LECTURE 8: BROWNIAN MOTION

§1. A LITTLE HISTORY

(1.1) Robert Brown (1828). In 1828, an English botanist by the name of Robert Brown discovered that if you grains of pollen suspended in water, then each individual grain seems to undergo a rather erratic movement. He also posed the problem of describing this movement that has come to be known as "molecular motion" or "diffusion." This work was largely ignored by the scientific community for some time.

(1.2) Louis Bachelier (1900). Independently from Brown's work, in his 1900 Ph.D. thesis at the University of Paris, and under the guidance of the great French mathematician H. Poincaré, Louis Bachelier worked out a theory for the fluctuations of the stock market that involved the development of a stochastic process that is now called "Brownian motion." [See L. Bachelier (1900). Théorie de la spéculation, *Annales de l'Ecole Normale Superiure*, *Ser.* 3, **17**, 21–86. See also the English translation: L. Bachelier (1964). *The Random Character of Stock Market Prices*, P. H. Cootner editor, MIT Press, Cambridge.] Unfortunately, Bachelier's work went largely unnoticed for nearly a century, since his arguments contained flaws nearly all of which are now known to be minor. However, amongst the accomplishments of Bachelier's thesis were his discovery of two deep and fundamental facts about the Brownian motion: One, that it has a Markovian character (in words, given the position at time t, you do not need the prior positions to predict or simulate the future behavior of the process); and two, that it has the reflection property: If $W(s)$ denotes the position of the Brownian motion at time s, then the maximal displacement by time t (i.e., $\max_{s\leq t} W(s)$) has the same distribution as the absolute displacement at time t (i.e., $|W(t)|$). The latter has a simple distribution and this leads to Bachelier's wonderful calculation:

$$P\left\{\max_{0\leq s\leq t} W(s) \leq \lambda\right\} = \sqrt{\frac{2}{\pi t}} \int_0^\lambda e^{-x^2/2t}\,dx. \tag{1.3}$$

(1.4) Albert Einstein (1905). In 1905 Albert Einstein came to the problem of Brownian motion independently (and unaware of) of Bachelier's work; his motivation was to answer Brown's question by proposing a mathematical model for molecular motion. [See A. Einstein (1956). Investigations on the theory of the Brownian movement, New York.] In particular, he used the connections between the Brownian motion and the diffusion equation to get the sharpest estimates of that time for the Avagadro's number and hence the diameter of a hydrogen atom. With hindsight, we now know that Bachelier went much further in his analysis than Einstein. However, it is easier to describe Einstein's prediction for what the Brownian motion should be. Actually proving that such an object exists and developing a calculus for it required a tremendous mathematical development to which I will come shortly. However, let me mention in passing that a good number of physicists continued Einstein's analysis and applications of Brownian motion in physics; some of the names that you should know about are Smoluchowski, Fokker and Planck, Uhlenbeck, and many others.

(1.5) Einstein's Predicates. Einstein predicted that the one-dimensional Brownian motion is a random function of time written as $W(t)$ for "time" $t \geq 0$, such that:

(a) At time 0, the random movement starts at the origin; i.e., $W(0) = 0$.

(b) At any given time $t > 0$, the position $W(t)$ of the particle has the normal distribution with mean 0 and variance t.

(c) If $t > s > 0$, then the displacement from time s to time t is independent of the past until time s; i.e., $W(t) - W(s)$ is independent of all the values $W(r)$; $r \leq s$.

(d) The displacement is time-homogeneous; i.e., the distribution of $W(t) - W(s)$ is the same as the distribution of $W(t-s)$ which is in turn normal with mean 0 and variance $t - s$.

(e) The random function W is continuous.

(1.6) Norbert Wiener (1923). In 1923, Norbert Wiener (a professor at MIT and a child prodigy) proved the existence of Brownian motion and set down a firm mathematical foundation for its further development and analysis. Wiener used the recently-developed mathematics of É. Borel and H. Steinhaus (the subject is called measure theory), and cleverly combined it with a nice idea from a different mathematical discpline (harmonic analysis) to show the in fact the following random series converges with probability one to an object that satisfies (nearly) all of Einstein's predicates: For all $0 \leq t \leq 1$,

$$W(t) = \sqrt{\frac{\pi}{2}}tX_0 + \frac{1}{\sqrt{\pi}} \sum_{j=1}^{\infty} \left[\frac{\sin(\pi j t)}{j} X_j + \frac{\cos(\pi j t)}{j} X_{-j} \right], \tag{1.7}$$

where $X_0, X_{\pm 1}, X_{\pm 2}, \ldots$ are independent standard normal random variables. [See the last two chapters of R. E. A. C. Paley and N. Wiener (1934). *Fourier Transforms in the Complex Plane*, New York.]

(1.8) Paul Lévy (1939). Finally, the classical development of Brownian motion was complete in a 1939 work of Paul Lévy who proved the following remarkable fact: If you replace the normal distribution by any other distribution in Einstein's predicate (cf. 1.5), then either there is no stochastic process that satisfies the properties (a)–(d), or (e) fails to hold! Lévy's work was closely related to the concurrent and independent work of A. I. Khintchine in Russia, and is nowadays called *The Lévy–Khintchine Formula.*

(1.9) Kiyosi Itô (1942/1946). The work of Paul Lévy started the modern age of random processes, and at its center, the theory of Brownian motion. The modern literature on this is truly vast. But all probabilists would (or should) agree that a center-piece of the classical literature is the 1942/1946 work of K. Itô who derived a calculus—and thereby a theory of stochastic differential equations—that is completely different from the ordinary nonstochastic theory. This theory is nowadays at the very heart of the applications of probability theory to mathematical finance. [See K. Itô (1942). On stochastic processes. 1. *Japanese J. Math.*, **18**, 261–301; K. Itô (1946). On a stochastic integral equation, *Proc. Jap. Aca.*, **22**, 32–25.]

(1.10) Monroe Donsker (1951). For us, the final important step in the analysis of Brownian motion was the 1951 work of Donsker who was a Professor of mathematics at The

New York University. [See M. Donsker (1951). An invariance principle for certain probability limit theorems, *Memoires of the American Math. Society*, **6**, and M. Donsker (1952). Justififcation and extension of Doob's heuristic approach to the Kolmogorov–Smirnov theorem, *The Annals of Math. Stat.*, **23**, 277–281.] Amongst other things, Donsker verified a 1949 conjecture of the great American mathematician J. L. Doob by showing that once you run them for a long time, all mean-zero variance-one random walks look like Brownian motion! [The said conjecture appears in J. L. Doob (1949). Heuristic approach to the Kolmogorov–Smirnov statistic, *The Annals of Math. Stat.*, **20**, 393–403].

§2. BROWNIAN MOTION

(2.1) Donsker's Theorem. As I mentioned in (1.10), Donsker's theorem states that once you run them for a long time, all mean-zero variance-one random walks look like Brownian motion. Here is a slightly more careful description: *Let $X_1, X_2, \ldots$ denote independent, identically distributed random variables with mean zero and variance one. The random walk is then the random sequence $S_n := X_1 + \cdots + X_n$, and for all n large, the random graph of $S_1/\sqrt{n}, S_2/\sqrt{n}, \ldots S_n/\sqrt{n}$ (linearly interpolate inbetween the values as Matlab does automatically), is close to the graph of Brownian motion run until time one.*

(2.2) Algorithm for Running a Brownian Motion W. Choose a large value of n and a starting value x, and perform the following. It uses Donsker's theorem above, and wwill plot the path of a one-dimensional Brownian motion run until time 1.

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– For i=1 to n;
    ◦ W(i) = x;
– end;        % Initialize the Brownian motion to
                 have all values equal to the starting point x.
– Plot (0,W(1))        % This plots the starting point.
– For i=2 to n;   % When i=1, W(i)=x already.
    ◦ Generate a random variable Z := ±1 with probability 1/2.
    ◦ Set W(i) = Z/√n + W(i − 1);
    ◦ Plot (i/n,W(i));
– end;
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(2.3) Application: Bachelier's Reflection Principle. Recall the reflection principle of D. Andre from (3.1, Lecture 2): If S_n is the simple walk, then

$$P\left\{\max_{1\le k\le n} S_n \ge \lambda\right\} = 2P\left\{S_n \ge \lambda\right\}. \tag{2.4}$$

But the same number of these simple-walk paths are over λ as they are under $-\lambda$. Thus,

$$P\left\{\max_{1\le k\le n} S_n \ge \lambda\right\} = P\left\{|S_n| \ge \lambda\right\}. \tag{2.5}$$

What this says is that the distribution of the maximum displacement of the walk is the same as the distribution of the absolute displacement. Replace λ by $\sqrt{n}\lambda$, let $n \to \infty$, and appeal to Donsker's theorem to deduce the following: *If W denotes Brownian motion, then for all $\lambda > 0$,*

$$P\left\{\max_{0\le s\le 1} W(s) \ge \lambda\right\} = P\left\{|W(s)| \ge \lambda\right\}. \tag{2.6}$$

But the distribution of $W(s)$ is a normal with mean zero and variance s. From this, one readily obtains Bachelier's reflection principle (cf. equation 1.3) with $t = 1$. The general case $t > 0$ is handled similarly. ♣