Solution to Quiz 3 – Math 3150-2
Summer 2006

- $g(x) = 0 \implies b_n^* = 0.$

- $c = L = 1$

- $b_n = 2\int_0^1 \cos(\pi x)\sin(n\pi x)\,dx \qquad n = 1, 2, \ldots$

Either integrate by parts, twice, or use the identity,

$$\cos\alpha \sin\beta = \frac{\sin(\alpha+\beta) - \sin(\alpha-\beta)}{2}.$$

$$\therefore \cos(\pi x)\sin(n\pi x) = \frac{1}{2}\left\{\sin\left[(n+1)\pi x\right] + \sin\left[(n-1)\pi x\right]\right\} \qquad \text{(why?)}$$

If $n=1$ then $\sin[(n-1)\pi x] = 0$, and so

$$b_1 = \int_0^1 \sin(2\pi x)\,dx = -\frac{1}{2\pi}\left(\cos 2\pi - 1\right) = 0$$

If $n \geq 2$ then

$$b_n = \int_0^1 \sin\left[(n+1)\pi x\right]dx + \int_0^1 \sin\left[(n-1)\pi x\right]dx$$

$$= \frac{-1}{(n+1)\pi}\left[\cos((n+1)\pi) - 1\right]$$

$$- \frac{1}{(n-1)\pi}\left[\cos((n-1)\pi) - 1\right].$$

$$\boxed{u(x,1) = \sum_{n=2}^{\infty} b_n \cos(n\pi)\sin(n\pi x).}$$

We can simplify a little:

Recall $\cos(\theta+\pi) = -\cos(\theta)$

$\cos(\theta-\pi) = -\cos(\theta)$.

$\therefore$ $\cos[(n+1)\pi] = \cos(n\pi+\pi) = -\cos(n\pi)$

$\cos[(n-1)\pi] = \cos(n\pi-\pi) = -\cos(n\pi)$.

$$\therefore \quad b_n = \frac{-1}{(n+1)\pi}\left(-\cos(n\pi)-1\right) - \frac{1}{(n-1)\pi}\left(-\cos(n\pi)-1\right) \qquad n\geq 2$$

$$= \left[\cos(n\pi)+1\right] \times \left(\frac{1}{(n+1)\pi} + \frac{1}{(n-1)\pi}\right)$$

$$= \frac{\cos(n\pi)+1}{\pi} \times \frac{2n}{n^2-1}$$

$$\cos(n\pi)+1 = (-1)^n + 1 = \begin{cases} 0 & n \text{ odd} \\ 2 & n \text{ even} \end{cases}.$$

So $b_n = \dfrac{4n}{\pi(n^2-1)}$. $\qquad n\geq 2$ even

$$u(x,t) = \sum_{\substack{n=2 \\ \text{even}}}^{\infty} \frac{4}{\pi}\left(\frac{n}{n^2-1}\right)\cos(n\pi t)\sin(n\pi x).$$

Plug $t=1$ to finish.