Math 1070-2: Spring 2008

Lecture 9

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Confidence intervals

[recap]

- A **point estimate** is a single number that is our "best prediction" for the parameter of interest. [Natural]
- An **interval estimate** is an interval of numbers within which the parameter value is believed to fall. [Requires math]
- **Example:** Avg. ht. was 70" in a random sample of size 100. Point estimate for the actual "population height" is 70 inches
- **Example:** In a random sample of 500, 45% voted for Party X. Point estimate for the percentage of votes, in that population, for Party X is 45%
- **Goal:** Given a certain percentage (say 98%), find a random interval that contains the parameter with probability at least that percentage.

Confidence intervals for proportions

[recap]

- $\hat{p}$ = "p hat" = sample proportion
- Then,

$$\hat{p} \pm (\ell \times SE) = \hat{p} \pm \left(\ell \times \sqrt{\frac{p(1-p)}{n}} \right)$$

- What is ℓ? [for 95%, $\ell = 1.96$; for 99%, $\ell = 2.58$]
- A good approximation:

$$\hat{p} \pm \left(\ell \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right)$$

An example

[recap]

- A market survey organization took a simple random sample of 500 households in a certain town with 25,000 households: 79 of the sampled households had computers. Is it possible to find a 95% CI for the percentage of all 25,000 households with computers?
  - You bet it is possible!
  - $\hat{p} = 79/500 = 0.158$ [$\approx$ 15.8%; point estimate for p]
  - SE:

$$\text{SE} \approx \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \approx 0.0163$$

  - 95% CI for p:

$$0.158 \pm (1.96 \times 0.0163) = 0.158 \pm 0.031948$$

  - p is somewhere between $0.126 = 12.6\%$ and $0.19 = 19\%$ [95% confidence = ????]

An example

7.7, p. 321

- 1998 data: "Do you believe in heaven"? [$n = 1158$]
- $\hat{p} = 0.86$ [proportion of "yes."]
- SE:

$$\text{SE} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.86 \times (1-0.86)}{1158}} \approx 0.0102.$$

- 95% CI for the proportion of "yes" for "heaven"

$$\hat{p} \pm (1.96 \times \text{SE}) \approx 0.86 \pm (1.96 \times 0.0102) \approx 0.84 \text{ to } 0.88.$$

- What does this mean?
- Aside: "MOE" here is $1.96 \times 0.0102 \approx 0.02$.

Confidence intervals at large

- General philosophy:

$$\text{Confidence interval} = \text{point estimate} \pm \text{margin of error}$$

- Example: For proportions [at 95%]

$$\text{Confidence interval} = \hat{p} \pm 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

- “Therefore”: For means [at 95%]

$$\text{Confidence interval} = \bar{x} \pm 1.96\frac{\sigma}{\sqrt{n}}$$

- “Therefore”: For means with σ unknown but n large [at 95%]

$$\text{Confidence interval} = \bar{x} \pm 1.96\frac{s}{\sqrt{n}}$$

An example

Example 6, pp. 334–335

- Recent GSS [General Social Survey]: "On the average day, about how many hours do you personally watch television?"
- Some results (MINITAB):

Variable	n	mean	SD	SE mean	95% CI
TV	905	2.983	2.361	0.0785	(2.83 , 3.14)

- Some questions:
  - What do the sample mean and SD suggest about the shape of the population distribution?
  - How did the software compute stadard error? What does it mean?
  - Interpret the 95% CI reported by the software.

An example

#7.25, p. 343

- 2002 GSS: “What do you think is he ideal number of children for a family to have?”
- 497 women responded; median = 2; mean = 3.02; SD = 1.81
- Point estimate of the population mean = ?
  - 3.02
- SE mean = ?
  - $1.81/\sqrt{497} \approx 0.0811$
- 95% CI for the mean = ?
  - $3.02 \pm (1.96 \times 0.0811) = (2.86\,, 3.17)$ [Note the text’s discrepancy]
- Is it plausible that $\mu = 2$?
  - No [95% sure]

CI's for means via the t-distribution

- The problem: Find a CI for μ, as before:

$$\text{Confidence interval} = \bar{x} \pm \left(z \times \frac{\sigma}{\sqrt{n}} \right)$$

- If σ unknown but n large, then $\sigma \approx s$ with very high probab. Therefore,

$$\text{Confidence interval} = \bar{x} \pm \left(z \times \frac{s}{\sqrt{n}} \right)$$

- If n small, then hopeless in general ... except
- If the underlying population is normal, then

$$95\% \text{ confidence interval} = \bar{x} \pm \left(t_{0.025} \times \frac{s}{\sqrt{n}} \right)$$

The t-distribution

- The "t-distribution with $(n - 1)$ degrees of freedom [df]" looks like the normal distribution
- It is not a normal distribution, though
- Tabulated in **Table B** [A3 in Appendix A]
-  The [Student's] t-distribution was found by William S. Gosset [1876–1937; aka "Student"] in 1908
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Gosset was in charge of QC for *The Guinness Brewing Company* in Dublin ☺

The t-table

	Confidence Level					
	80%	90%	95%	98%	99%	99.8%
	Right-Tail Probability					
df	$t_{.100}$	$t_{.050}$	$t_{.025}$	$t_{.010}$	$t_{.005}$	$t_{.001}$
1	3.078	6.314	12.706	31.821	63.656	318.289
2	1.886	2.920	4.303	6.965	9.925	22.328
3	1.638	2.353	3.182	4.541	5.841	10.214
4	1.533	2.132	2.776	3.747	4.604	7.173
5	1.476	2.015	2.571	3.365	4.032	5.894
6	1.440	1.943	2.447	3.143	3.707	5.208
7	1.415	1.895	2.365	2.998	3.499	4.785
8	1.397	1.860	2.306	2.896	3.355	4.501
9	1.383	1.833	2.262	2.821	3.250	4.297
10	1.372	1.812	2.228	2.764	3.169	4.144
11	1.363	1.796	2.201	2.718	3.106	4.025
12	1.356	1.782	2.179	2.681	3.055	3.930
13	1.350	1.771	2.160	2.650	3.012	3.852
14	1.345	1.761	2.145	2.624	2.977	3.787
15	1.341	1.753	2.131	2.602	2.947	3.733
16	1.337	1.746	2.120	2.583	2.921	3.686
17	1.333	1.740	2.110	2.567	2.898	3.646
18	1.330	1.734	2.101	2.552	2.878	3.611
19	1.328	1.729	2.093	2.539	2.861	3.579
20	1.325	1.725	2.086	2.528	2.845	3.552
21	1.323	1.721	2.080	2.518	2.831	3.527
22	1.321	1.717	2.074	2.508	2.819	3.505
23	1.319	1.714	2.069	2.500	2.807	3.485
24	1.318	1.711	2.064	2.492	2.797	3.467
25	1.316	1.708	2.060	2.485	2.787	3.450
26	1.315	1.706	2.056	2.479	2.779	3.435
27	1.314	1.703	2.052	2.473	2.771	3.421
28	1.313	1.701	2.048	2.467	2.763	3.408
29	1.311	1.699	2.045	2.462	2.756	3.396
30	1.310	1.697	2.042	2.457	2.750	3.385
40	1.303	1.684	2.021	2.423	2.704	3.307
50	1.299	1.676	2.009	2.403	2.678	3.261
60	1.296	1.671	2.000	2.390	2.660	3.232
80	1.292	1.664	1.990	2.374	2.639	3.195
100	1.290	1.660	1.984	2.364	2.626	3.174
∞	1.282	1.645	1.960	2.326	2.576	3.091

Examples

- For a 95% CI:
  - If $n = 3$, then $\text{df} = 2$ and $t_{0.025} = 4.303$ $[\gg 1.96]$
  - If $n = 10$, then $\text{df} = 9$ and $t - 0.025 = 2.262$ $[> 1.96]$
  - If $n = 50$, then $\text{df} = 49 \approx 50$ and $t_{0.025} = 2.009$ $[> 1.96]$
  - If $n = 99$, then $\text{df} = 98 \approx 100$ and $t_{0.025} = 1.984$ $[\approx 1.96]$
- What is going on?
- What if we want a 99% CI, say?
- Ans: Find $t_{0.005}$.

An example

#7.6, p. 321

- Starting salary for $n = 3$ recent math. grads from NCSU
- Data: 45K, 35K, and 55K
- Point estimate for the mean starting salary of all recent math. grads from NCSU: $(45 + 35 + 55)/3 = 45$K
- SD of data = 10K
- If salaries are $\approx$ normal, then a 95% CI is [df = 2]

$$\text{CI} = 45 \pm \left(t_{0.025} \frac{s}{\sqrt{n}} \right) = 45 \pm \left(4.303 \times \frac{10}{\sqrt{3}} \right) \approx (20.16\,, 69.8)$$

- How reliable is the point estimate?

The normal assumption

- The t-confidence interval is valid exactly **only** when the underlying population is $\approx$ normal
- What if the underlying population is almost normal?
- Good news: The CI is "robust" under slight deviations from normality
- To check: plot the histogram, and think.
- Bad news: **None** of our methods are robust [or any good, for that matter] under "nonrandom" sampling
- Fact: If n is large, then the t-distribution is almost standard normal

Choosing the sample size

- Question: How big a sample size do we need in order to have a MOE of size m
- For proportions, the MOE for a 95% CI [for p] is

$$m = 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

  provided that n is large
- I.e.,

$$m^2 = 3.8416 \times \left(\frac{\hat{p}(1-\hat{p})}{n}\right)$$

- Solve:

$$n = 3.8416 \times \left(\frac{\hat{p}(1-\hat{p})}{m^2}\right).$$

- General formula [p. 348], but you are expected to do this on your own in the final exam

Choosing the sample size

- For means, the MOE for a 95% CI [for p] is

$$m = 1.96 \frac{\sigma}{\sqrt{n}}$$

  provided that n is large
- I.e.,

$$m^2 = 3.8416 \times \left(\frac{\sigma^2}{n}\right)$$

- Solve:

$$n = 3.8416 \times \left(\frac{\sigma^2}{m^2}\right)$$

- If this n is large and σ is not known, then $s \approx \sigma$
- General formula [p. 349], but you are expected to do this on your own in the final exam