac{\sqrt{1+\frac{1}{n}}-1}{2\sqrt{1+\frac{1}{n}}+2} \le \frac{1}{2}\left(\sqrt{1+\frac{1}{n}}-1\right).$$

Multiply and divide by $\sqrt{1+n^{-1}}+1$ to see that, for all $n \in \mathbb{N}$,

$$\left|\sqrt{n^2+n} - n - \frac{1}{2}\right| \le \frac{1}{2}\frac{1/n}{\sqrt{1+\frac{1}{n}}+1} \le \frac{1}{2n}.$$

In particular, for every $\varepsilon > 0$,

$$\left|\sqrt{n^2+n} - n - \frac{1}{2}\right| < \varepsilon \qquad \forall n > N_\varepsilon = \frac{1}{2\varepsilon}.$$

This completes the proof.

2.2. #6. First, let us show that

$$1 + \frac{1}{n} \to 1 \qquad \text{as } n \to \infty.$$

Indeed, for every $\varepsilon > 0$,

$$\left|\left(1+\tfrac{1}{n}\right) - 1\right| = \frac{1}{n} < \varepsilon \qquad \forall n > N_\varepsilon := \frac{1}{\varepsilon}.$$

Claim. *If $a_n \to a$ and $b_n \to b$ then $a_n b_n \to ab$.*

Proof. Apply the triangle inequality to see that

$$\begin{aligned}|a_nb_n - ab| &= |a_nb_n - a_nb + a_nb - ab| \le |a_nb_n - a_nb| + |a_nb - ab| \\ &= |a_n||b_n - b| + |b||a_n - a|.\end{aligned}$$

Since $a_n \to a$, the sequence $a_1, a_2, \cdots$ is bounded [Corollary 2.2.4]. In other words, there exists a finite constant K such that $|a_n| \le K$ for all $n \in \mathbb{N}$. This yields,

$$|a_nb_n - ab| \le K|b_n - b| + |b||a_n - a|.$$

Let $L := \max\{K\,,|b|\}$ to see that

$$|a_n b_n - ab| \le L\left\{|b_n - b| + |a_n - a|\right\}.$$

For every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|b_n - b| < \varepsilon$ and $|a_n - a| < \varepsilon$ for all $n > N$. Therefore, it follows that

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}:\ |a_n b_n - ab| \le 2L\varepsilon \qquad \forall n > N.$$

This proves the claim. □

With the Claim under way, it is easy to finish the exercise: We verified that $1+n^{-1} \to 1$. The claim shows that $(1+n^{-1})^2 = (1+n^{-1})(1+n^{-1}) \to 1 \times 1 = 1$, and another appeal to the claim shows that

$$\left(1+\tfrac{1}{n}\right)^3 = \left(1+\tfrac{1}{n}\right)^2\left(1+\tfrac{1}{n}\right) \to 1 \times 1 = 1.$$

2.3. #8. By default, for all $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $|b_n - b| < \varepsilon$. Since $b > 0$, we can choose $\varepsilon := b/2$, specifically, in order to see that

$$|b_n - b| < \frac{b}{2} \qquad \forall n > N_{b/2}.$$

Now $|b_n - b| < b/2$ is equivalent to

$$\frac{b}{2} = b - \frac{b}{2} < b_n < b + \frac{b}{2} = \frac{3b}{2}.$$

This proves that $b_n > b/2$ for all $n > N_{b/2}$. Let $M := \min_{1 \le i \le N_{b/2}} b_i$ to see that: (i) $M > 0$; and (ii) $b_n > \min\{M\,, b/2\}$ for all n. This does the job with $m := \min\{M\,, b/2\}$.

2.3. #11. Because $b_n = n^{1/n} - 1$, we can solve to find that $n^{1/n} = b_n + 1$, equivalently,

$$b_n = n^{1/n} - 1 \qquad \text{for all } n \in \mathbb{N}.$$

Claim. $b_n \ge 0\ \forall n \in \mathbb{N}$.

Proof of Claim. $n^{1/n} - 1 \ge 0$ if and only if $n^{1/n} \ge 1$, which is equivalent to $n \ge 1^n = 1$. Since $n \ge 1$, the Claim follows. □

Since $n^{1/n} = b_n + 1$, it follows that $n = (1 + b_n)^n$. Therefore, by the Binomial theorem,

$$n = \sum_{i=0}^{n} \binom{n}{i} b_n^i \ge \binom{n}{2} b_n^2 = \frac{n(n+1)}{2} b_n^2.$$

I have used the Claim to justify the inequality. Solve to obtain

$$b_n \le \sqrt{\frac{2}{n+1}} \qquad \forall n \in \mathbb{N}.$$

2.3. #12. By the "squeeze argument," since $0 \le b_n = n^{1/n} - 1 \le \sqrt{2/(n+1)} \to 0$, it follows that $b_n \to 0$ as $n \to \infty$. That is, $n^{1/n} \to 1$ as $n \to \infty$.