$\boxed{\text{Solutions}}$

Midterm #3
Mathematics 3150–2, Summer 2006
Department of Mathematics, University of Utah
July 17, 2006

Name: ______________________________
Student ID Number: ______________________________

* This is a closed-book, closed-notes examination.
* This exam begins at 11:00 a.m. and ends at 12:00 p.m. sharp.
* This exam is made up of 2 questions for a total of 20 points.
* Write your answers clearly. **If you show merely a numerical answer, then you are likely to receive zero partial credit.** So show and explain your work.
* **Confine your work to this worksheet.** You may use both sides of the paper. There is also an extra sheet of paper per problem for you to write on if you wish.
* There is a formula sheet in the very back of this worksheet which you may use at will.

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1. (10 points) Compute $B_{m,n}$, for all $m, n \geq 1$, where

$\boxed{a=b=1}$.

$$x + y = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{m,n} \sin(m\pi x)\sin(n\pi y), \qquad 0 < x < 1,\ 0 < y < 1.$$

$$B_{m,n} = 4\int_0^1 \underbrace{\left(\int_0^1 (x+y)\sin(m\pi x)\,dx\right)}_{\text{Call this } I.} \sin(n\pi y)\,dy$$

$$I = \int_0^1 x\sin(m\pi x)\,dx + y\int_0^1 \sin(m\pi x)\,dx.$$

The 1^{st} integral is computed by parts:

$$\left.\begin{aligned} u &= x & v' &= \sin(m\pi x) \\ u' &= 1 & v &= \frac{-1}{m\pi}\cos(m\pi x) \end{aligned}\right] \text{ This yields:}$$

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$$(*) \quad \boxed{\int_0^1 x \sin(m\pi x)\,dx = -\frac{\cos(m\pi)}{m\pi} = \frac{(-1)^{m+1}}{m\pi}.}$$

Also,

$$(**) \quad \boxed{\int_0^1 \sin(m\pi x)\,dx = \frac{1-\cos(m\pi)}{m\pi} = \frac{1-(-1)^m}{m\pi}}$$

$$\Rightarrow \quad I = \frac{(-1)^{m+1}}{m\pi} + y\,\frac{1-(-1)^m}{m\pi}.$$ Plug back into $B_{m,n}$:

$$\Rightarrow \quad B_{m,n} = 4\int_0^1 \left[\frac{(-1)^{m+1}}{m\pi} + y\,\frac{1-(-1)^m}{m\pi}\right]\sin(n\pi y)\,dy$$

$$= \frac{4(-1)^{m+1}}{m\pi}\int_0^1 \sin(n\pi y)\,dy + \frac{4\left[1-(-1)^m\right]}{m\pi}\int_0^1 y \sin(n\pi y)\,dy$$

$$= \underbrace{\frac{4(-1)^{m+1}\left[1-(-1)^n\right]}{mn\pi^2}}_{(\text{by } **)} + \underbrace{\frac{4\left[1-(-1)^m\right](-1)^{n+1}}{mn\pi^2}}_{(\text{by } *)}$$

$$= \frac{4}{mn\pi^2}\left\{(-1)^{m+1}\left[1-(-1)^n\right] + (-1)^{n+1}\left[1-(-1)^m\right]\right\}$$

$$= \begin{cases} 0 & \text{if } m \text{ and } n \text{ even,} \\ 8/mn\pi^2 & \text{if } m \text{ odd, } n \text{ even; or } m \text{ even, } n \text{ odd,} \\ -16/mn\pi^2 & \text{if } m \text{ and } n \text{ odd.} \end{cases}$$

$c=2$ $a=1$ $b=1$

2. (10 points) Solve the two-dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = 4\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad 0 < x < 1,\ 0 < y < 1,\ t > 0,$$

with **boundary conditions**

$$u(0, y, t) = u(1, y, t) = 0 \qquad 0 < y < 1,\ t > 0,$$
$$u(x, 0, t) = u(x, 1, t) = 0 \qquad 0 < x < 1,\ t > 0,$$

and **initial condition**

$$u(x, y, 0) = 100 \qquad 0 < x < 1,\ 0 < y < 1.$$

$B^*_{mn} = 0$ in the formula.

$f(x,y) = 100.$

$\lambda_{mn} = 2\pi\sqrt{m^2+n^2}$

$$\Rightarrow \quad B_{m,n} = 400 \int_0^1 \sin(m\pi x)\,dx \int_0^1 \sin(n\pi y)\,dy$$

$$= 400\, \frac{1-(-1)^m}{m\pi}\, \frac{1-(-1)^n}{n\pi} \qquad (\text{by } {**})$$

$$= \begin{cases} \dfrac{1600}{mn\pi^2} & \text{if } m \text{ and } n = \text{odd}, \\[2ex] 0 & \text{otherwise.} \end{cases}$$

$$\therefore \quad u(x,y,t) = \frac{1600}{\pi^2} \sum_{\substack{m=1 \\ \text{both odd}}}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn} \cos\left(2\pi\sqrt{m^2+n^2}\, t\right).$$