Math 3210–1, Summer 2016
Solutions to Assignment 10

5.3. #3. Let $f(x) := \int_0^x \sin(t^2)\,\mathrm{d}t$ and $g(x) = x^2$ for all $x \in \mathbb{R}$. We are asked to find

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2} \sin(t^2)\,\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}x}(f \circ g)(x).$$

Apply the chain rule: $(f \circ g)' = (f' \circ g)g'$. Now, $g'(x) = 2x$ for all x; and $f'(x) = \sin(x^2)$ by the fundamental theorem of calculus [this is because $\sin(t^2)$ is a differentiable function on $\mathbb{R}$]. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}x}(f \circ g)(x) = f'(g(x))g'(x) = 2x\sin(x^4).$$

5.3. #4. Let $f(x) := \int_0^x \exp(-t^2)\,\mathrm{d}t$ for all $t \in \mathbb{R}$ in order to see that

$$f'(x) = \mathrm{e}^{-x^2},$$

owing to the fundamental theorem of calculus.

We are asked to find

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{1/x}^{x} \mathrm{e}^{-t^2}\,\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}x}\left(f(x) - f(1/x)\right) = f'(x) - f'(1/x)\left[-\frac{1}{x^2}\right] = f'(x) + \frac{f'(1/x)}{x^2}.$$

Combine the above two facts to see that

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{1/x}^{x} \mathrm{e}^{-t^2}\,\mathrm{d}t = \exp(-x^2) + x^{-2}\exp(-1/x^2).$$

5.3. #5. $f(x) = 1/x^2$ is not integrable near 0; therefore, the mentioned theorem does not apply.

5.3. #6. Define $g(x) = f(x)$ for all x. Then, for all $-\infty < a \le b < \infty$,

$$\int_a^b f(x)f'(x)\,\mathrm{d}x = f^2(b) - f^2(a) - \int_a^b f(x)f'(x)\,\mathrm{d}x.$$

Solve for the integral:

$$\int_a^b f(x)f'(x)\,\mathrm{d}x = \frac{f^2(b) - f^2(a)}{2}.$$

5.3. #9. Let $g(x) = f(x)$ for all $x \neq c$ and $g(c) \neq f(c)$. Define $h(x) := g(x) - f(x)$ for all $x \in \mathbb{R}$. Then, $h(x) = 0$ for all $x \neq c$ and $h(c) := g(c) - f(c) \neq 0$. Our goal is to prove that $\int_a^b h(x)\,\mathrm{d}x = b - a$.

Method 1. Perhaps the simplest proof is this: Define P_n to be the following partition of $[a\,,b]$: $P_n = \{x_0\,,x_1\,,x_2\,,x_3\}$, where

$$x_0 = a, \quad x_1 = c - n^{-1}, \quad x_2 = c + n^{-1}, \quad x_3 = b.$$

Of course, this makes sense only if $x_1, x_3 \in (1\,,b)$. That is, $c+n^{-1} < b$ and $c-n^{-1} > a$; equivalently, $n > N := \max\{(b-c)^{-1}\,,(c-a)^{-1}\}$.

Now, for every $n > N$, $U(f\,,P_n) = M_1(x_1 - x_0) + M_2(x_2 - x_1) + M_3(x_3 - x_2) = 2/n$ and $L(f\,,P_n) = 0$, manifestly. Therefore, for every $\varepsilon > 0$,

$$U(h\,,P_n) = U(h\,,P_n) - L(h\,,P_n) \leq \varepsilon \qquad \forall n > \max\left(\varepsilon^{-1}\,,N\right).$$

It follows that $\overline{\int}_a^b f = \underline{\int}_a^b h = 0$. Therefore, $\int_a^b h = 0$.

Method 2. Let $F(z) := \int_a^z h$ for all $z \in [a\,,b]$. Then, F is continuous on $(a\,,b)$ [by the fundamental theorem of calculus]. If $z \in (a\,,c)$, then

$$F(c) = F(c) - F(a) = (F(c) - F(z)) + (F(z) - F(a))\,.$$

Since $f(x) = g(x)$ every $x \in [a\,,z]$, we can deduce that $h = 0$ on $[a\,,z]$ and hence $F(z) - F(a) = \int_a^z h = 0$. Therefore, $F(c) - F(a) = F(c) - F(z)$ for all $z \in (a\,,c)$. Let $z \to c$ and appeal to continuity to see that $F(c) - F(a) = F(c) - \lim_{z\to c} F(z) = F(c) - F(c) = 0$. Consequently, $F(c) = F(a)$, which is zero.

5.4. #9. This is an improper integral; that is,

$$\int_1^\infty \frac{\mathrm{d}x}{x^p} = \lim_{a\to\infty} \int_1^a \frac{\mathrm{d}x}{x^p}.$$

If $p \neq 1$, then for all $a > 1$,

$$\int_1^a \frac{\mathrm{d}x}{x^p} = \frac{a^{1-p}}{1-p} - \frac{1}{1-p}.$$

If $p > 1$, then $a^{1-p} \to 0$ as $a \to \infty$; and hence

$$\int_1^\infty \frac{\mathrm{d}x}{x^p} = -\frac{1}{1-p} = \frac{1}{p-1} \qquad \text{if } p > 1.$$

On the other hand, $a^{1-p} \to \infty$ as $a \to \infty$; therefore, $\int_a^\infty x^{-p}\mathrm{d}x$ diverges. Finally, if $p = 1$, then

$$\int_1^a \frac{\mathrm{d}x}{x} = \ln(a) \to \infty \qquad \text{as } a \to \infty.$$

Therefore, the integral diverges in that case, as well.

Actually, it is not so easy to give an honest proof that $\lim_{a\to\infty}\ln(a)=\infty$. So let me do that here as well. [It is not so easy to prove this because we have not rigorously proved that its inverse $\exp(a)$ goes to infinity as $a\to\infty$.] If $k>1$ is an integer, then

$$\ln(\mathrm{e}^k)=\int_1^{\exp(k)}\frac{\mathrm{d}x}{x}=\sum_{j=0}^{k-1}\int_{\exp(j)}^{\exp(j+1)}\frac{\mathrm{d}x}{x},$$

thanks to the additive nature of the Riemann integral. If x is between e^j and e^{j+1}, then $1/x\ge \mathrm{e}^{-(j+1)}$, and hence by Theorem 5.2.4,

$$\ln(\mathrm{e}^k)\ge\sum_{j=0}^{k-1}\int_{\exp(j)}^{\exp(j+1)}\exp(-j-1)\,\mathrm{d}x=\sum_{j=0}^{k-1}\left[\mathrm{e}^{j+1}-\mathrm{e}^j\right]\mathrm{e}^{-j-1}=\sum_{j=0}^{k-1}\left(1-\mathrm{e}^{-1}\right)=k\left(1-\mathrm{e}^{-1}\right).$$

Therefore, $\ln(\mathrm{e}^k)\to\infty$ as $k\to\infty$. In particular, for all $N\in\mathbb{N}$ there exists $K\in\mathbb{N}$ such that $\ln(\mathrm{e}^k)\ge N$ for all $k\ge K$. Since ln is increasing, $\ln(a)\ge N$ for all $a\ge \mathrm{e}^K$ as well. Therefore, $\lim_{a\to\infty}\ln(a)=\infty$.

5.4. #10. This is an improper integral: $\int_0^1 x^{-p}\mathrm{d}x=\lim_{a\downarrow 0}\int_a^1 x^{-p}\mathrm{d}x$. Now, for all $a\in(0\,,1)$,

$$\int_a^1\frac{\mathrm{d}x}{x^p}=\begin{cases}-\ln(a) & \text{if } p=1,\\ \dfrac{1-a^{1-p}}{1-p} & \text{if } p\neq 1.\end{cases}$$

Therefore, $\int_0^1 x^{-p}\mathrm{d}x$ converges if and only if $0<p<1$, in which case,

$$\int_0^1\frac{\mathrm{d}x}{x^p}=\frac{1}{1-p}\qquad\text{when } 0<p<1.$$

5.4. #12. Let $f(x):=\ln x$ and $g(x):=x$ for all $0<x<1$. Then, for all $a\in(0\,,1)$,

$$\begin{aligned}\int_a^1\ln(x)\,\mathrm{d}x&=\int_a^1 f(x)g'(x)\,\mathrm{d}x=f(1)g(1)-f(a)g(a)-\int_a^1 f'(x)g(x)\,\mathrm{d}x\\&=-a\ln(a)-(1-a).\end{aligned}$$

By the l'Hôpital's rule,

$$\lim_{a\downarrow 0}a\ln(a)=\lim_{a\downarrow 0}\frac{\ln(a)}{1/a}=\lim_{a\downarrow 0}\frac{1/a}{-1/a^2}=0.$$

Therefore, $\int_0^1\ln(x)\,\mathrm{d}x=\lim_{a\downarrow 0}\int_a^1\ln(x)\,\mathrm{d}x=-1$, and in particular converges.