1. A truth serum has the property that 80% of the guilty suspects are properly judged. Moreover, innocent suspects are misjudged 2% of the time. If the suspect was selected from a group of suspects of which only 5% have ever committed a crime, and if the serum indicates that he is guilty, what is the probability that he is innocent? Show your work.

Solution Let $G = \{\text{guilty}\}$, and $J = \{\text{judged guilty}\}$ to see that $G' = \{\text{innocent}\}$ and $J' = \{\text{judged innocent}\}$. Moreover, the problem tells us that $P\{G\} = 0.05$, $P\{J \mid G\} = 0.8$, and $P\{J \mid G'\} = 0.02$. We are after $P\{G' \mid J\}$.

First, we compute $P\{J\}$: by Bayes' rule,

$$P\{J\} = P\{J \cap G\} + P\{J \cap G'\}$$

$$= P\{J \mid G\}P\{G\} + P\{J \mid G'\}P\{G'\}$$

$$= 0.8 \times 0.05 + 0.02 \times 0.95$$

$$= 0.059.$$

Thus,

$$P\{G' \mid J\} = \frac{P\{J \mid G'\}P\{G'\}}{P\{J\}}$$

$$= \frac{0.02 \times 0.95}{0.059}$$

$$\approx 0.322.$$

2. Three fair coins are tossed independently from one another. Let $X$ denote the total number of heads, and $Y$ denote the number of tails thus obtained.

(a) Find the joint probability distribution of $X$ and $Y$.

(b) What is $P\{X \geq 2\}$?

Solution to (a): The joint distribution is given by the following.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1/8</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>3/8</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>3/8</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1/8</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Solution to (b): $P\{X \geq 2\} = 3/8 + 1/8 = 4/8 = 1/2$. 

3. Suppose the joint density function of $X$ and $Y$ is
\[
f(x, y) = \begin{cases} 
4xy, & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1, \\
0, & \text{otherwise.}
\end{cases}
\]

(a) Compute the marginal density functions of $X$ and $Y$.
(b) Are $X$ and $Y$ independent? Explain clearly.

Solution to (a): To find the marginal $g(x)$ of $X$, integrate $f(x, y)$ against $dy$, i.e.,
\[
g(x) = \begin{cases} 
\int_0^1 4xy \, dy = 4x \int_0^1 y \, dy = 2x, & \text{if } 0 \leq x \leq 1, \\
0, & \text{otherwise.}
\end{cases}
\]
Similarly,
\[
h(y) = \begin{cases} 
2y, & \text{if } 0 \leq y \leq 1, \\
0, & \text{otherwise.}
\end{cases}
\]

Solution to (b): Yes, since $f(x, y) = g(x)h(y)$.

4. SAT verbal scores are known to be approximately distributed according to a normal curve. This year, 10,000 people took the test, and the average verbal SAT score was 550 points, and the standard deviation was 100 points. A graduate program in creative writing will only admit students whose SAT verbal score was in the top 2% of the applicant pool. What is the minimum score needed for admission in to this program? Show your work.

Solution: The $z$-score for top 2% is between 2.05 and 2.06, which we take as $z = 2.055$, i.e.,
\[
\frac{\text{Min.Score} - \mu}{\sigma} = 2.055.
\]
Plug $\sigma = 100$ and $\mu = 550$ to get $\text{Min.Score} = 2.055 \times 100 + 550 = 755.5$ points.

5. Consider the random variables $X$ and $Y$ whose joint probability distribution is
\[
\begin{array}{c|ccc}
\hline
& 1 & 2 & 3 \\
\hline
x & 0.05 & 0.05 & 0.1 \\
1 & 0.05 & 0.1 & 0.35 \\
2 & 0.1 & 0.2 & 0.1 \\
\end{array}
\]

One can show that $E[X] = 2.45$, $E[Y] = 2.1$, $\text{Var}(X) \approx 0.5$, and $\text{Var}(Y) \approx 0.50$. Use this information to compute the correlation between $X$ and $Y$. 


Solution: We need the covariance, which is in turn found by first calculating $E[XY]$.

\[
\begin{align*}
E[XY] &= (1 \times 1 \times 0.05) + (2 \times 1 \times 0.05) + (3 \times 1 \times 0.1) + (1 \times 2 \times 0.05) + (2 \times 2 \times 0.01) \\
&\quad + (3 \times 2 \times 0.35) + (1 \times 3 \times 0) + (2 \times 3 \times 0.2) + (3 \times 3 \times 0.1) = 4.79.
\end{align*}
\]

So the covariance is
\[
\sigma_{XY} = E[XY] - \mu_X \mu_Y = 4.79 - (2.45 \times 2.1) = -0.355.
\]

Thus, the correlation is
\[
\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \cdot \sigma_Y} = \frac{-0.355}{\sqrt{0.5} \times \sqrt{0.5}} \approx -0.71.
\]