Fourier Series Summary

Suppose f is a $2p$-periodic function that is piecewise smooth. Then,

$$\frac{f(x+)+f(x-)}{2} = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right)\right],$$

where:

1. $f(x+) = \lim_{y\downarrow x} f(y)$ is the right-limit at x;
2. $f(x-) = \lim_{y\uparrow x} f(y)$ is the left-limit at x;
3. the Fourier coefficients are described by:

$$a_0 = \frac{1}{2p}\int_{-p}^{p} f(y)\,dy;$$
$$a_n = \frac{1}{p}\int_{-p}^{p} f(y)\cos\left(\frac{n\pi y}{p}\right)\,dy \qquad n \geq 1;$$
$$b_n = \frac{1}{p}\int_{-p}^{p} f(y)\sin\left(\frac{n\pi y}{p}\right)\,dy \qquad n \geq 1.$$

4. If f is continuous at x, then $f(x-) = f(x+)$ and

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right)\right].$$

5. (Parseval's identity)

$$\frac{1}{2p}\int_{-p}^{p}\left(f(x)\right)^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

Even Functions

Suppose f is a $2p$-periodic function that is piecewise smooth and *even*. [Even means $f(-x) = f(x)$.] Then,

$$\begin{aligned}
a_0 &= \frac{1}{p} \int_{-p}^{p} f(y)\, dy; \\
a_n &= \frac{2}{p} \int_{0}^{p} f(y) \cos\left(\frac{n\pi y}{p}\right)\, dy \qquad n \geq 1; \\
b_n &= 0 \qquad n \geq 1.
\end{aligned}$$

Odd Functions

Suppose f is a $2p$-periodic function that is piecewise smooth and *odd*. [Odd means $f(-x) = -f(x)$.] Then,

$$\begin{aligned}
a_n &= 0 \qquad n \geq 0; \\
b_n &= \frac{2}{p} \int_{0}^{p} f(y) \sin\left(\frac{n\pi y}{p}\right)\, dy \qquad n \geq 1.
\end{aligned}$$