Final Examination
Mathematics 3150–2, Summer 2006
Department of Mathematics, University of Utah
August 4, 2006

Name: Solutions
Student ID Number:

* This is a closed-book, closed-notes examination.
* This exam begins at 10:00 a.m. and ends at 12:00 p.m. sharp.
* This exam is made up of 4 questions for a total of 20 points.
* Write your answers clearly. **If you show merely a numerical answer, then you are likely to receive zero partial credit.** So show and explain your work.
* **Confine your work to this worksheet.** You may use both sides of the paper. There is also an extra sheet of paper per problem for you to write on if you wish.
* There is a formula sheet in the very back of this worksheet which you may use at will.

1. (15 points total) Let

$$f(x) = e^{-x^2/2}.$$

(a) (5 points) Compute $\hat{f}(\omega)$.

$$\hat{f}(\omega) = e^{-\omega^2/2}$$

(b) (10 points) Compute $f * f$.

$$\widehat{f*f}(\omega) = e^{-\omega^2/2}\, e^{-\omega^2/2} = e^{-\omega^2}$$

Let $g = f*f$. To determine g so we need to find g so that $\hat{g}(\omega) = e^{-\omega^2}$. Let $g(x) = \frac{1}{\sqrt{2}} e^{-x^2/4}$. Then $g(x) = \frac{1}{\sqrt{2}} e^{-ax^2/2}$ for $a = \frac{1}{2}$, so $\hat{g}(\omega) = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{a}} e^{-\omega^2/2a} = e^{-\omega^2}$. That is,

$$\boxed{(f*f)(x) = \frac{e^{-x^2/4}}{\sqrt{2}}}$$

2. (10 points) Solve the one-dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < 1,\ t > 0,$$

with **boundary conditions**

$$u(0\,,t) = u(1\,,t) = 0 \qquad t > 0,$$

and **initial conditions**

$$u(x\,,0) = 1,\ \frac{\partial}{\partial t}u(x\,,0) = 0 \qquad 0 < x < 1.$$

$b_n^* = 0.$ $\quad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$$= 2\int_0^1 1 \sin(n\pi x)\,dx$$

$$= \frac{2}{n\pi}(1-\cos n\pi).$$

Plug :

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(1-\cos n\pi)\cos(n\pi t)\sin(n\pi x).$$

3. (5 points) Solve the PDE

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \qquad -\infty < x < \infty,\ t > 0,$$

subject to

$$u(x,0) = e^{-x^2/2} \quad — \quad \hat{u}(w,0) = e^{-w^2/2} \qquad -\infty < x < \infty.$$

Let $\hat{u}(w,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iwx}\, u(x,t)\, dx$ so that

$$0 = \frac{\partial}{\partial t}\hat{u} + \widehat{\frac{\partial u}{\partial x}}$$

$$= \frac{\partial}{\partial t}\hat{u} + iw\hat{u}$$

$$\hat{u}(w,t) = A(w)\, e^{-iwt} \rightarrow \quad A(w) = e^{-w^2/2} \rightarrow$$

$$\hat{u}(w,t) = e^{-iwt - w^2/2}$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iw(x-t)}\, e^{-w^2/2}\, dw$$

$$= \widehat{e^{-x^2/2}}\,(t-x) = e^{-(t-x)^2/2}.$$

4. (10 points) Compute the coefficients $B_{n,m}$, for all $n, m \geq 1$, where

$$1 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{n,m} \sin(n\pi x) \sin(m\pi y) \qquad -\pi < x < \pi, \ -\pi < y < \pi.$$

can't be true because $x = y = 0$ makes the

right-hand side 0.