# On Sines and Cosines
# Math 3150–Summer 2006

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## 1 Trigonometry via Complex Numbers

The right definition of sin and cos is as follows:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Because cos is an even function and sin is odd, we have also the identity,

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta.$$

Add the preceding together to find that $e^{i\theta} + e^{-i\theta} = 2\cos\theta$; i.e.,

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}. \tag{1.1}$$

Also, subtract to find that $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$; i.e.,

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \tag{1.2}$$

Apply (1.1) with $\theta := \alpha$ and (1.2) with $\theta := \beta$ to find that

$$\begin{aligned}
\cos\alpha\,\sin\beta &= \left(\frac{e^{i\alpha}+e^{-i\alpha}}{2}\right)\left(\frac{e^{i\beta}-e^{-i\beta}}{2i}\right) \\
&= \frac{e^{i(\alpha+\beta)}-e^{i(\alpha-\beta)}+e^{-i(\alpha-\beta)}-e^{-i(\alpha+\beta)}}{4i} \\
&= \frac{e^{i(\alpha+\beta)}-e^{-i(\alpha+\beta)}}{4i}-\frac{e^{i(\alpha-\beta)}-e^{-i(\alpha-\beta)}}{4i} \\
&= \frac{\sin(\alpha+\beta)}{2}-\frac{\sin(\alpha-\beta)}{2} \\
&= \frac{\sin(\alpha+\beta)-\sin(\alpha-\beta)}{2}.
\end{aligned} \tag{1.3}$$

Similarly,

$$\begin{aligned}
\cos\alpha\,\cos\beta &= \left(\frac{e^{i\alpha}+e^{-i\alpha}}{2}\right)\left(\frac{e^{i\beta}+e^{-i\beta}}{2}\right) \\
&= \frac{e^{i(\alpha+\beta)}+e^{i(\alpha-\beta)}+e^{-i(\alpha-\beta)}+e^{-i(\alpha+\beta)}}{4} \\
&= \frac{e^{i(\alpha+\beta)}+e^{-i(\alpha+\beta)}}{4}+\frac{e^{i(\alpha-\beta)}+e^{-i(\alpha-\beta)}}{4} \\
&= \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}.
\end{aligned} \tag{1.4}$$

Finally, because $i^2 = -1$,

$$\begin{aligned}
\sin\alpha\,\sin\beta &= \left(\frac{e^{i\alpha}-e^{-i\alpha}}{2i}\right)\left(\frac{e^{i\beta}-e^{-i\beta}}{2i}\right) \\
&= -\frac{e^{i(\alpha+\beta)}-e^{i(\alpha-\beta)}-e^{-i(\alpha-\beta)}+e^{-i(\alpha+\beta)}}{4} \\
&= -\frac{e^{i(\alpha+\beta)}+e^{-i(\alpha+\beta)}}{4}+\frac{e^{i(\alpha-\beta)}+e^{-i(\alpha-\beta)}}{4} \\
&= \frac{-\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}.
\end{aligned} \tag{1.5}$$