Zeros of a two-parameter random walk*

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Abstract

We prove that the number γ_N of the zeros of a two-parameter simple random walk in its first $N \times N$ time steps is almost surely equal to $N^{1+o(1)}$ as $N \to \infty$. This is in contrast with our earlier joint effort with Z. Shi [4]; that work shows that the number of zero crossings in the first $N \times N$ time steps is $N^{(3/2)+o(1)}$ as $N \to \infty$. We prove also that the number of zeros on the diagonal in the first N time steps is $((2\pi)^{-1/2} + o(1)) \log N$ almost surely.

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1 Introduction

Let $\{X_{i,j}\}_{i,j=1}^{\infty}$ denote i.i.d. random variables, taking the values ± 1 with respective probabilities 1/2, and consider the two-parameter random walk $\mathbf{S} := \{S(n\,,m)\}_{n,m \geq 1}$ defined by

$$S(n\,,m) := \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j} \qquad \text{for } n, m \geq 1. \tag{1.1}$$

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A lattice point $(i\,,j)$ is said to be a *vertical crossing* for the random walk $\mathbf{S}$ if $S(i\,,j)S(i\,,j+1) \le 0$. Let $Z(N)$ denote the total number of vertical crossings in the box $[1\,,N]^2 \cap \mathbf{Z}^2$. A few years ago, together with Zhan Shi [4] we proved that with probability one,

$$Z(N) = N^{(3/2)+o(1)} \qquad \text{as } N \to \infty. \tag{1.2}$$

We used this result to describe an efficient method for plotting the zero set of the two-parameter walk S; this was in turn motivated by our desire to find good simulations of the level sets of the Brownian sheet.

The goal of the present paper is to describe the rather different asymptotic behavior of two other "contour-plotting algorithms." Namely, we consider the total number of zeros in $[1\,,N]^2 \cap \mathbf{Z}^2$:

$$\gamma_N := \sum\sum_{(i,j)\in[0,N]^2} \mathbf{1}_{\{S(i,j)=0\}}, \tag{1.3}$$

together with the total number of on-diagonal zeros in $[1\,,2N]^2 \cap \mathbf{Z}^2$:

$$\delta_N := \sum_{i=1}^{N} \mathbf{1}_{\{S(2i,2i)=0\}}. \tag{1.4}$$

The main results are listed next.

Theorem 1.1. *With probability one,*

$$\gamma_N = N^{1+o(1)} \qquad \textit{as } N \to \infty. \tag{1.5}$$

Theorem 1.2. *With probability one,*

$$\lim_{N\to\infty} \frac{\delta_N}{\log N} = \frac{1}{(2\pi)^{1/2}}, \tag{1.6}$$

*where "*log*" denotes the natural logarithm.*

The theorems are proved in reverse order of difficulty, and in successive sections.

2 Proof of Theorem 1.2

Throughout, we need ordinary random-walk estimates. Therefore, we use the following notation: Let $\{\xi_i\}_{i=1}^{\infty}$ be i.i.d. random variables, taking the values ± 1 with respective probabilities $1/2$, and consider the one-parameter random walk $\mathbf{W} := \{W_n\}_{n=1}^{\infty}$ defined by

$$W_n := \xi_1 + \cdots + \xi_n. \tag{2.1}$$

We begin by proving a simpler result.

Lemma 2.1. *As $N \to \infty$,*[1]

$$\mathrm{E}\delta_N = \frac{1}{(2\pi)^{1/2}} \log N + O(1). \tag{2.2}$$

Before we prove this, we recall some facts about simple random walks.

We are interested in the function,

$$p(n) := \mathrm{P}\{W_{2n} = 0\}. \tag{2.3}$$

First of all, we have the following, which is a consequence of the inversion formula for Fourier transforms:

$$p(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} [\cos(t)]^{2n}\, dt. \tag{2.4}$$

Therefore, according to Wallis' formula [1, eq. **6.1.49**, p. 258], as $n \to \infty$,

$$p(n) = \frac{1}{(\pi n)^{1/2}} \left[1 - \frac{1}{8n} + \frac{1}{128n^2} - \cdots \right], \tag{2.5}$$

in the sense of formal power series.[2]

Next, we present a "difference estimate."

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[1] We always write $a_N = O(1)$ to mean that $\sup_N |a_N| < \infty$. Note the absolute values.

[2] Suppose $a_1, a_2, \ldots$ are non-negative series which $a_1(n) \le a_2(n) \le \cdots$. Then please recall that "$p(n) = a_1(n) - a_2(n) + a_3(n) - \cdots$" is short-hand for "$a_1(n) - a_2(n) \le p(n) \le a_1(n) - a_2(n) + a_3(n)$," etc.

Lemma 2.2. *For all integers $n \geq 1$,*

$$0 \leq p(n) - p(n+1) = O(n^{-3/2}). \tag{2.6}$$

Proof. Because $0 \leq \cos^2 t \leq 1$, (2.4) implies that $p(n) \geq p(n+1)$. The remainder follows from (2.5) and a few lines of computations. $\square$

Proof of Lemma 2.1. Because $S(2i\,,2i)$ has the same distribution as W_{4i^2}, it follows that $\mathrm{E}\delta_N = \sum_{1\leq i\leq N} p(2i^2)$. The result follows readily from this and (2.5). $\square$

Next, we bound the variance of δ_N.

Proposition 2.3. *There exist finite constants $C_0, C_1 > 1$ such that*

$$C_0^{-1} \log N \leq \mathrm{Var}\delta_N \leq C_0 \log N \qquad \textit{for all } N \geq C_1. \tag{2.7}$$

Proof. Evidently,

$$\mathrm{E}[\delta_N^2] = \mathrm{E}\delta_N + 2 \sum\sum_{1\leq i<j\leq N} P(i\,,j), \tag{2.8}$$

where

$$P(i\,,j) := \mathrm{P}\left\{ S(2i\,,2i) = 0\ ,\ S(2j\,,2j) = 0 \right\}, \tag{2.9}$$

for $1 \leq i < j < \infty$. But $S(2j\,,2j) = S(2i\,,2i) + W_{i,j}$, where $W_{i,j}$ is a sum of $4(j^2-i^2)$-many i.i.d. Rademacher variables, and is independent of $S(2i\,,2i)$. Therefore,

$$P(i\,,j) = p(2i^2)p\left(2(j^2-i^2)\right). \tag{2.10}$$

According to Lemma 2.2, $P(i\,,j) \geq p(2i^2)p(2j^2)$. Therefore, by (2.8),

$$\begin{aligned} \mathrm{E}[\delta_N^2] &\geq \mathrm{E}\delta_N + 2 \sum\sum_{1\leq i<j\leq N} p(2i^2)p(2j^2) \\ &= \mathrm{E}\delta_N + (\mathrm{E}\delta_N)^2 - \sum_{1\leq i\leq N} p^2(2i^2). \end{aligned} \tag{2.11}$$

Thanks to (2.5), the final sum is $O(1)$. Therefore, Lemma 2.1 implies that

$$\operatorname{Var}\delta_N \geq \frac{1}{(2\pi)^{1/2}}\log N + O(1). \tag{2.12}$$

In order to bound the converse bound, we use Lemma 2.2 to find that

$$\begin{aligned} p\left(2(j^2-i^2)\right) - p(2j^2) &= \sum_{2(j^2-i^2)\leq \ell < 2j^2} \left[p(\ell) - p(\ell+1)\right] \\ &\leq c \sum_{2(j^2-i^2)\leq \ell < 2j^2} \frac{1}{\ell^{3/2}}, \end{aligned} \tag{2.13}$$

where c is positive and finite, and does not depend on $(i\,,j)$. Note that if $1 \leq k < K$, then

$$\begin{aligned} \sum_{k\leq \ell\leq K} \frac{1}{\ell^{3/2}} \leq \int_k^{K+1} \frac{\mathrm{d}l}{l^{3/2}} &= 2\left(\frac{1}{k^{1/2}} - \frac{1}{(K+1)^{1/2}}\right) \\ &= 2\cdot\frac{(K+1)^{1/2} - k^{1/2}}{k^{1/2}(K+1)^{1/2}} \\ &\leq 2\cdot\frac{K-k+1}{k^{1/2}K} \leq 4\cdot\frac{K-k}{k^{1/2}K}. \end{aligned} \tag{2.14}$$

We can deduce from the preceding two displays that

$$\begin{aligned} p\left(2(j^2-i^2)\right) - p(2j^2) &\leq \text{const}\cdot\frac{i^2}{j^2\cdot(j^2-i^2)^{1/2}} \\ &\leq \text{const}\cdot\frac{i^2}{j^{5/2}\cdot(j-i)^{1/2}}, \end{aligned} \tag{2.15}$$

and the implied constants do not depend on $(i\,,j)$. Thus, (2.5) implies that

$$\begin{aligned} \sum\sum_{1\leq i<j\leq N} p(2i^2)\left[p(2(j^2-i^2)) - p(2j^2)\right] &\leq \text{const}\cdot\sum\sum_{1\leq i<j\leq N}\frac{i}{j^{5/2}\cdot(j-i)^{1/2}} \\ &= \text{const}\cdot(Q_1 - Q_2), \end{aligned} \tag{2.16}$$

where

$$Q_1 := \sum\sum_{1\le i<j\le N} \frac{1}{j^{3/2}\cdot(j-i)^{1/2}} \quad \text{and} \quad Q_2 := \sum\sum_{1\le i<j\le N} \frac{(j-i)^{1/2}}{j^{5/2}}. \tag{2.17}$$

Direct computation shows now that Q_1 and Q_2 are both $O(\log N)$ as $N \to \infty$, whence

$$\begin{aligned}\sum\sum_{1\le i<j\le N} P(i\,,j) &\le \sum\sum_{1\le i<j\le N} p(2i^2)p(2j^2) + O(\log N)\\ &= (\mathrm{E}\delta_N)^2 - \sum_{1\le i\le N} p^2(2i^2) + O(\log N)\\ &= (\mathrm{E}\delta_N)^2 + O(\log N);\end{aligned} \tag{2.18}$$

see (2.5). This and (2.8) together imply that the variance of δ_N is $O(\log N)$. Therefore, (2.12) finishes the proof. □

Proof of Theorem 1.2. Thanks to Proposition 2.3 and the Chebyshev inequality, we can write the following: For all $\epsilon > 0$,

$$\mathrm{P}\left\{|\delta_N - \mathrm{E}\delta_N| \ge \epsilon \log N\right\} = O\left(\frac{1}{\log N}\right). \tag{2.19}$$

Set $n_k := [\exp(q^k)]$ for an arbitrary but fixed $q > 1$, and apply the Borel–Cantelli lemma to deduce that

$$\lim_{k\to\infty} \frac{\delta_{n_k}}{\log n_k} = \frac{1}{(2\pi)^{1/2}}. \tag{2.20}$$

Let $m \to \infty$ and find $k = k(m)$ such that $n_k \le m < n_{k+1}$. Evidently, $\delta_{n_k} \le \delta_m \le \delta_{n_{k+1}}$. Also, $\log n_k \le \log n_{k+1} = (q + o(1)) \log n_k$. Therefore, a.s.,

$$\limsup_{m\to\infty} \frac{\delta_m}{\log m} \le \limsup_{k\to\infty} \frac{\delta_{n_{k+1}}}{\log n_k} = \frac{q}{(2\pi)^{1/2}}. \tag{2.21}$$

Similarly, a.s.,

$$\liminf_{m\to\infty} \frac{\delta_m}{\log m} \geq \liminf_{k\to\infty} \frac{\delta_{n_k}}{\log n_{k+1}} \geq \frac{1}{q(2\pi)^{1/2}}. \tag{2.22}$$

Let $q \downarrow 1$ to finish. □

3 Proof of Theorem 1.1

We begin by proving the easier half of Theorem 1.1; namely, we first prove that with probability one, $\gamma_N \leq N^{1+o(1)}$.

Proof of Theorem 1.1: First Half. We apply (2.5) to deduce that as $N \to \infty$,

$$\mathrm{E}\gamma_N = \sum_{i=1}^{N}\sum_{j=1}^{N} \mathrm{P}\{S(i\,,j)=0\} = \sum_{i=1}^{N}\sum_{j=1}^{N} p(ij/2) \leq \text{const} \cdot \left(\sum_{i=1}^{N} i^{-1/2}\right)^2,$$

and this is $\leq \text{const} \cdot N$. By Markov's inequality,

$$\mathrm{P}\{\gamma_N \geq N^{1+\epsilon}\} \leq \text{const} \cdot N^{-\epsilon}, \tag{3.1}$$

where the implied constant is independent of $\epsilon > 0$ and $N \geq 1$. Replace N by 2^k and apply the Borel–Cantelli lemma to deduce that with probability one, $\gamma_{2^k} < 2^{k(1+\epsilon)}$ for all k sufficiently large. If $2^k \leq N \leq 2^{k+1}$ is sufficiently large [how large might be random], then a.s.,

$$\gamma_N \leq \gamma_{2^{k+1}} < 2^{(k+1)(1+\epsilon)} \leq 2^{k(1+2\epsilon)} \leq N^{1+2\epsilon}. \tag{3.2}$$

Since $\epsilon > 0$ is arbitrary, this proves half of the theorem. □

The proof of the converse half is more delicate, and requires some preliminary estimates.

For all $i \geq 1$ define

$$\begin{aligned}
\rho_1(i) &:= \min\{j \geq 1 :\ S(i\,,j)S(i\,,j+1) \leq 0\}\,,\\
\rho_2(i) &:= \min\{j \geq \rho_1(i) :\ S(i\,,j)S(i\,,j+1) \leq 0\}\,,\\
&\ \vdots\\
\rho_\ell(i) &:= \min\{j \geq \rho_{\ell-1}(i) :\ S(i\,,j)S(i\,,j+1) \leq 0\}\,,\ldots\,.
\end{aligned}\tag{3.3}$$

These are the successive times of "vertical upcrossings over time-level i." For all integers $i \geq 1$ and all real numbers $t \geq 1$, let us consider

$$f(i\,;t) := \max\{k \geq 1 :\ \rho_k(i) \leq t\}\,. \tag{3.4}$$

Then, it should be clear that

$$\sum_{i=1}^{N} f(i\,;N) = Z(N). \tag{3.5}$$

where $Z(N)$ denotes the total number of vertical upcrossings in $[1\,,N]^2$; see the introduction.

Lemma 3.1. *With probability one, if N is large enough, then*

$$\max_{1\leq i\leq N} f(i\,;N) \leq N^{1/2+o(1)}. \tag{3.6}$$

Remark 3.2. It is possible to improve the "$\leq$" to an equality. In fact, one can prove that $f(1\,;N) = N^{1/2+o(1)}$ a.s., using the results of Borodin [2]; for further related results see [3]. We will not prove this more general assertion, as we shall not need it in the sequel. □

Proof. Choose and fix two integers $N \geq 1$ and $i \in \{1\,,\ldots\,,N\}$.

We plan to apply estimates from the proof of Proposition 4.2 of [4], whose $\zeta_i(0\,,N)$ is the present $f(i\,;N)$.

After Komlós, Major, and Tusnády [6], we can—after a possible enlargement of the underlying probability space—find three finite and positive constants c_1, c_2, c_3 and construct a standard Brownian motion $\mathbf{w} := \{w(t)\}_{t\geq 0}$ such that for all $z > 0$,

$$\max_{1\leq j\leq N} \mathrm{P}\left\{|S(i\,,j) - w(ij)| > c_1 \log(ij) + z\right\} \leq c_2 \mathrm{e}^{-c_3 z}. \tag{3.7}$$

The Brownian motion $\mathbf{w}$ depends on the fixed constant i, but we are interested only in its law, which is of course independent of i. In addition, the constants c_1, c_2, c_3 are universal.

Fix $\epsilon \in (0\,,1/2)$ and $\delta \in (0\,,\epsilon/2)$, and consider the event

$$\mathcal{E}_N := \left\{ \max_{1\leq j\leq N} |S(i\,,j) - w(ij)| \leq N^\delta \right\}. \tag{3.8}$$

[We are suppressing the dependence of $\mathcal{E}_N$ on i, as i is fixed.] By (3.7), we can find a constant c_4—independent of N and i—such that

$$\mathrm{P}(\mathcal{E}_N) \geq 1 - c_4 N^{-4}. \tag{3.9}$$

Let $S(i\,,0) := 0$ for all i. Then, almost surely on $\mathcal{E}_N$, we have

$$\begin{aligned}
&\sum_{j=0}^{N-1} \mathbf{1}_{\{S(i,j)\geq 0\ ,\ S(i,j+1)\leq 0\}} \\
&\qquad\leq \sum_{j=0}^{N-1} \mathbf{1}_{\{w(ij)\geq -N^\delta\ ,\ w(i(j+1))\leq N^\delta\}} \\
&\qquad\leq \sum_{j=0}^{N-1} \mathbf{1}_{\{w(ij)\geq 0\ ,\ w(i(j+1))\leq 0\}} + 2\sup_{a\in\mathbf{R}} \sum_{j=0}^{N} \mathbf{1}_{\{a\leq w(ij)\leq a+N^\delta\}}.
\end{aligned} \tag{3.10}$$

This is equation (6.6) of [4]. Now we use eq. (1.13) of Borodin [2] to couple

$\mathbf{w}$ with another Brownian motion $\mathbf{B} := \{B(t)\}_{t\geq 0}$ such that

$$\mathrm{P}\left\{\left|\sum_{j=0}^{N-1} \mathbf{1}_{\{w(ij)\geq 0\ ,\ w(i(j+1))\leq 0\}} - \mu(N/i)^{1/2} L_1^0(B)\right| \geq c_5 N^{1/4}\log N\right\} \leq (c_5 N)^{-4}, \tag{3.11}$$

where $\mu := \mathrm{E}(\lfloor B^+(1)\rfloor)$, $c_5 \in (0\,,1)$ does not depend on $(i\,,N)$, and $L_1^0(\mathbf{B}) := \lim_{\eta\downarrow 0}(2\eta)^{-1}\int_0^1 \mathbf{1}_{\{|B(s)|\leq\eta\}}\,\mathrm{d}s$ denotes the local time of $\mathbf{B}$ at time 1 at space value 0. See also the derivation of [4, eq. (6.10)] for some detailed technical comments.

It is well known that $\mathrm{P}\{L_1^0(\mathbf{B}) \geq \lambda\} \leq 2\mathrm{e}^{-\lambda^2/2}$ for all $\lambda > 0$ [7]. In particular, $\mathrm{P}\{L_1^0(\mathbf{B}) \geq N^\delta\} \leq 2\exp(-N^\delta/2)$. Since $\delta < 1/4$, this, (3.9), and (3.11) together imply that

$$\mathrm{P}\left\{\sum_{j=0}^{N-1} \mathbf{1}_{\{w(ij)\geq 0\ ,\ w(i(j+1))\leq 0\}} \geq \frac{N^{(1/2)+\delta}}{i^{1/2}}\right\} \leq c_6 N^{-4}, \tag{3.12}$$

where $c_6 \in (1\,,\infty)$ is independent of N and i. On the other hand, eq. (6.20) of [4] tells us that we can find a constant $c_7 \in (1\,,\infty)$—independent of N and i—such that

$$\mathrm{P}\left\{2\sup_{a\in\mathbf{R}}\sum_{j=0}^{N} \mathbf{1}_{\{a\leq w(ij)\leq a+N^\delta\}} \geq \frac{N^{(1/2)+\delta}}{i^{1/2}}\right\} \leq c_7 N^{-4} + 2\mathrm{e}^{-N^{2\delta}}. \tag{3.13}$$

Since $i \geq 1$ and $\delta < 1/4 < 1/2$, this implies that

$$\mathrm{P}\left\{2\sup_{a\in\mathbf{R}}\sum_{j=0}^{N} \mathbf{1}_{\{a\leq w(ij)\leq a+N^\delta\}} \geq N^{(1/2)+\delta}\right\} \leq c_7 N^{-4} + 2\mathrm{e}^{-N^{2\delta}}. \tag{3.14}$$

Now we combine (3.10), (3.12), and (3.14) to deduce the following:

$$\begin{aligned}
&\sum_{N=1}^{\infty} \mathrm{P}\left(\max_{1\le i\le N} \sum_{j=0}^{N-1} \mathbf{1}_{\{S(i,j)\ge 0\ ,\ S(i,j+1)\le 0\}} \ge 2N^{(1/2)+\delta}\ ;\ \mathcal{E}_N \right) \\
&\qquad \le \sum_{N=1}^{\infty} \sum_{i=1}^{N} \mathrm{P}\left(\sum_{j=0}^{N-1} \mathbf{1}_{\{S(i,j)\ge 0\ ,\ S(i,j+1)\le 0\}} \ge 2N^{(1/2)+\delta}\ ;\ \mathcal{E}_N \right) \\
&\qquad \le \sum_{N=1}^{\infty} \left(c_6 N^{-3} + c_7 N^{-3} + 2N \exp(-N^{2\delta}) \right) \\
&\qquad < \infty.
\end{aligned} \tag{3.15}$$

This and (3.9), in turn, together imply that

$$\sum_{N=1}^{\infty} \mathrm{P}\left\{ \max_{1\le i\le N} \sum_{j=0}^{N-1} \mathbf{1}_{\{S(i,j)\ge 0\ ,\ S(i,j+1)\le 0\}} \ge 2N^{(1/2)+\delta} \right\} < \infty. \tag{3.16}$$

Since $-S$ is another simple walk on $\mathbf{Z}$, it follows that

$$\sum_{N=1}^{\infty} \mathrm{P}\left\{ \max_{1\le i\le N} f(i\,;N) \ge 2N^{(1/2)+\delta} \right\} < \infty. \tag{3.17}$$

The lemma follows the Borel–Cantelli lemma, because ϵ, and hence δ, can be made arbitrarily small. □

Consider the following random set of times for all $N \ge 1$ and $\alpha \in (0\,,1/2)$:

$$\mathcal{H}_N(\alpha) := \left\{ 1 \le i \le N :\ f(i\,;N) > N^{(1/2)-\alpha} \right\}. \tag{3.18}$$

Lemma 3.3. *Choose and fix a positive constant $\alpha < 1/2$. Then, the following happens a.s.: For all but a finite number of values of N,*

$$|\mathcal{H}_N(\alpha)| \ge N^{1-\alpha}, \tag{3.19}$$

where $|\cdots|$ denotes cardinality.

Proof. We apply (1.2), via (3.5) and Lemma 3.1, to see that with probability one, the following holds for all but a finite number of values of N:

$$\begin{aligned}2N^{(3-\alpha)/2} &\leq \sum_{1\leq i\leq N} f(i\,;N)\\ &= \sum_{i\in\mathcal{H}_N(\alpha)} f(i\,;N) + \sum_{\substack{1\leq i\leq N:\\ f(i,N)\leq N^{(1-2\alpha)/2}}} f(i\,;N)\\ &\leq |\mathcal{H}_N(\alpha)|\cdot N^{(1+\alpha)/2} + N^{(3-2\alpha)/2}.\end{aligned} \tag{3.20}$$

The lemma follows. □

Define

$$U(i\,;\ell) := \mathbf{1}_{\{S(i,\rho_\ell(i))S(i,1+\rho_\ell(i))=0\}}. \tag{3.21}$$

The following is a key estimate in our proof of Theorem 1.1.

Proposition 3.4. *There exists a finite constant $c > 0$ such that for all integers $i, M \geq 1$,*

$$\mathrm{P}\left\{\sum_{\ell=1}^{M} U(i\,;\ell) \leq \frac{cM}{i^{1/2}}\right\} \leq \exp\left(-\frac{cM}{4i^{1/2}}\right). \tag{3.22}$$

Our proof of Proposition 3.4 begins with an estimate for the simple walk.

Lemma 3.5. *There exists a constant K such that for all $n \geq 1$ and positive even integers $x \leq 2n$,*

$$\mathrm{P}\left(W_{2n} = x \mid W_{2n} \geq x\right) \geq \frac{K}{n^{1/2}}. \tag{3.23}$$

Proof. Let $\mathcal{P}_n(x)$ denote the conditional probability in the statement of the lemma. Define the stopping times $\nu(x) := \min\{j \geq 1 : W_{2j} = x\}$, and write

$$\mathcal{P}_n(x) = \sum_{j=x/2}^{n} \frac{\mathrm{P}\left(W_{2n} = x \mid \nu(x) = 2j\right)\cdot \mathrm{P}\{\nu(x) = 2j\}}{\mathrm{P}\{W_{2n} \geq x\}}. \tag{3.24}$$

We first recall (2.3), and then apply the strong markov property to obtain $\mathrm{P}(W_{2n} = x \mid \nu(x) = 2j) = p(n-j)$. Thanks to (2.5), we can find two constants K_1 and K_2 such that $p(n-j) \geq K_1(n-j)^{-1/2} \geq K_1 n^{-1/2}$ if $n-j \geq K_2$. On the other hand, if $n-j < K_2$, then $p(n-j) \geq K_3 \geq K_3 n^{-1/2}$. Consequently,

$$\begin{aligned} \mathcal{P}_n(x) &\geq \frac{K_4}{n^{1/2}\mathrm{P}\{W_{2n} \geq x\}} \cdot \sum_{j=x/2}^{n} \mathrm{P}\{\nu(x) = 2j\} \\ &= \frac{K_4}{n^{1/2}} \cdot \frac{\mathrm{P}\{\nu(x) \leq 2n\}}{\mathrm{P}\{W_{2n} \geq x\}}, \end{aligned} \tag{3.25}$$

and this last quantity is at least $K_4 n^{-1/2}$ since $\{\nu(x) \leq 2n\} \supseteq \{W_{2n} \geq x\}$. □

Here and throughout, let $\mathcal{F}(i\,;\ell)$ denote the σ-algebra generated by the random variables $\{\rho_i(j)\}_{j=1}^{\ell}$ and $\{S(i\,,m)\}_{m=1}^{\rho_i(\ell)}$ [interpreted in the usual way, since $\rho_i(\ell)$ is a stopping time for the infinite-dimensional walk $i \mapsto S(i\,,\bullet)$]. Then we have the following.

Lemma 3.6. *For all $i, \ell \geq 1$,*

$$\mathrm{P}\left(S(i\,,1+\rho_\ell(i)) = 0 \mid \mathcal{F}(i\,;\ell) \right) \geq \frac{K}{i^{1/2}}, \tag{3.26}$$

where K was defined in Lemma 3.5.

Proof. Let $\xi := -S(i\,,\rho_\ell(i))$, for simplicity. According to the definition of the $\rho_\ell(i)$'s,

$$S(i\,,1+\rho_\ell(i)) \geq 0 \qquad \text{almost surely on } \{\xi > 0\}. \tag{3.27}$$

Consequently,

$$\Delta_{i,\ell} := S(i\,,1+\rho_\ell(i)) - S(i\,,\rho_\ell(i)) \geq \xi \qquad \text{almost surely on } \{\xi > 0\}. \tag{3.28}$$

Clearly, the strong markov property of the infinite dimensional random walk

$i \mapsto S(i\,;\bullet)$ implies that with probability one,

$$\begin{aligned}\mathrm{P}\left(S(i\,, 1+\rho_\ell(i)) = 0 \mid \mathcal{F}(i\,;\ell)\right) &= \mathrm{P}\left(\Delta_{i,\ell} = \xi \mid \mathcal{F}(i\,;\ell)\right) \\ &\geq \mathrm{P}\left(\Delta_{i,\ell} = \xi \mid \Delta_{i,\ell} \geq \xi\,;\,\xi\right) \mathbf{1}_{\{\xi>0\}}.\end{aligned} \tag{3.29}$$

Therefore, we can apply Lemma 3.5 together with to deduce that (3.26) holds a.s. on $\{\xi > 0\}$. Similar reasoning shows that the very same bound holds also a.s. on $\{\xi < 0\}$. □

We are ready to derive Proposition 3.4.

Proof of Proposition 3.4. We recall the following form of Bernstein's inequality, as found, for example, in [5, Lemma 3.9]: *Suppose* $J_1, \ldots, J_n$ *are random variables, on a common probability space, that take values zero and one only. If there exists a nonrandom* $\eta > 0$ *such that* $\mathrm{E}(J_{k+1} \mid J_1\,, \ldots, J_k) \geq \eta$ *for all* $k = 1, \ldots, n-1$*. Then, that for all* $\lambda \in (0\,,\eta)$*,*

$$\mathrm{P}\left\{\sum_{i=1}^n J_i \leq \lambda n\right\} \leq \exp\left(-\frac{n(\eta-\lambda)^2}{2\eta}\right). \tag{3.30}$$

We apply the preceding with $J_\ell := U(i\,;\ell)$; Lemma 3.6 tells us that we can use (3.30) with $\eta := Ki^{-1/2}$ and $\lambda := \eta/2$ to deduce the Proposition with $c := K/2$. □

Lemma 3.7. *Choose and fix two constants* $a, b > 0$ *such that* $1 > a > 2b$*. Then with probability one,*

$$\min_{1 \leq i \leq N^{1-a}} \sum_{1 \leq \ell \leq N^{(1/2)-b}} U(i\,;\ell) \geq cN^{(a/2)-b}, \tag{3.31}$$

for all N *sufficiently large, where* c *is the constant in Proposition 3.4.*

Proof. Proposition 3.4 tells us that

$$\begin{aligned}
\mathrm{P}\left\{\min_{1\le i\le N^{1-a}} \sum_{1\le \ell\le N^{(1/2)-b}} U(i\,;\ell) \le cN^{(a/2)-b}\right\}
&\le \mathrm{P}\left\{\sum_{1\le \ell\le N^{(1/2)-b}} U(i\,;\ell) \le \frac{cN^{(1/2)-b}}{i^{1/2}} \text{ for some } i\le N^{1-a}\right\}\\
&\le \sum_{1\le i\le N^{1-a}} \exp\left(-\frac{cN^{(1/2)-b}}{4i^{1/2}}\right)\\
&\le N^{1-a}\exp\left(-\frac{cN^{(a/2)-b}}{4}\right). \qquad (3.32)
\end{aligned}$$

An application of the Borel–Cantelli lemma finishes the proof. □

We are ready to complete the proof of our first theorem.

Proof of Theorem 1.1: Second Half. Let us begin by choosing and fixing three small constants α, a, and $b$that satisfy

$$0<\alpha<\frac{1}{2} \quad\text{and}\quad 0<2b<a<1. \qquad (3.33)$$

Clearly,

$$\begin{aligned}
&\bigcap_{1\le i\le N^{1-a}} \left\{\sum_{1\le \ell\le N^{(1/2)-b}} U(i\,;\ell) > cN^{(a/2)-b}\right\}\\
&\quad\subseteq \left\{\sum_{i\in\mathcal{H}_{N^{1-a}}(\alpha)} \sum_{1\le \ell\le N^{(1/2)-b}} U(i\,;\ell) \ge cN^{(a/2)-b}\,|\mathcal{H}_{N^{1-a}}(\alpha)|\right\}.
\end{aligned} \qquad (3.34)$$

According to Lemma 3.3, $|\mathcal{H}_{N^{1-a}}(\alpha)| \ge N^{(1-\alpha)(1-a)}$ for all N large. The preceding and Lemma 3.7 together imply that with probability one,

$$\sum_{i\in\mathcal{H}_{N^{1-a}}(\alpha)} \sum_{1\le \ell\le N^{(1/2)-b}} U(i\,;\ell) \ge c\,N^{(1-\alpha)(1-a)+(a/2)-b}, \qquad (3.35)$$

for all N sufficiently large. Because

$$\begin{aligned}\gamma_N &= \sum_{i=1}^{N} \sum_{\ell=1}^{f(i,N)} U(i\,;\ell)\\ &\geq \sum_{i\in\mathcal{H}_{N^{1-a}}(\alpha)} \sum_{1\leq\ell\leq N^{(1/2)-\alpha}} U(i\,;\ell),\end{aligned} \tag{3.36}$$

it follows from (3.35) that for all but finitely-many values of N:

$$\liminf_{N\to\infty} \frac{\log\gamma_N}{\log N} \geq (1-\alpha)(1-a) + \frac{a}{2} - b, \tag{3.37}$$

for all α, a, and b that satisfy (3.33). Let $\alpha, a \downarrow 0$ and then $b \downarrow 0$, in this order, to find that $\liminf_{N\to\infty} \log\gamma_N / \log N \geq 1$. That is, we have proved that $\gamma_N \geq N^{1+o(1)}$. This completes our proof. □

4 Questions on the distribution of zeros

We conclude this paper by asking a few open questions:

1. Let us call a point $(i\,,j) \in \mathbf{Z}_+^2$ *even* if ij is even. Define Q_N to be the largest square in $[0\,,N]^2$ such that $S(i\,,j)=0$ for every even point $(i\,,j)$ in Q_N. What is the asymptotic size of the cardinality of $Q_N \cap \mathbf{Z}^2$, as $N \to \infty$ along even integers? The following shows that this is a subtle question: One can similarly define $\tilde{Q}_N$ to be the largest square in $[0\,,N]^2$—with one vertex equal to $(N\,,N)$–such that $S(i\,,j)=0$ for all even $(i\,,j) \in \tilde{Q}_N$. [Of course, N has to be even in this case.] In the present case, we estimate the size of $\tilde{Q}_N$ by first observing that if N is even, then

$$\begin{aligned}&\mathrm{P}\left\{S(N\,,N) = S(N+2\,,N+2) = 0\right\}\\ &= \mathrm{P}\left\{S(N\,,N)=0\right\} \cdot \mathrm{P}\left\{S(N+2\,,N+2) - S(N\,,N) = 0\right\}\\ &= (\mathrm{const} + o(1)) N^{-3/2} \quad \text{as } N \to \infty \text{ along evens.}\end{aligned} \tag{4.1}$$

Since the preceding defines a summable sequence, the Borel–Cantelli lemma tells us that $\#\tilde{Q}_N \leq 1$ for all sufficiently-large even integers N.

2. Consider the number $D_N := \sum_{i=1}^N \mathbf{1}_{\{S(i,N-i)=0\}}$ of "anti-diagonal" zeros. It it the case that with probability one,

$$0 < \limsup_{N\to\infty} \frac{\log D_N}{\log\log N} < \infty? \tag{4.2}$$

At present, we can prove that $D_N \leq (\log N)^{1+o(1)}$.

3. The preceding complements the following, which is not very hard to prove:

$$\liminf_{N\to\infty} D_N = 0 \qquad \text{almost surely.} \tag{4.3}$$

Here is the proof: According to the local central limit theorem, and after a line or two of computation, $\lim_{N\to\infty} \mathrm{E}(D_{2N}) = (\pi/8)^{1/2}$. Therefore, by Fatou's lemma, $\liminf_{N\to\infty} D_{2N} \leq (\pi/8)^{1/2} < 1$ with positive probability, whence almost surely by the Kolmogorov zero-one law [applied to the sequence-valued random walk $\{S(i\,,\bullet)\}_{i=1}^\infty$]; (4.3) follows because D_N is integer valued. We end by proposing a final question related to (4.3): Let $\{\mathcal{S}(s\,,t)\}_{s,t\geq 0}$ denote two-parameter Brownian sheet; that is, $\mathcal{S}$ is a centered gaussian process with continuous sample functions, and $\mathrm{E}[\mathcal{S}(s\,,t)\mathcal{S}(u\,,v)] = \min(s\,,u)\min(t\,,v)$ for all $s,t,u,v \geq 0$.

Define "anti-diagonal local times,"

$$\mathcal{D}_t := \lim_{\epsilon\to 0} \frac{1}{2\epsilon}\int_0^t \mathbf{1}_{\{|\mathcal{S}(s,t-s)|\leq\epsilon\}}\,\mathrm{d}s \qquad \text{for } t>0. \tag{4.4}$$

(a) Does $\{\mathcal{D}_t\}_{t>0}$ exist? Is it continuous?

(b) Is it true that $\mathcal{Z} := \{t>0 : \mathcal{D}_t = 0\}$ is almost surely nonempty? That is, does the continuum-limit analogue of (4.3) hold? If $\mathcal{Z}$ is nonempty, then what is its Hausdorff dimension?

4. For all $\epsilon \in (0\,,1)$ and integers $N \geq 1$ define

$$E(\epsilon\,,N) := \left\{(i\,,j) \in [\epsilon N\,,N]^2 : \, S(i\,,j) = 0\right\}. \tag{4.5}$$

   It is not hard to verify that if $\epsilon \in (0\,,1)$ is fixed, then $E(\epsilon\,,N) = \varnothing$ for infinitely-many $N \geq 1$. This is because there exists $p \in (0\,,1)$—independent of N—such that for all N sufficiently large,

$$\mathrm{P}\left\{S(\epsilon N\,,N) \geq 2N\,, \max_{\epsilon N \leq i,j \leq N} |S(i\,,j) - S(\epsilon N\,,N)| \leq N\right\} > p. \tag{4.6}$$

   Is there a good way to characterize which positive sequences $\{\epsilon_k\}_{k=1}^{\infty}$, with $\lim_{k\to\infty} \epsilon_k = 0$, have the property that $E(\epsilon_N\,,N) \neq \varnothing$ eventually?

5. Let γ'_N denote the number of points $(i\,,j) \in [0\,,N]^2$ such that $S(i\,,j) = 1$. What can be said about $\gamma_N - \gamma'_N$?

6. A point $(i\,,j)$ is a *twin zero* if it is even and there exists $(a\,,b) \in \mathbf{Z}_+^2$ such that: (i) $0 < |i-a| + |j-b| \leq 100$ [say]; and (ii) $S(a\,,b) = 0$. Let $d(\epsilon\,,N)$ denote the number of twin zeros that lie in the following domain:

$$D(\epsilon\,,N) := \left\{(i\,,j) \in \mathbf{Z}_+^2 : \, \epsilon i < j < i/\epsilon\,, 1 < i < N\right\}. \tag{4.7}$$

   Is it true that $\lim_{N\to\infty} d(\epsilon\,,N) = \infty$ a.s. for all $\epsilon \in (0\,,1)$?

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