Math 3210–1, Summer 2016
Solutions to Assignment 11

6.1. #7. The sum $\sum_{k=1}^{\infty}(-2/3)^k$ converges absolutely because $\sum_{k=1}^{\infty}(2/3)^k < \infty$, thanks to elementary properties of geometric series.

6.2. #8. Since $a_k = \sqrt{k}\mathrm{e}^{-\sqrt{k}}$ is not decreasing, one has to be careful, if one were to use the integral test. [It *can* be done this way though.] Instead, let us just observe that $k^2 a_k$ is bounded, say there exists a finite constant M such that $k^2 a_k \le M$ for all $k \ge 1$. If so, then this would imply that $a_k \le M/k^2$ in which case, $\sum_k |a_k| = \sum_k a_k \le M \sum_k k^{-2} < \infty$. This proves that that $\sum_k a_k$ converges.

Let's prove the more general fact that $f(x) = x^{5/2}\mathrm{e}^{-\sqrt{x}}$ is bounded on $[1\,,\infty)$. [This is more general because $k^2 a_k = f(k)$.] Now, f is differentiable and

$$f'(x) = \tfrac{5}{2}x^{3/2}\mathrm{e}^{-\sqrt{x}} - \tfrac{1}{2}x^2\mathrm{e}^{-\sqrt{x}} = \tfrac{1}{2}x^{3/2}\mathrm{e}^{-\sqrt{x}}\left[5 - \sqrt{x}\right].$$

Set this expression equal to zero to see that the critical point of f is at $x = 25$ with $f(25) = (5/\mathrm{e})^5$. If $x > 25$ then $f'(x) < 0$ and if $x < 25$ then $f'(x) > 0$. This proves that f is maximized at $x = 25$ and $f(x) \le (5/\mathrm{e})^5$. This does the job with $M = (5/\mathrm{e})^5$.

6.2. #11. We know that there exists a finite constant M such that $|b_k| \le M$ for all $k \ge 1$. Then, the partial sum of the absolute sum satisfies

$$\sum_{k=1}^{n} |a_k b_k| \le M \sum_{k=1}^{n} |a_k| \le M \sum_{k=1}^{\infty} |a_k|.$$

Therefore, $s_n := \sum_{k=1}^{n} |a_k b_k|$ is bounded, and hence $\sum_k a_k b_k$ converges absolutely.

6.3. #1. Since $a_k := 1/k^{1/3}$ decreases to zero, $\sum_k (-1)^k a_k$ converges. But $\sum_k a_k = \infty$ [p-series]. Therefore, $\sum_k a_k$ converges conditionally.

6.3. #11. Let $a_k = b_k = 2^{-k}$ to see that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} 2^{-n} = \sum_{n=0}^{\infty}(n+1)2^{-n}.$$

By the product formula, this quantity is equal to

$$\left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{k=0}^{\infty} b_k\right) = 2 \times 2 = 4,$$

thanks to the fact that $\sum_k a_k = \sum_k b_k = 2$, due to general facts about geometric series.

6.5. #1. Because

$$e^{|x|} = \sum_{n=0}^{\infty} \frac{|x|^n}{n!}$$

defines an absolutely convergence series, the summands must tend to zero; that is, $\lim_{n\to\infty} |x|^n/n! = 0$.

6.5. #4. The sum in question is

$$\sum_{k=0}^{n} \frac{1}{k!}.$$

According to Taylor's theorem [with $a = 0$], for all $x > 0$ there exists $c \in (0\,, x)$ such that

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{e^c x^{n+1}}{(n+1)!}.$$

Therefore, set $x = 1$ to see that there exists $c \in (0\,, 1)$ such that

$$\left| \sum_{k=0}^{n} \frac{1}{k!} - e \right| = \frac{e^c}{(n+1)!} < \frac{e}{(n+1)!} < \frac{3}{(n+1)!}.$$

It remains to choose n large enough to ensure that $c_n := 3/(n+1)! \le 0.001$. Now,

$$c_6 = \frac{3}{7!} = \frac{1}{1 \times 2 \times 4 \times 5 \times 6 \times 7} \approx 0.0006 < 0.001.$$

So, $n = 6$ works.

6.5. #12. Note that

$$\ln \left| \frac{e^{-1/x^2}}{x^n} \right| = -\frac{1}{x^2} - n \ln |x| = -\frac{1}{x^2} \left[1 + nx^2 \ln |x| \right].$$

By l'Hôpital's rule,

$$\lim_{x\downarrow 0} x^2 \ln x = \lim_{x\downarrow 0} \frac{\ln x}{1/x^2} = \lim_{x\downarrow 0} \frac{1/x}{-2/x^3} = 0.$$

Similarly,

$$\lim_{x\uparrow 0} x^2 \ln |x| = \lim_{x\downarrow 0} x^2 \ln x = 0.$$

Therefore,

$$\lim_{x\downarrow 0} \ln \left| \frac{e^{-1/x^2}}{x^n} \right| = -\infty,$$

which is another way to say that

$$\lim_{x\to 0} \left| \frac{e^{-1/x^2}}{x^n} \right| = 0.$$