Math 3210-1, Final Exam

Thursday August 4, 2016

---

1. Prove that $\sqrt{5}$ is an irrational number.

   Solution. By Theorem 1.3.9, if $k \in \mathbb{Z}$, then all rational solutions to the equation $x^2 = k$ are integers. in particular, set $k = 5$ to see that if $\sqrt{5}$ were rational then it would have to be an integer. Now it is easy to see that $2 < \sqrt{5} < 3$, whence $\sqrt{5} \notin \mathbb{Z}$.

---

2. Suppose that $a_1, a_2, \ldots$ is a sequence of real numbers that satisfy $a_n < 1$ for all $n \in \mathbb{N}$.

   (a) (5 points) Prove that $\limsup_{n\to\infty} a_n \le 1$.

   (b) (5 points) Can one improve the preceding to $\limsup_{n\to\infty} a_n < 1$? Justify your answer.

   Solution. (a) Because 1 is an upper bound for $\{a_n\}_{n=1}^{\infty}$, 1 is greater than or equal to the smallest upper bound for $\{a_n\}_{n=1}^{\infty}$. In other words, $\sup_{n\ge 1} a_n \le 1$. Therefore in particular, $s_k = \sup_{n\ge k} a_n \le 1$ for all $k \ge 1$. Let $k \to \infty$ to see that $\limsup_{n\to\infty} a_n = \lim_{k\to\infty} s_k \le 1$, thanks to the definition of lim sup.

   (b) No. For instance, consider the sequence $a_n = 1 - \mathrm{e}^{-n}$ for all $n \ge 1$.

---

3. Let $\alpha > 0$ be fixed.

   (a) Prove that if $\alpha \le 1$, then $(1+x)^\alpha \le 1 + \alpha x \ \forall x \ge 0$.

   (b) Prove that if $\alpha > 1$, then $(1+x)^\alpha \ge 1 + \alpha x \ \forall x \ge 0$.

   Solution. Define $f(x) = (1+x)^\alpha - \alpha x \quad \forall x \ge 0$. Then f is differentiable on $(0\,,\infty)$ and continuous on $[0\,,1]$; moreover, $f'(x) = \alpha(1+x)^{\alpha-1} - \alpha \quad \forall x > 0$. If $\alpha \in (0\,,1]$, then $(1+x)^{\alpha-1} \le 1$, whence $f'(x) \le 0$ for all $x > 0$. In particular, $f(x) \le f(0) = 1$ for all x. This proves (a).

   Conversely, if $\alpha > 1$ then $(1+x)^{\alpha-1} > 1$, whence $f'(x) > 0$ for all $x > 0$. In particular, $f(x) \ge f(0) = 1$ for all x. This proves (b).

---

4. Let $f : [0\,,1] \to \mathbb{R}$ be an integrable function, and define $g(x) = f(x)$ for all $x \in [0\,,1]$ except $x = 1/2$. Define $g(1/2) = c$ for an arbitrary $c \ne f(1/2)$. Prove that g is integrable and $\int_0^1 g = \int_0^1 f$.

   Solution. Let $g(x) = f(x)$ for all $x \ne 1/2$ and $g(1/2) \ne f(1/2)$. Define $h(x) := g(x) - f(x)$ for all $x \in \mathbb{R}$. Then, $h(x) = 0$ for all $x \ne 1/2$ and $h(1/2) := g(1/2) - f(1/2) \ne 0$. Our goal is to prove that h is integrable and $\int_a^b h(x)\,\mathrm{d}x = 0$. Define P_n to be the following partition of $[a\,,b]$: $P_n = \{x_0\,, x_1\,, x_2\,, x_3\}$, where

$$x_0 = 0, \quad x_1 = \tfrac{1}{2} - n^{-1}, \quad x_2 = \tfrac{1}{2} + n^{-1}, \quad x_3 = 1.$$

Of course, this makes sense only if $x_1, x_3 \in (0\,,1)$. That is, $\frac{1}{2}+n^{-1} < 1$ and $\frac{1}{2}-n^{-1} > 0$; equivalently, $n > 2$. Now, for every $n > 2$,

$$U(f\,,P_n) = M_1(x_1 - x_0) + M_2(x_2 - x_1) + M_3(x_3 - x_2) = 2/n \qquad \text{and} \qquad L(f\,,P_n) = 0.$$

Therefore, for every $\varepsilon > 0$,

$$U(h\,,P_n) = U(h\,,P_n) - L(h\,,P_n) = \frac{2}{n} \le \varepsilon \qquad \forall n > \max\left(\varepsilon^{-1}\,,2\right).$$

It follows that $\overline{\int}_a^b f = \underline{\int}_a^b h = 0$. Therefore, h is integrable and $\int_a^b h = 0$.

---

5. Suppose $f, g, h : [a\,,b] \to \mathbb{R}$ are three integrable functions on a bounded and closed interval $[a\,,b]$. Prove that $F(x) := f(x)g(x)h(x)$ defines an integrable function on $[a\,,b]$. If you use any results from the textbook and/or homework then you should recall the result carefully. There is no need to prove those results here though.

   Solution. $q = fg$ is integrable by a theorem in the textbook. Therefore, $F = qh$ is integrable.

---

6. Let $f(x) = \mathrm{e}^{-1/|x|}$ for all nonzero $x \in \mathbb{R}$, and $f(0) := 0$.

   (a) (10 points) Prove that f differentiable at $x = 0$ and $f'(0) = 0$.

   (b) (10 points; extra credit) Does f' exist at any other $x \neq 0$? Justify your answer.

   Solution. (a) For all $x \neq 0$,

$$\left|\frac{f(x) - f(0)}{x} - 0\right| = \frac{\exp(-1/|x|)}{|x|}.$$

   We saw in a problem session that $r^{-1}\exp(-1/r) \to 0$ as $r \downarrow 0$. Similarly, $(-r)\exp(1/r) \to 0$ as $r \uparrow 0$. This proves that $\lim_{x\uparrow 0} \frac{f(x)-f(0)}{x-0} = \limsup_{x\downarrow 0} \frac{f(x)-f(0)}{x} = 0$. Therefore, $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0$, which in turn implies that f is differentiable at $x = 0$ and $f'(0) = 0$.

   (b) Yes, and for simple reasons: If $x > 0$, then $f(x) = \exp(-1/x)$, which is a composition of a function that is differentiable everywhere [the exponential function] with a function that is differentiable at every $x > 0$ [the function $-1/x$]. Therefore, f is differentiable at every point $x > 0$, and $f'(x) = x^{-2}\exp(-1/x)$ for all $x > 0$. Similarly, $f(x) = \exp(1/x)$ $[x < 0]$ is differentiable at every point $x < 0$ and the derivative is $f'(x) = -x^{-2}\exp(1/x)$ for all $x < 0$. In other words,

$$f'(x) = \begin{cases} \dfrac{\mathrm{sign}(x)\mathrm{e}^{-1/|x|}}{x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$