LECTURE 1: THE SIMPLE WALK

The *simple random walk* is a mathematical motion for one-dimensional molecular motion, and is defined as follows: At time $n = 0$, the particle's position is $S_0 = 0$. Then you toss a fair coin to go left or right with probability $\frac{1}{2}$ each. Let S_1 denote the position of the particle at time 1 obtained in this way. Now repeat the process, making sure that everytime you toss a coin, it is tossed independently of the coin preceding it. This gives you a random (or stochastic) process $S := \{S_n\}_{n\geq 1}$.

You can think of the process S as a random "dynamical system." It is a dynamical system roughly because you apply the same procedure at time n to determine the value at time $n+1$; it is random since this procedure involves random tosses of coins.

§1. A COMBINATORIAL INTERPRETATION

Suppose you want to know the probability that the random process S has "done something before time n." For instance, what is the probability that some time before time n, the random walk passed the point k. (In symbols, $P\{\max_{1\leq j\leq n} S_j \geq k\} =?$) Or, what is the probability that you never hit zero before time n (In symbols, $P\{\min_{1\leq j\leq n} S_j > 0\} =?$)

Combinatorics (or counting) give us one way to make such calculations. Let us say that $\pi_0, \pi_1, \pi_2, \ldots, \pi_n$ is a *path of length n* if $\pi_0 = 0$, and for all $1 \leq i \leq n$, $|\pi_{i+1} - \pi_i| = 1$. Note that each realization of the random walk by time n gives a path of length n.

(1.1) Observation. *There are 2^n paths of length n. Moreover, if $\pi_0, \ldots, \pi_n$ is any given path of length n, then*

$$P\{S_1 = \pi_1, \ldots, S_n = \pi_n\} = 2^{-n}.$$

In other words, all paths are equally likely to be the random walk path. This is an easy exercise.

§2. A PROBABILISTIC INTERPRETATION

For $i = 1, 2, \cdots$ define $X_i := S_i - S_{i-1}$. The values $X_1, X_2, \ldots$ are the displacement values at times $1, 2, \cdots$. In other words, if the coin at time j told us to go to the right, then $X_j = +1$, else $X_j = -1$. Since the coins were independent, the X_i's are independent random variables. Finally, they all have the same distribution which is given by $P\{X = -1\} = P\{X = +1\} = \frac{1}{2}$. Finally, note that $S_n = X_1 + \cdots + X_n$.

Notation. *Any process of the form $T_n = Y_1 + \cdots + Y_n$, where the Y_i's are independent and identically distributed, is called a random walk. In particular, the simple walk is a random walk.*

§3. PRELIMINARY CALCULATIONS

Let us compute a few momonts to get a feeling for the behavior of the simple walk S. First,

$$E\{S_n\} = E\{X_1\} + \cdots + E\{X_n\}.$$

But the X_i's are have the same distribution, and so they all have the same expectation, which is $E\{X\} = 1 \times P\{X = 1\} + (-1) \times P\{X = -1\} = 1 \times \frac{1}{2} + (-1) \times \frac{1}{2} = 0$. Therefore, we have

(3.1) Expected Value. *For each n, $E\{S_n\} = 0$.*

Suppose you are playing a fair game many times in succession. Everytime you play, the probability of winning a dollar is the same as that of losing (i.e., $= \frac{1}{2}$), and you play the game independently each time. Then, S_n is the fortune (if > 0 and loss if ≤ 0) that you have amassed by time n. The above tells us that you expect to come out even in a fair game. Not a surprise. But there are fluctuations and the expected fluctuation is the standard deviation, i.e., the square root of the variance.

(3.2) Variance. *For each n, $\mathrm{Var}(S_n) = n$.*

Proof: In order to make this computation, recall that for any random variable Y, $\mathrm{Var}(Y) = E(Y^2) - |E\{Y\}|^2$. Therefore, $\mathrm{Var}(S_n) = E\{S_n^2\}$. We compute this as follows: First note that

$$S_n^2 = (X_1 + \cdots + X_n)^2 = \sum_{j=1}^{n} X_j^2 + \sum\sum_{i \neq j} X_i X_j.$$

When $i \neq j$, X_i and X_j are independent, so $E\{X_i X_j\} = E\{X_i\}E\{X_j\}$, which is 0. Therefore, $E\{S_n^2\} = \sum_{j=1}^{n} E\{X_j^2\} = nE\{X^2\}$. But $E\{X^2\} = 1^2 \times P\{X + 1\} + (-1)^2 \times P\{X = -1\} = 1$, which shows us that the variance of S_n is indeed n. ♣

On the other hand, we could get an even better idea of the size of S_n by computing higher moments. Note that $E\{S_n^4\} = E\{|S_n - E(S_n)|^4\}$.

(3.3) Fourth Moment. *For each n, $E\{S_n^4\} = 3n^2 - 2n$.*

Proof: We proceed as before and expand S_n^4:

$$\begin{aligned} S_n^4 = & \sum_{i=1}^{n} X_i^4 + \binom{4}{2} \cdot \frac{1}{2} \sum\sum_{i \neq j} X_i^2 X_j^2 \\ & + \binom{4}{3} \sum\sum_{i \neq j} X_i X_j^3 + \frac{4!}{1! \cdot 1! \cdot 2!} \cdot \frac{1}{2} \sum\sum\sum_{i \neq j \neq k} X_i X_j X_k^2 \\ & + \frac{4!}{1! \cdot 1! \cdot 1! 1!} \sum\sum\sum\sum_{i \neq j \neq k \neq l} X_i X_j X_k X_l. \end{aligned}$$

By the independence of the X's, and since their means are 0, after we take expectations, only the first two terms contribute, i.e.,

$$E\{S_n^4\} = nE\{X^4\} + \frac{4!}{2! \cdot 2!} \frac{n(n-1)}{2} \left(E\{X^2\}\right)^2 = nE\{X^4\} + 3n(n-1) \left(E\{X^2\}\right)^2.$$

But we have already seen that $E\{X^2\} = 1$, and one computes just as easily that $E\{X^4\} = 1$. The calculation of the fourth moment follows. ♣

§4. CHEBYSHEV'S AND MARKOV'S INEQUALITIES

The Markov, and more generally, the Chebyshev inequality are inequalities that state that for random variables that have sufficiently many moments are large with very little probability.

(4.1) Markov's Inequality. *Suppose X is a nonnegative random variable. Then for all $\lambda > 0$,*

$$P\{X \geq \lambda\} \leq \frac{E\{X\}}{\lambda}.$$

Proof: For any number (random not) $X \geq 0$, we have $X \geq X\mathbf{1}_{\{X \geq \lambda\}} \geq \lambda \mathbf{1}_{\{X \geq \lambda\}}$, where $\mathbf{1}_A$ is the indicator of the event A, i.e.,

$$\mathbf{1}_A = \begin{cases} 1, & \text{if } A \text{ happens,} \\ 0, & \text{if } A^c \text{ happens.} \end{cases} \tag{4.2}$$

Therefore, we take expectations to deduce that

$$E\{X\} \geq \lambda E\left(\mathbf{1}_{\{X \geq \lambda\}}\right) = \lambda P\{X \geq \lambda\}, \tag{4.3}$$

since for any random event A, $E(\mathbf{1}_A) = 1 \times P\{A\} + 0 \times P\{A^c\} = P\{A\}$. Divide (4.3) by $\lambda > 0$ to get Markov's inequality. ♣

Markov's inequality states that if $X \geq 0$ has a finite mean, then the probability that X is large is very small. If X has more moments, this probability is even smaller in sense.

(4.4) Chebyshev's Inequality. *Suppose X is a random variable that has a finite variance, and let $\mu := E\{X\}$ denote its means. Then for all $\lambda > 0$,*

$$P\{|X - \mu| \geq \lambda\} \leq \frac{\operatorname{Var}(X)}{\lambda^2}.$$

Proof: Let $Y := |X - \mu|^2$ and note that $P\{|X - \mu| \geq \lambda\} = P\{Y \geq \lambda^2\}$. Since $E\{Y^2\} = \operatorname{Var}(X)$, apply Markov's inequality to finish. ♣

There are higher-moment versions of Chebyshev's inequality. Here is one. I will omit the proof, since it is the same as that of (4.4).

(4.5) Chebyshev's Inequality for Fourth Moments. *Suppose X is a random variable that has a finite fourth moment, and suppose $E\{X\} = 0$. Then for all $\lambda > 0$,*

$$P\{|X| \geq \lambda\} \leq \frac{E\{X^4\}}{\lambda^4}.$$