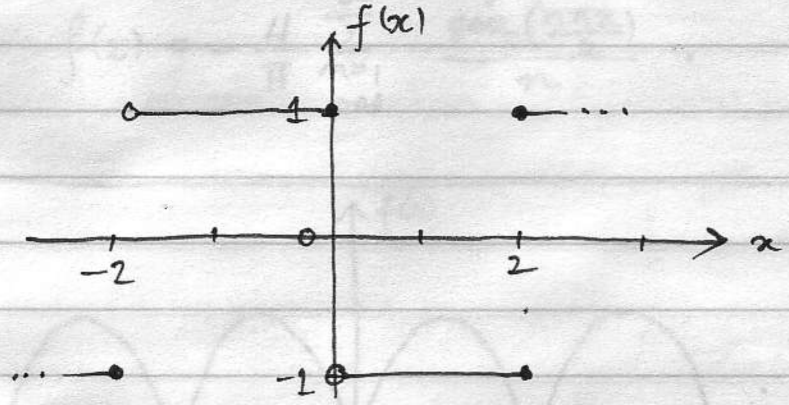
Solutions to Midterm #1 : Math 3150-2

(1a)



$\therefore \quad f(1-) = -1 \qquad\qquad f(2+) = +1.$

(1b) Odd function

(1c) • f is odd $\Rightarrow$ $a_0 = a_1 = \dots = 0$

• f is 4-period $\Rightarrow$ $p=2$

• for all $n \geq 1$,

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^2 (-1)\sin\left(\frac{n\pi x}{2}\right)dx$$

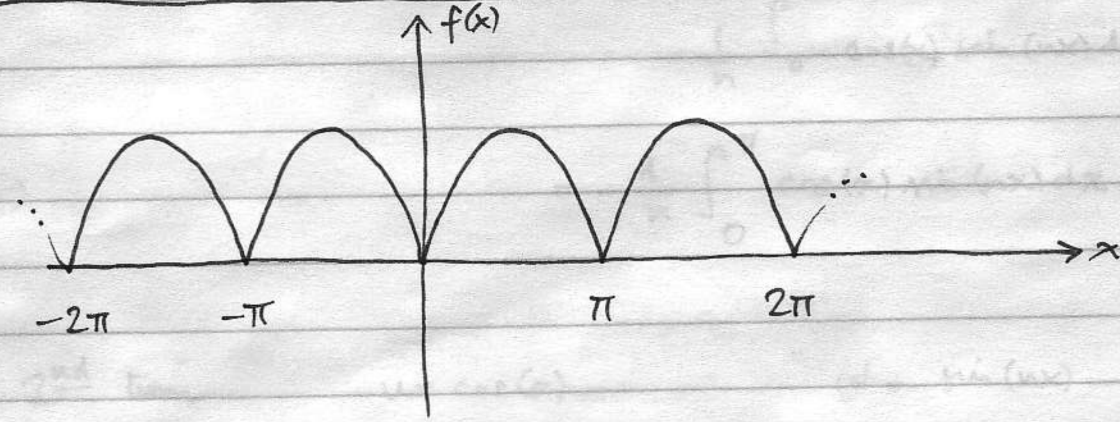
$$= \frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Bigg]_0^2 = \frac{2}{n\pi}\left[(-1)^n - 1\right]$$

$$= \begin{cases} -4/n\pi & \text{if } n \text{ odd,} \\ 0 & \text{if } n \text{ even.} \end{cases}$$

Therefore, for all $x \in (-2,2) \setminus \{0\}$,

$$f(x) = -\frac{4}{\pi} \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{\sin\left(\frac{n\pi x}{2}\right)}{n} .$$

(2a)



(2b) f is even, so $b_n = 0$ for all $n \geq 1$. $p = \pi$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi} \int_0^{\pi} \sin x \, dx$$

$$= -\frac{1}{\pi} \cos x \Big|_0^{\pi} = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\,dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \sin(x) \cos(nx)\,dx .$$

Integrate by parts, <u>twice</u>.

1st time $\quad u = \sin x \qquad v' = \cos(nx)$

$u' = \cos x \qquad v = \frac{1}{n}\sin(nx).$

$$\therefore \quad \int_0^{\pi} \sin(x)\cos(nx)\,dx = \frac{1}{n}\sin(x)\sin(nx)\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\cos(x)\sin(nx)\,dx,$$

$$\circledast \qquad = -\frac{1}{n}\int_0^{\pi}\cos(x)\sin(nx)\,dx.$$

2nd time $\quad u = \cos(x) \qquad v' = \sin(nx)$

$u' = -\sin(x) \qquad v = -\frac{1}{n}\cos(nx)$

$$\therefore \quad \int_0^{\pi}\cos(x)\sin(nx)\,dx = -\frac{1}{n}\cos(x)\cos(nx)\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin x\cos(nx)\,dx$$

$$= \frac{1}{n}\left[\cos(n\pi)+1\right] - \frac{1}{n}\int_0^{\pi}\sin(x)\cos(nx)\,dx.$$

Plug into $\circledast$:

$$\int_0^{\pi}\sin x\cos nx\,dx = -\frac{1}{n^2}\left[\cos(n\pi)+1\right] + \frac{1}{n^2}\int_0^{\pi}\sin x\cos(nx)\,dx$$

$$\therefore \quad \int_0^{\pi}\sin x\cos(nx)\,dx = \frac{n^2}{n^2-1}\left[-\frac{1}{n^2}\left(\cos(n\pi)+1\right)\right]$$

$$= -\frac{1+\cos(n\pi)}{n^2-1}.$$

Therefore,

$$a_n = \frac{-2}{\pi}\,\frac{1+\cos(n\pi)}{n^2-1}$$ (see p.2)

$$= \begin{cases} \dfrac{-4}{\pi}\cdot\dfrac{1}{n^2-1}, & \text{if } n \text{ even,} \\ 0\,, & \text{if } n \text{ odd.} \end{cases}$$

$$\Rightarrow \qquad f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{\substack{n=1 \\ \text{even}}}^{\infty} \frac{\cos(nx)}{n^2-1}$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos(2kx)}{(2k)^2-1},$$

as asserted.