LECTURE 9: BROWNIAN MOTION AND DIFFUSION

§1. MONTE-CARLO SIMULATION

(1.1) The St.-Petersbourg Paradox. Let W denote Brownian motion, and for any number $\lambda > 0$ (say $\lambda = 1$ for the sake of concreteness) define

$$T_\lambda := \min\{s \geq 0 : \ W(s) = \lambda\}, \tag{1.2}$$

which is the first time Brownian motion attains the level λ. For this random variable, one has the property that $P\{T_\lambda < \infty\} = 1$, and yet $E\{T_\lambda\} = +\infty$. That is, although Brownian motion will eventually reach λ, it is never expected to! This is a variant of the St.-Petersbourg paradox of gambling. That $P\{T_\lambda < +\infty\} = 1$ is not hard to see on a simulation. But how does one verify that $E\{T_\lambda\} = +\infty$? One starts with a formula from measure theory:

$$E\{T_\lambda\} = \int_0^\infty P\{T_\lambda > x\}\, dx. \tag{1.3}$$

So if we could show that as $x \to \infty$, $P\{T_\lambda > x\} \sim Cx^{-1/2}$ (say), it would follow that for large n, $\int_n^\infty P\{T_\lambda > x\}\, dx \sim C\int_n^\infty x^{-1/2}\, dx = +\infty$. This is indeed the case:

(1.5) Theorem. *For each $\lambda > 0$, there exists some uninteresting constant C such that as $x \to \infty$, $P\{T_\lambda > x\} \sim Cx^{-1/2}$.*

(1.6) Simulation Verification. How does one verify this theorem by simulation methods? Note that $P\{T_\lambda > x\}$ is an expectation and hence can be simulated by Monte–Carlo simulation (cf. Lecture 6, Exercise 1.10). Indeed, $P\{T_\lambda > x\} = E\{\mathbf{1}_{\{T_\lambda > x\}}\}$. So, one can Monte-Carlo-simulate this by generating a large number (N) of independent Brownian motions $W^1, \ldots, W^N$, each until the first time they hit λ. Let $T_\lambda^1, \ldots, T_\lambda^N$ denote their respective hitting times to λ and note that $T_\lambda^1, \ldots, T_\lambda^N$ are independent and identically distributed. Thus, by Kolmogorov's strong law of large numbers (Theorem 0.1, Lecture 2),

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\ell=1}^N \mathbf{1}_{T_\lambda^\ell > x\}} = P\{T_\lambda > x\}. \tag{1.7}$$

In words, generate N independent Brownian motions and see how many of them take up at least x units of time to reach λ. If N is large, then this should be close to $P\{T_\lambda x\}$. Now conjecture that for some $\alpha > 0$, $P\{T_\lambda > x\} \approx x^{-\alpha}$. If so, then $\log P\{T_\lambda > n\} \sim -\alpha \log n$ for large n, and this means that if you plot the log-plot of the function $P\{T_\lambda > x\}$, you will see a constant function; the constant is $-\alpha$ and, thanks to Theorem (1.5), it should be equal to $-\frac{1}{2}$. ♣

(1.8) Random Walk Projects. Suppose S denotes the two-dimensional simple walk; fix some nice set $A \subset \mathbf{R}^2$, and let $T_A := \min\{n \geq 0 : \ S_n \in A\}$ be the first time that you hit that set. For instance, A could be a single point $\{(1,1)\}$ (say), a curve, etc. Can you find the extent to which the geometry of A affects the rate of decay of $P\{T_A > x\}$ as $x \to \infty$? For instance, can you detect a discernable difference between the two cases $A := \{(1,1)\}$ and $A :=$ the square $\{(x,y) : |x| = 1,\ |y| = 1\}$? (There is a huge difference.)

(1.9) Brownian Motion Projects. Continuing with our discussion of (1.8), we may ask, "what if the set A changes with time?" The most interesting case is if A is replaced by $\sqrt{n}A$. To be conrete, consider the example of (1.6) but replace λ by $\sqrt{x}\lambda$; let us also write n for x to remind ourselves that it is an integer. So, to summarize: Let $S_n :=$ the simple walk on $\mathbf{Z}$, and more generally consider $P\{\mathcal{T}_{\sqrt{n}\lambda} > nx\}$, where $\mathcal{T}_m$ denotes the first time the random walk hits m for any m. Then, by Donsker's theorem (Theorem 2.1, Lecture 8),

$$\lim_{n\to\infty} P\{\mathcal{T}_{\sqrt{n}\lambda} > nx\} = P\{T_\lambda > x\}, \tag{1.10}$$

and recall that T_λ is the first time Brownian motion hits λ. If instead of $\sqrt{n}\lambda$ you write $n^\alpha\lambda$ for $\alpha \neq \frac{1}{2}$, then nothing interesting happens. Either the probabilities are too small, or they converge to positive constants.

§2. ITÔ DIFFUSIONS

(2.1) A Model in Discrete Time. Suppose that you want to model the random-walk-like movement of a particle in space, but now the space is inhomogeneous, so that in some parts, the walk moves rapidly, and in others very slowly. (Think of a random walk in space that is in part filled with air and in part with oil.) We will restrict "space" to one-dimensions since it is easier to imagine what is going on.

One way to proceed is to construct independent molecular fluctuations, $X_1, X_2, \ldots$. These are—as before—equal to ± 1 with probability $\frac{1}{2}$ each, and are independent random variables. Our "diffusion" (or random walk in inhomogeneous media) will be denoted by the process $Y_0.Y_1,\ldots$, where Y_0 is wherever the process starts (say at the origin.) So, $Y_0 := 0$, and having constructed $Y_0, \ldots, Y_k$, define $Y_{k+1} := Y_k + a(Y_k)X_{k+1}$, where the function a tells us how much to alter the usual fluctuations of the ordinary walk (based on X's), depending on where the diffusion Y is at time k. For instance, by sure that you understand that if the function $a(x) := 2$ for all x, then the diffusion Y is just a simple walk times 2; i.e., a simple walk that fluctuates twice as wildly. We can add a drift term to this diffusion as well to model the effect of a push. That is, $Y_{k+1} = a(Y_k)X_{k+1} + b(Y_k)$.

(2.2) Itô Diffusions in Continuous Time. Just as Brownian motion was obtained as limits of random walks, we can proceed to construct continuous-time diffusions by discrete-time approximations. Here is the simulation algorithm; it will construct an Itô diffusion in continuous time whose fluctuation are guided by some function a and whose drift is by some function b:

$$Y(0) := 0, \qquad Y\left(\frac{k+1}{n}\right) := Y\left(\frac{k}{n}\right) + a\left(Y\left(\frac{k}{n}\right)\right)\cdot\frac{X_{k+1}}{\sqrt{n}} + b\left(Y\left(\frac{k}{n}\right)\right)\cdot\frac{1}{n}. \tag{2.3}$$

The $1/\sqrt{n}$ term is just central limit theorem scaling as in Donsker's theorem for Brownian motion. Indeed, if $a(x) := 1$ and $b(x) := 0$, the process Y is Brownian motion. Another way to write this is to bring the term $Y(k/n)$ to the left-hand side to convince yourselves that Y "solves" the following "stochastic differential equation:"

$$dY(t) = a(Y(t))dW(t) + b(Y(t))dt, \tag{2.4}$$

where W is Brownian motion.

(2.5) Warning. The above stochastic differential equation has very different properties (as well as a different meaning) than ordinary differential equations of the calculus of real functions. For instance, Paley, Wiener, and Zygmund proved that with probability one, the Brownian motion W is nowhere differentiable, so that $dW(t)$ is not the usual "differential." [See R. E. A. C. Paley, N. Wiener, and A. Zygmund (1933). Notes on random functions, *Math. Zeit.*, **37**, 647–668.] The difference is best seen when trying to understand Itô's formula that is next.

(2.6) Itô's Formula. If you consider two differentiable functions f and g, then by the chain rule of the calculus of real functions,

$$(f(g))' = f'(g) \times g'. \tag{2.7}$$

If g is the random function W instead (i.e., Brownian motion), it is nowhere differentiable (cf. 2.6 above), and hence $W'(s)$ does not exist at any s. Itô's formula tells us what happens to chain rule in this case: For a twice continuously differentiable function f,

$$f(W(t)) = f(W(0)) + \int_0^t f'(W(s))\, dW(s) + \frac{1}{2}\int_0^t f''(W(s))\, ds, \tag{2.8}$$

where the "stochastic integral" $\int f' dW$ needs to be defined. It can be shown to satisfy the following natural approximation, but the choice of the so-called left-point rule is absolutely essential now:

$$\int_0^t g(W(s))\, dW(s) = \lim_{n\to\infty} \sum_{j=0}^{n} g\left(W\left(\frac{jt}{n}\right)\right) \times \left\{W\left(\frac{(j+1)t}{n}\right) - W\left(\frac{jt}{n}\right)\right\}, \tag{2.9}$$

where "limit" needs to be understood in some carefully stated sense. What is important about this approximation is that it shows quite clearly that the stochastic integral will have mean zero always! Indeed, note that $g(W(jt/n))$ and $\{W((j+1)t/n) - W(jt/n)$ are independent thanks to Einstein's predicate (1.5c, Lecture 8). Now elementary probability theory tells us that whenever ξ and ζ are independent random variables, then $E\{\xi\zeta\} = E\{\xi\}E\{\zeta\}$. Since $E\{W(t) - W(s)\} = 0$, this shows that stochastic integrals are always mean-zero processes; i.e.,

$$E\left\{\int_0^t g(W(s))\, dW(s)\right\} = 0. \tag{2.10}$$

(2.11) Itô's Formula for the Diffusion Y. The diffusion Y also has an Itô formula; it is the following more complicated one:

$$f(Y(t)) = f(Y(0)) + \int_0^t f'(Y(s))\, dY(s) + \frac{1}{2}\int_0^t f''(Y(s))\, [a(Y(s)]^2\; ds. \tag{2.12}$$

Plug the value of $dY(s)$ from (2.4) and we obtain the Itô formula,

$$\begin{aligned} f(Y(t)) =& f(Y(0)) + \int_0^t f'(Y(s))a(Y(s))\, dW(s) + \int_0^t f'(Y(s))b(Y(s))\, ds \\ &+ \frac{1}{2}\int_0^t f''(Y(s))\, [a(Y(s)]^2\; ds. \end{aligned} \tag{2.13}$$

The point is that this formulation has a stochastic integral in terms of dW which we have already seen is mean-zero.

(2.14) Existence of the Diffusion. Unfortunately, the simulation algorithm of (2.3) will produce something that yields nonsense unless the functions a and b are "nice." By this I mean that the Itô equation (2.4) will have solutions only if a and b are nice. One such condition is that a' and b' exist and are bounded functions. Under this condition, with probability one, (2.4) can be shown to have a unique solution process Y.