Math 3210–1, Summer 2016
Solutions to Assignment 2

This solution key includes problem 9 from §1.5. But it also has solutions to the other problems from that homework set so that you can how I expect you to write mathematics when you turn it in. Hopefully by studying this solution key you can be better prepared for the midterm, as well.

Good luck and see you tomorrow,
Davar Khoshnevisan (May 31, 2016)

1.4. #1. (a) Let A denote the set of all odd integers.

Claim. A does not have an upper bound.

The claim shows that the set of the upper bounds of A is empty.

Proof of Claim. Suppose, to the contrary, that A has an upper bound n. Recall that this means that $n \in \mathbb{N}$ satisfies the following:

$$n > x \quad \text{for every } x \in A. \tag{1}$$

In particular, this means that $n \notin A$, which in turn implies that n is not an odd number. Therefore, n is an even number, which makes $n+1$ odd. Consequently, $n+1 \in A$. Because $n+1 \not< n$, we have found a contradiction to (1).

(b) Let $A = \{1 - 1/n : \ n \in \mathbb{N}\}$. If $x \in A$ then by definition there exists $n \in \mathbb{N}$ such that $x = 1 - 1/n$. This implies that if $x \in A$ then $x \leq 1$, whence 1 is an upper bound for A. Of course, any number greater than 1 is also an upper bound.

Claim. If $b < 1$ then b is not an upper bound for A.

We can conclude from this Claim that the collection of all upper bounds of A is exactly the interval $[1\,,\infty)$.

Proof of Claim. The strategy of proof is to show that if $b < 1$ then there exists $x \in A$ such that $x \not< b$, equivalently, $x \geq b$. Equivalently still, we need to show that if $b < 1$ then there exists $n \in \mathbb{N}$ such that $1 - 1/n \geq b$, equivalently, $\frac{1}{n} \leq 1 - b$. This follows from the Archimedean property of $\mathbb{R}$ (Example 1.4.9 on page 25).

(c) Let $A := \{r \in \mathbb{Q} : r^3 < 8\}$. If $r \in A$ then $r^3 < 8$, equivalently $r < 2$. Therefore, 2 is an upper bound, and so is every number ≥ 2.

Claim. If $b < 2$ then b is not an upper bound for A.

The Claim implies that the collection of the upper bounds of A coincides with $[2\,,\infty)$.

Proof of Claim. Choose and fix some number $b < 2$. Since $b^8 < 8$, it follows that $b \in A$. This means that b is a rational number that satisfies $b < 2$. Since b and 2 are both rational, we can find another rational number $c \in (b\,,2)$ [see, for example, Exercise 7 in this exercise set]. Since $c \in \mathbb{Q}$ and $c^3 < 8$, we can deduce that $c \in A$ and $c > b$, whence b is not an upper bound for A.

(d) Let $A := \{\sin x : x \in \mathbb{R}\}$. Because $\sin x \leq 1$ for all $x \in \mathbb{R}$, 1 is an upper bound for A, and hence so is every real number ≥ 1. We will prove that 1 is the least upper bound for A; that is, the set of all upper bounds of A is $[1\,,\infty)$.

If $b \in [-1\,,1)$, then there exist a real number $x \in [0\,,\pi/2)$ such that $b = \sin x$. This is because the sine function is strictly increasing, and hence invertible, on $[0\,,\pi/2)$, mapping $[0\,,\pi/2)$ onto $[-1\,,1)$. Now choose an arbitrary $c \in (b\,,1)$. The monotonicity of the sine function implies that: (i) There exists $y \in [0\,,\pi/2)$ such that $c = \sin y$; and (ii) $c > b = \sin x$. Consequently, $c \in A$ and hence b is not an upper bound for A.

Finally, if $b < -1$, then b is a lower bound for A and cannot be an upper bound for A. Therefore, in all cases, we see that if $b < 1$ then b is not an upper bound for A, as was claimed earlier.

1.4. #4. The curve $y = x^2 + x - 1$ has two real roots: $\frac{1}{2}[\sqrt{5} \pm 1]$. Since $x^2 + x - 1$ is unbounded as x becomes very large or very small, we see that

$$x \in A \quad \text{if and only if} \quad \frac{\sqrt{5}-1}{2} < x < \frac{\sqrt{5}+1}{2}.$$

In other words,

$$A = \left(\frac{\sqrt{5}-1}{2}\,,\frac{\sqrt{5}+1}{2}\right),$$

whose upper bound is $(\sqrt{5}+1)/2$.

1.4. #8. Since $r \in \mathbf{Q}$ and $r \neq 0$ we can find $n, m \in \mathbb{Z}$, $n, m \neq 0$, such that $r = n/m$. Now suppose to the contrary that $x + r$ is rational. In this case there must exist $p, q \in \mathbb{Z}$, $q \neq 0$, such that $x + r = p/q$. Equivalently,

$$x + \frac{n}{m} = \frac{p}{q}.$$

Solve to see that

$$x = \frac{p}{q} - \frac{n}{m} = \frac{mp - nq}{mq},$$

which is the ratio of two integers. This proves that x must be rational, which is a contradiction.

Similarly, we can suppose to the contrary that xr is rational. In that case there must exist $a, b \in \mathbb{Z}$, $b \neq 0$, such that $xr = a/b$.

$$x \times \frac{n}{m} = \frac{a}{b} \Leftrightarrow x = \frac{am}{nb},$$

which is the ratio of two integers. This implies that $x \in \mathbb{Q}$, which contradicts the hypothesis that x is irrational.

1.4. #9. Since $1 < \sqrt{2} < 2$ and $\sqrt{2}/2 = 1/\sqrt{2}$,

$$\frac{1}{2} < \frac{1}{\sqrt{2}} < 1.$$

Now $1/\sqrt{2}$ is an irrational number [for if it weren't then we could write it as n/m for $n, m \in \mathbb{Z}$ with $n, m \neq 0$, in which case $\sqrt{2} = m/n$ would have to be rational]. Multiply the preceding display by $s - r > 0$ then add r:

$$r + \frac{s-r}{2} < r + \frac{s-r}{\sqrt{2}} < r + (s - r).$$

Equivalently,

$$\frac{s+r}{2} < r + \frac{s-r}{\sqrt{2}} < s.$$

Because $(s + r)/2 > (r + r)/2 = r$, this yields

$$r < r + \frac{s-r}{\sqrt{2}} < s.$$

Since r and $s-r$ are rational and $1/\sqrt{2}$ is irrational, it follows that $x := r+(s-r)/\sqrt{2}$ is irrational [proof by contradiction, once again]. This x is an example of such an irrational number.

1.5. #2. (a) $\sup(-2\,,8] = \max(-2\,,8] = 8$, $\inf(-2\,,8] = -2$. Since $-2 \notin (-2\,,8]$, the latter inf is not a min.

(b) The sequence $(n + 2)/(n^2 + 1)$ decreases as n increases [this was proved in class]. Therefore, the supremum is the maximum, and is achieved at $n = 1$. That is, the supremum is $3/2$. At the same time, $(n + 2)/(n^2 + 1) \geq 0$ and tends to 0 as n becomes arbitrarily large. Therefore, the infimum of the set is 0, though 0 is not in the set; there is no minimum as result.

(c) As was shown in class, $\sqrt{5}$ is the supremum. Since $\sqrt{5}$ is not rational, it is not in the set itself. Therefore, the maximum does not exist. Similarly, the infimum is $-\sqrt{5}$ and there is no minimum.

1.5. #8. If $x \in A \cup B$ then $x \in A$ whence $x \leq \sup A$. Similarly, $x \leq \sup B$. Therefore, $\max\{\sup A\,,\sup B\}$ is an upper bound for $A \cup B$, whence

$$\sup(A \cup B) \leq \max\{\sup A\,,\sup B\}.$$

In order to prove the converse, note that if $x \in A$ then certainly $x \in A \cup B$ whence $x \leq \sup(A \cup B)$. This proves that $\sup A \leq \sup(A \cup B)$. Similarly, $\sup B \leq \sup(A \cup B)$, whence $\max\{\sup A\,,\sup B\} \leq \sup(A \cup B)$. Together with the previous displayed inequality, this shows that $\sup(A \cup B) = \max\{\sup A\,,\sup B\}$.

The results for the infima are similar and I will leave a careful proof to you [as a second go at this exercise].

1.5. #9. To be perfectly rigorous we need to avoid calculus in this problem, though knowledge of calculus allows for much simpler arguments here. ["Calculus" has not been described rigorously yet.]

(a) The maximum of f on I is 1; the minimum is 0. [Plot f!]

(b) Because

$$f(x) = \frac{x+1}{x-1} = \frac{x-1+2}{x-1} = 1 + \frac{2}{x-1} \qquad \text{for } x \in (1\,,2),$$

f is decreasing. Therefore, the supremum of $f(x)$ is at $x = 1$ and the infimum is at $x = 2$. There are no max/min however since $x = 1$ and $x = 2$ are not in I.

(c) $f(x) = x(2-x) > 0$ for all $x \in (0\,,1)$ and $f(0) = 0$. Therefore, f has a minimum at $x = 0$.

Claim. f is increasing on $[0\,,1)$.

If this Claim were true, then it would follow that the supremum of f over I is 1, though $f(x) < 1$ for all $x \in [0\,,1)$. This would imply that there is no maximum. It remains to prove the Claim.

Proof of Claim. We want to show that if $x, y \in [0\,,1)$ and $x < y$, then $f(x) \leq f(y)$. Equivalently, we want to prove that

$$2x - x^2 \leq 2y - y^2 \quad \Leftrightarrow \quad y^2 - x^2 - 2(y-x) \leq 0. \tag{2}$$

Yet equivalently, we want to prove that

$$(y-x)(y+x) - 2(y-x) \leq 0.$$

Since $y - x > 0$ we can divide both sides by $y - x$ without changing the inequality's direction in order to see that our goal is to prove that

$$y + x - 2 \leq 0 \quad \Leftrightarrow \quad y + x \leq 2.$$

But it is easy to see that $y + x \leq 2$ since both x and y are < 1.