Weak nonmild solutions to some SPDEs*

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Abstract

We study the nonlinear stochastic heat equation driven by space-time white noise in the case that the initial datum u_0 is a (possibly signed) measure. In this case, one cannot obtain a mild random-field solution in the usual sense. We prove instead that it is possible to establish the existence and uniqueness of a weak solution with values in a suitable function space. Our approach is based on a construction of a generalized definition of a stochastic convolution via Young-type inequalities.

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1 Introduction

Let us consider the nonlinear stochastic heat equation

$$\frac{\partial}{\partial t} u_t(x) = (\mathcal{L}u_t)(x) + \sigma(u_t(x))\dot{W}(t\,,x) \qquad (t \geq 0, x \in \mathbf{R}), \qquad (1.1)$$

where: (i) $\mathcal{L}$ is the generator of a real-valued Lévy process $\{X_t\}_{t \geq 0}$ with Lévy exponent Ψ, normalized so that $\mathrm{E}e^{i\xi X_t} = e^{-t\Psi(\xi)}$ for every $\xi \in \mathbf{R}$ and

$t \geq 0$; (ii) $\sigma : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous with Lipschitz constant Lip_σ; (iii) $\dot{W}$ is space-time white noise; and (iv) The initial datum u_0 is a signed Borel measure on $\mathbf{R}$.

Equation (1.1) arises in many different contexts; three notable examples are Bertini and Cancrini [1], Gyöngy and Nualart [15], and Carmona and Molchanov [6].

In the case that $u_0 : \mathbf{R} \to \mathbf{R}_+$ is a bounded measurable function, the theory of Dalang [10] shows that there exists a unique random-field mild solution $\{u_t(x)\}_{t \geq 0, x \in \mathbf{R}}$ provided that

$$\Upsilon(\beta) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}\xi}{\beta + 2\mathrm{Re}\Psi(\xi)} < \infty \quad \text{for some, hence all, } \beta > 0. \quad (1.2)$$

In general, Dalang's Condition (1.2) cannot be improved upon [10, 19].

Dalang's condition (1.2) implies also that the Lévy process X has transition functions $p_t(x)$ [14, Lemma 8.1]; i.e., for all measurable $f : \mathbf{R} \to \mathbf{R}_+$,

$$\mathrm{E}f(X_t) = \int_{-\infty}^{\infty} p_t(z)f(z)\,\mathrm{d}z \qquad \text{for all } t > 0. \quad (1.3)$$

A *mild solution* in this setting is a random field $\{u_t(x)\}_{t \geq 0, x \in \mathbf{R}}$ that satisfies

$$u_t(x) = (P_t u_0)(x) + \int_{[0,t] \times \mathbf{R}} p_{t-s}(y-x)\sigma(u_s(y))\,W(\mathrm{d}s\,\mathrm{d}y) \quad \text{a.s.}, \quad (1.4)$$

for all $t \geq 0$ and $x \in \mathbf{R}$, where $\{P_t\}_{t \geq 0}$ denotes the semigroup associated to the process X, and the stochastic integral is understood as a Walsh martingale-measure stochastic integral [20]. Notice that (1.4) can be rewritten in the following form: For all $(t, x) \in \mathbf{R}_+ \times \mathbf{R}$,

$$u_t(x) = (P_t u_0)(x) + \left(\tilde{p} * (\sigma \circ u)\dot{W}\right)_t (x) \quad \text{a.s.}, \quad (1.5)$$

where "$*$" denotes "stochastic convolution"; see (2.1) below and $\tilde{p}_t(x) = p_t(-x)$ for all $x \in \mathbf{R}$.

In the case that u_0 is not a bounded and measurable function, but instead a (possibly signed) Borel measure on $\mathbf{R}$, the solution u cannot be defined as

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a random field, but has to be considered as a process of function-space type. As a consequence, the stochastic convolution in (1.4) is not well defined in the sense of Walsh. Section 2 below is devoted to extending the definition of the stochastic convolution of a process Γ with respect to $Z\dot{W}$ in the case that Z is in a suitable Banach space $\boldsymbol{B}_{\beta,\eta}^{k}$ of random processes. The key step of this extension involves developing a kind of "stochastic Young inequality" (Proposition 2.1). Such an inequality appeared earlier in [9], in a different context, in order to obtain intermittency properties for equation (1.1) in the case that u_0 is a lower semicontinuous bounded function of compact support.

In Section 3 we establish the existence and uniqueness of a weak solution to (1.1). Namely, we prove that Dalang's condition (1.2) implies that if $u_0 = \mu$ is a (possibly signed) Borel measure on $\mathbf{R}$ that satisfies a suitable integrability condition (3.5), then there exists a unique $u \in \boldsymbol{B}_{\beta,\eta}^{k}$ such that u almost surely satisfies (1.5) for almost every $t \geq 0$ and $x \in \mathbf{R}$. This solution is *not* a random field; but it *is* a function-space-valued solution.

In Section 4 we prove that our condition for existence and uniqueness is unimprovable. And in Section 5 we mention briefly examples of initial data u_0 that lead to the existence and uniqueness of a weak solution to (1.1), together with further remarks that explain what happens if we study the 1-D stochastic wave equation in place of (1.1).

2 Generalized stochastic convolutions

Let $\Gamma : (0\,,\infty) \times \mathbf{R} \to \mathbf{R}$ be measurable, and $Z := \{Z_t(x)\}_{t>0, x\in\mathbf{R}}$ be a predictable random field in the sense of Walsh [20, p. 292]. Let us define the *stochastic convolution* $\Gamma * Z\dot{W}$ of the process Γ with the noise $Z\dot{W}$ as the predictable random field

$$(\Gamma * Z\dot{W})_t(x) := \int_{[0,t]\times\mathbf{R}} \Gamma_{t-s}(x-y) Z_s(y)\, W(\mathrm{d}s\,\mathrm{d}y). \qquad (2.1)$$

The preceding is defined as a stochastic integral with respect to the martingale measure $Z\dot{W}$ in the sense of Walsh [20, Theorem 2.5], and is well defined in the sense of Walsh [20, Chapter 2] provided that the following

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condition holds for all $t > 0$ and $x \in \mathbf{R}$:

$$\left\| (\Gamma * Z\dot{W})_t(x) \right\|_2^2 = \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \ [\Gamma_{t-s}(x-y)]^2 \, \|Z_s(y)\|_2^2 < \infty. \quad (2.2)$$

Let $\boldsymbol{W}^2$ denote the collection of all predictable random fields Z that satisfy the following: $Z_t(x) \in L^2(\mathrm{P})$ and

$$\int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \ [\Gamma_{\tau-s}(x-y)]^2 \, \|Z_s(y)\|_2^2 < \infty, \quad (2.3)$$

for all $0 < t \le \tau$ and $x \in \mathbf{R}$.

We may think of the elements of $\boldsymbol{W}^2$ as *Walsh-integrable random fields*. And because (2.3) implies (2.2), the preceding discussion tells us that the stochastic convolution $\Gamma * Z\dot{W}$ is a well-defined predictable random field for every $Z \in \boldsymbol{W}^2$.

Our present goal is to extend the definition of the stochastic convolution of Z so that it includes more general random processes Z. Other extensions of this stochastic convolutions have been developed for other purposes as well [8, 10, 11, 17].

Let us choose and fix a real number $k \in [2, \infty)$, and define $\boldsymbol{L}^k$ to be the collection of all predictable random fields $\{Z_t(x)\}_{t>0, x\in\mathbf{R}}$ such that $Z_t(x) \in L^k(\mathrm{P})$ for all $t > 0$ and $x \in \mathbf{R}$. Let $M(\mathbf{R})$ be the space of σ-finite Borel measures on $\mathbf{R}$. For every $\beta > 0$ and $\eta \in M(\mathbf{R})$ we can define a norm on $\boldsymbol{L}^k$ as follows: For every $Z \in \boldsymbol{L}^k$,

$$\mathcal{N}_{\beta,\eta}^k(Z) := \left(\int_0^{\infty} \mathrm{e}^{-\beta t} \, \mathrm{d}t \, \sup_{z\in\mathbf{R}} \int_{-\infty}^{\infty} \eta(\mathrm{d}x) \ \|Z_t(x-z)\|_k^2 \right)^{1/2}. \quad (2.4)$$

Here and throughout, we use implicitly the following observation: If $Z, Z' \in \boldsymbol{L}^k$ satisfy $\mathcal{N}_{\beta,\eta}^k(Z - Z') = 0$, then Z and Z' are modifications of one another. There is an obvious converse as well: If Z and Z' are modifications of one another, then $\mathcal{N}_{\beta,\eta}^k(Z - Z') = 0$. We omit the elementary proof.

Our next proposition is a "stochastic Young's inequality," and plays a key role in our extension of Walsh-type stochastic convolutions. But first let us introduce some notation that will be used here and throughout.

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Throughout this paper, z_k denotes the optimal constant in Burkholder's $L^k(\mathrm{P})$-inequality for continuous square-integrable martingales; its precise value involves zeroes of Hermite polynomials, and has been computed by Davis [13]. By the Itô isometry, $z_2 = 1$. Carlen and Kree [5, Appendix] have shown that $z_k \le 2\sqrt{k}$ for all $k \ge 2$, and moreover $z_k = (2 + o(1))\sqrt{k}$ as $k \to \infty$.

Proposition 2.1 (A stochastic Young's inequality). *For every $k \in [2, \infty)$, $Z \in \boldsymbol{W}^2 \cap \boldsymbol{L}^k$, $\eta \in M(\mathbf{R})$, and $\beta > 0$,*

$$\mathcal{N}_{\beta,\eta}^k(\Gamma * Z\dot{W}) \le z_k \left(\int_0^\infty \mathrm{e}^{-\beta t} \|\Gamma_t\|_{L^2(\mathbf{R})}^2 \, \mathrm{d}t \right)^{1/2} \cdot \mathcal{N}_{\beta,\eta}^k(Z), \qquad (2.5)$$

where z_k is the optimal constant in Burkholder's inequality.

Remark 2.2. We emphasize that $\boldsymbol{W}^2 \cap \boldsymbol{L}^2 = \boldsymbol{W}^2$. □

Before we prove Proposition 2.1 let us first describe how it can be used to extend stochastic convolutions. Proposition 2.1 will be proved after that extension is described.

Let $\boldsymbol{B}_{\beta,\eta}^k$ denote the completion of $\boldsymbol{W}^2 \cap \boldsymbol{L}^k$ under the norm $\mathcal{N}_{\beta,\eta}^k$.[1] It follows then that $\boldsymbol{B}_{\beta,\eta}^k$ is a Banach space of predictable processes [identified up to evanescence].

Proposition 2.1 immediately implies that if

$$\Upsilon(\beta) := \int_0^\infty \mathrm{e}^{-\beta t} \|\Gamma_t\|_{L^2(\mathbf{R})}^2 \, \mathrm{d}t < \infty, \qquad (2.6)$$

then $Z \mapsto \Gamma * Z\dot{W}$ has a unique extension to all $Z \in \boldsymbol{B}_{\beta,\eta}^k$, and the resulting extension—written still as $Z \mapsto \Gamma * Z\dot{W}$—defines a bounded linear operator from $\boldsymbol{B}_{\beta,\eta}^k$ into itself. And the operator norm is at most the square root of the *Dalang integral* $\Upsilon(\beta)$. [In the case that $\Gamma_t(x)$ denotes the transition density of a Lévy process with Lévy exponent Ψ, Plancherel's theorem implies that $\Upsilon(\beta)$ is the same Dalang integral as in (1.2); see (3.1) below as well.]

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[1] The latter is of course a norm on equivalence classes of modifications of random fields and not on random fields themselves. But we abuse notation as it is standard.

From now on, we deal solely with this extension of the stochastic convolution. However, we point out the there is a great deal of variability in this extension, as the parameters $\beta > 0$, $k \in [2, \infty)$, and $\eta \in M(\mathbf{R})$ can take on many different values.

Let us conclude this section by establishing our stochastic Young's inequality.

The proof of Proposition 2.1 relies on an elementary estimate for Walsh-type stochastic integrals.

Lemma 2.3. *For all real numbers $t > 0$, $x \in \mathbf{R}$, and $k \in [2, \infty)$, and for every $Z \in \mathbf{W}^2 \cap \mathbf{L}^k$,*

$$
\left\| (\Gamma * Z\dot{W})_t(x) \right\|_k^k \le z_k^k \left(\int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \ [\Gamma_{t-s}(x-y)]^2 \|Z_s(y)\|_k^2 \right)^{k/2}. \quad (2.7)
$$

Proof. Condition (2.3) implies that if $0 < t \le \tau$, then

$$
(\Gamma * Z\dot{W})_{t,\tau}(x) := \int_{[0,t] \times \mathbf{R}} \Gamma_{\tau-s}(x-y) Z_s(y) \, W(\mathrm{d}s \, \mathrm{d}y) \quad (2.8)
$$

is well defined and in $L^2(\mathrm{P})$. Moreover,

$$
\left\| (\Gamma * Z\dot{W})_{t,\tau}(x) \right\|_2 = \left(\int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \ [\Gamma_{\tau-s}(x-y)]^2 \, \|Z_s(y)\|_2^2 \right)^{1/2}. \quad (2.9)
$$

Walsh's theory of martingale measures [20, Theorem 2.5] tells us that the stochastic process $(0, \tau] \ni t \mapsto (\Gamma * Z\dot{W})_{t,\tau}(x)$ is a continuous $L^2(\mathrm{P})$-martingale. Therefore, Davis's refinement [13] of Burkholder's inequality [2, 3, 16] implies that

$$
\left\| (\Gamma * Z\dot{W})_{t,\tau}(x) \right\|_k^k \le z_k^k \left\| \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \ [\Gamma_{\tau-s}(x-y)]^2 \, [Z_s(y)]^2 \right\|_{k/2}^{k/2}. \quad (2.10)
$$

And it follows from Minkowski's inequality that

$$\left\| (\Gamma * Z\dot{W})_{t,\tau}(x) \right\|_k^k \leq z_k^k \int_0^t \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}y \ [\Gamma_{\tau-s}(x-y)]^2 \, \|Z_s(y)\|_k^2. \quad (2.11)$$

The lemma follows from this upon setting $\tau := t$. $\qquad\square$

Proof of Proposition 2.1. The original construction of Walsh implies that $\|(\Gamma * Z\dot{W})_t(x)\|_k$ defines a Borel-measurable function of $(t,x) \in (0,\infty) \times \mathbf{R}$. Indeed, it suffices to verify this measurability assertion in the case that Z is a simple function in the sense of Walsh [20, p. 292], in which case the said measurability follows from a direct computation.

We may apply Lemma 2.3 with $x-z$ in place of the variable x, and then integrate $[\mathrm{d}\eta]$ to obtain

$$
\begin{aligned}
\int_{-\infty}^\infty & \eta(\mathrm{d}x) \ \left\| (\Gamma * Z\dot{W})_t(x-z) \right\|_k^2 \\
&\leq z_k^2 \int_{-\infty}^\infty \eta(\mathrm{d}x) \int_0^t \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}y \ [\Gamma_{t-s}(x-z-y)]^2 \|Z_s(y)\|_k^2 \\
&= z_k^2 \int_{-\infty}^\infty \eta(\mathrm{d}x) \int_0^t \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}y \ [\Gamma_{t-s}(y)]^2 \|Z_s(x-z-y)\|_k^2 \\
&\leq z_k^2 \int_0^t \mathrm{d}s \ \|\Gamma_{t-s}\|_{L^2(\mathbf{R})}^2 \sup_{v\in\mathbf{R}} \int_{-\infty}^\infty \eta(\mathrm{d}x) \ \|Z_s(x-v)\|_k^2.
\end{aligned}
\tag{2.12}
$$

Or equivalently,

$$
\begin{aligned}
\sup_{z\in\mathbf{R}} \int_{-\infty}^\infty & \eta(\mathrm{d}x) \ \left\| (\Gamma * Z\dot{W})_t(x-z) \right\|_k^2 \\
&\leq z_k^2 \int_0^t \mathrm{d}s \ \|\Gamma_{t-s}\|_{L^2(\mathbf{R})}^2 \sup_{z\in\mathbf{R}} \int_{-\infty}^\infty \eta(\mathrm{d}x) \ \|Z_s(x-z)\|_k^2.
\end{aligned}
\tag{2.13}
$$

Multiply both sides by $\exp(-\beta t)$, integrate $[\mathrm{d}t]$ and use Laplace transforms properties for convolutions to obtain the result. $\qquad\square$

Proposition 2.4. *Suppose $\sigma : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous and $Z, Z^* \in$*

$\boldsymbol{B}_{\beta,\eta}^k$ *for some* $k \in [2, \infty)$, $\beta > 0$, *and* $\eta \in M(\mathbf{R})$. *Then,*

$$\mathcal{N}_{\beta,\eta}^k(\sigma \circ Z - \sigma \circ Z^*) \le \mathrm{Lip}_\sigma \cdot \mathcal{N}_{\beta,\eta}^k(Z - Z^*). \qquad (2.14)$$

Proof. If $Z, Z^* \in \boldsymbol{W}^2 \cap \boldsymbol{L}^k$, then this is immediate. In the general case we proceed by approximation: Let $Z^1, Z^2, \ldots, Z^{1,*}, Z^{2,*}, \ldots$ be in $\boldsymbol{W}^2 \cap \boldsymbol{L}^k$ such that $Z^n \to Z$ and $Z^{n,*} \to Z^*$ in $\boldsymbol{B}_{\beta,\eta}^k$, as $n \to \infty$. By going to a subsequence, if necessary, we can [and will!] also assume that

$$\mathcal{N}_{\beta,\eta}^k(Z^n - Z^{n+1}) + \mathcal{N}_{\beta,\eta}^k(Z^{n,*} - Z^{n+1,*}) \le 2^{-n} \quad \text{for all } n \ge 1. \qquad (2.15)$$

It follows also that for all $n \ge 1$,

$$\mathcal{N}_{\beta,\eta}^k(\sigma \circ Z^n - \sigma \circ Z^{n+1}) + \mathcal{N}_{\beta,\eta}^k(\sigma \circ Z^{n,*} - \sigma \circ Z^{n+1,*}) \le \mathrm{Lip}_\sigma \cdot 2^{-n}. \qquad (2.16)$$

Of course, this implies immediately that $\sigma \circ Z^n$ and $\sigma \circ Z^{n,*}$ converge in $\boldsymbol{B}_{\beta,\eta}^k$. It suffices to prove that the mentioned limits are respectively $\sigma \circ Z$ and $\sigma \circ Z^*$. But (2.16) implies that

$$\int_0^\infty \mathrm{e}^{-\beta t} \, \mathrm{d}t \sum_{n=1}^\infty \sup_{z \in \mathbf{R}} \int_{-\infty}^\infty \eta(\mathrm{d}x) \, \|\Delta_t^n(x - z)\|_k^2 < \infty, \qquad (2.17)$$

where $\Delta_t^n(x)$ stands for any one of the following four quantities:

- $Z_t^n(x) - Z_t^{n+1}(x)$;

- $Z_t^{n,*}(x) - Z_t^{n+1,*}(x)$;

- $\sigma(Z_t^n(x)) - \sigma(Z_t^{n+1}(x))$; or

- $\sigma(Z_t^{n,*}(x)) - \sigma(Z_t^{n+1,*}(x))$.

Because $\sum_{n=1}^\infty \sup_{z \in \mathbf{R}}(\cdots) \le \sup_{z \in \mathbf{R}} \sum_{n=1}^\infty (\cdots)$ in (2.17), it follows readily that for almost every pair $(t, x) \in \mathbf{R}_+ \times \mathbf{R}$:

- $\lim_{n \to \infty} Z_t^n(x) = Z_t(x)$ almost surely;

- $\lim_{n \to \infty} Z_t^{n,*}(x) = Z_t^*(x)$ almost surely;

- $\lim_{n \to \infty} \sigma(Z_t^n(x)) = \sigma(Z_t(x))$ almost surely; and

- $\lim_{n \to \infty} \sigma(Z_t^{n,*}(x)) = \sigma(Z_t^*(x))$ almost surely.

[Note the order of the quantifiers!] We showed earlier that $\lim_{n \to \infty} \sigma \circ Z^n$ and $\lim_{n \to \infty} \sigma \circ Z^{n,*}$ exist in $\boldsymbol{B}_{\beta,\eta}^k$. The preceding shows that those limits are respectively $\sigma \circ Z$ and $\sigma \circ Z^*$. This completes the proof. $\qquad \square$

3 Existence and Uniqueness

This section is devoted to the statement and proof of the existence and uniqueness of a weak solution to (1.1). We will make use of the generalized stochastic convolution developed in Section 2.

Before we proceed further, let us observe that from now on $\Gamma_t(x)$ of the previous section is chosen to be equal to the modified transition functions $\tilde{p}_t(x)$, in which case Dalang's integral can be computed from Plancherel's formula as follows:

$$\Upsilon(\beta) = \int_0^\infty e^{-\beta t} \|p_t\|_{L^2(\mathbf{R})}^2 \, \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{\mathrm{d}\xi}{\beta + 2\mathrm{Re}\Psi(\xi)}. \qquad (3.1)$$

In particular, the Υ of (2.6) and that of (1.2) are equal in the present setting.

Next we identify our notion of "solution" to (1.1) in the case that $u_0 = \mu$ is a measure.

Suppose first that u_0 is a nice *function* and (1.1) has a mild solution u with initial datum u_0. Then for all $t > 0$ and $x \in \mathbf{R}$,

$$\mathrm{P}\left\{ u_t(x) = (P_t u_0)(x) + \left(\tilde{p} * (\sigma \circ u)\dot{W} \right)_t (x) \right\} = 1. \qquad (3.2)$$

Consequently, Fubini's theorem tells us that every mild solution u to (1.1) with initial function $\mu := u_0$ is a *weak solution* in the sense that the following holds with probability one [note the order of the quantifiers!]:

$$u_t(x) = (P_t \mu)(x) + \left(\tilde{p} * (\sigma \circ u)\dot{W} \right)_t (x) \quad \text{for a.e. } (t, x) \in \mathbf{R}_+ \times \mathbf{R}. \qquad (3.3)$$

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It is easy to see that the preceding agrees with Walsh's definition of a weak solution [20, p. 309].

Now we consider (1.1) in the case that $u_0 = \mu$ is a possibly-signed Borel measure on $\mathbf{R}$.

Let us suppose that Dalang's condition (2.6) holds, and consider an arbitrary $u \in \boldsymbol{B}_{\beta,\eta}^k$. Since σ is Lipschitz continuous, it follows that $\sigma \circ u \in \boldsymbol{B}_{\beta,\eta}^k$. Therefore, we can conclude that the stochastic convolution $\tilde{p} * (\sigma \circ u) \dot{W}$ is a well-defined mathematical object, as was shown in the previous section. Consequently, we can try to find a solution u to (1.1) with $u_0 = \mu$ by seeking to find $u \in \boldsymbol{B}_{\beta,\eta}^k$ such that

$$u = P_\bullet \mu + \tilde{p} * (\sigma \circ u) \dot{W}, \tag{3.4}$$

where the equality is understood as equality of elements of $\boldsymbol{B}_{\beta,\eta}^k$. Of course, we implicitly are assuming that $P_\bullet \mu \in \boldsymbol{B}_{\beta,\eta}^k$ as well. That condition is clearly satisfied if

$$\int_0^\infty \mathrm{e}^{-\beta s} \, \mathrm{d}s \sup_{z \in \mathbf{R}} \int_{-\infty}^\infty \eta(\mathrm{d}x) \, |(P_s \mu)(x-z)|^2 < \infty. \tag{3.5}$$

Then, u is a solution of function-space type to (1.1) with $u_0 = \mu$. But it has more structure than that. Indeed, suppose that: (i) (3.5) holds; and (ii) There exists $u \in \boldsymbol{B}_{\beta,\eta}^k$ that satisfies (3.4). Then the preceding discussion shows also that u is a weak solution to (1.1) in the sense of Walsh [20, p. 309]. And it would be hopeless to try to prove that such a u is a mild solution, as there is no natural way to define $u_t(x)$ for all $t > 0$ and $x \in \mathbf{R}$.

Throughout the remainder of this section we choose $\eta \in M(\mathbf{R})$. In the case that $\sigma(0) \neq 0$, then we assume additionally that η is a finite measure.

Theorem 3.1. *Consider* (1.1) *subject to* $u_0 = \mu$, *where* μ *is a signed measure that satisfies* (3.5). *If* (1.2) *holds and*

$$\Upsilon(\beta) < \frac{1}{(z_k \mathrm{Lip}_\sigma)^2}, \tag{3.6}$$

then there exists a solution $u \in \boldsymbol{B}_{\beta,\eta}^k$ *that satisfies* (3.4). *Moreover,* u *is*

unique in $\boldsymbol{B}_{\beta,\eta}^k$; i.e., if there exists another weak solution v that is in $\boldsymbol{B}_{\beta,\eta}^k$ for some $k \geq 2$, then v is a modification of u.

Proof. First, we argue that we can always choose β such that (3.6) holds.

Indeed, Condition (3.5) implies that $\mathcal{N}_{\beta,\eta}^k(P_\bullet u_0) < \infty$ for all $\beta > 0$ and $k \in [2,\infty)$. Also, because of Dalang's condition (1.2), and by the monotone convergence theorem, $\lim_{\alpha \to \infty} \Upsilon(\alpha) = 0$. Therefore, we can combine these two observations to deduce that (3.6) holds for all β large, where $1/0 := \infty$. Throughout the remainder of the proof, we hold fixed a β that satisfies (3.6).

Set $u_t^{(0)} := 0$, and iteratively define

$$u^{(n+1)} := P_\bullet \mu + \tilde{p} * \left([\sigma \circ u^{(n)}] \dot{W} \right). \tag{3.7}$$

These $u^{(n+1)}$'s are all well defined elements of $\boldsymbol{B}_{\beta,\eta}^k$. In fact, it follows from Proposition 2.1 that for all $n \geq 0$,

$$\mathcal{N}_{\beta,\eta}^k \left(u^{(n+1)} \right) \leq \mathcal{N}_{\beta,\eta}^k \left(P_\bullet \mu \right) + z_k \sqrt{\Upsilon(\beta)} \mathcal{N}_{\beta,\eta}^k \left(\sigma \circ u^{(n)} \right). \tag{3.8}$$

And because $|\sigma(z)| \leq |\sigma(0)| + \mathrm{Lip}_\sigma |z|$ for all $z \in \mathbf{R}$,

$$
\begin{aligned}
&\mathcal{N}_{\beta,\eta}^k \left(u^{(n+1)} \right) \\
&\leq \mathcal{N}_{\beta,\eta}^k \left(P_\bullet \mu \right) + z_k \sqrt{\Upsilon(\beta)} \left[|\sigma(0)| \cdot \mathcal{N}_{\beta,\eta}^k(\mathbf{1}) + \mathrm{Lip}_\sigma \cdot \mathcal{N}_{\beta,\eta}^k \left(u^{(n)} \right) \right],
\end{aligned}
\tag{3.9}
$$

where $\mathbf{1}_t(x) := 1$ for all $t > 0$ and $x \in \mathbf{R}$. In particular, $u^{(l)} \in \boldsymbol{B}_{\beta,\eta}^k$ for all $l \geq 0$, by induction. This is clear if $\sigma(0) = 0$; and if $\sigma(0) \neq 0$, then it is also true because $\mathcal{N}_{\beta,\eta}^k(\mathbf{1}) = \sqrt{\eta(\mathbf{R})/\beta} < \infty$, thanks to the finiteness assumption on η [for the case $\sigma(0) \neq 0$]. Moreover, (3.6) and induction together show more; namely, that $\sup_{n \geq 0} \mathcal{N}_{\beta,\eta}^k(u^{(n)}) < \infty$.

A similar computation, this time using also Proposition 2.4, shows that for all $n \geq 1$,

$$\mathcal{N}_{\beta,\eta}^k \left(u^{(n+1)} - u^{(n)} \right) \leq z_k \mathrm{Lip}_\sigma \sqrt{\Upsilon(\beta)} \cdot \mathcal{N}_{\beta,\eta}^k \left(u^{(n)} - u^{(n-1)} \right). \tag{3.10}$$

And (3.6) implies that $\sum_{n=0}^\infty \mathcal{N}_{\beta,\eta}^k(u^{(n+1)} - u^{(n)}) < \infty$, therefore, $\{u^{(n)}\}_{n=0}^\infty$

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is a Cauchy sequence in $\boldsymbol{B}_{\beta,\eta}^k$. Let $u := \lim_{n\to\infty} u^{(n)}$, where the limit takes place in $\boldsymbol{B}_{\beta,\eta}^k$. According to Proposition 2.1,

$$\mathcal{N}_{\beta,\eta}^k \left(\tilde{p} * u^{(n)} \dot{W} - \tilde{p} * u \dot{W} \right) \le z_k \sqrt{\Upsilon(\beta)} \cdot \mathcal{N}_{\beta,\eta}^k \left(u^{(n)} - u \right)$$
$$\to 0 \qquad \text{as } n \to \infty. \tag{3.11}$$

It follows readily from these remarks that $\mathcal{N}_{\beta,\eta}^k (u - P_\bullet \mu + \tilde{p} * (\sigma \circ u)\dot{W}) = 0$. That is, u satisfies (3.3) for almost all $(t,x) \in \mathbf{R}_+ \times \mathbf{R}$; see also (3.4). This proves part (1) of the theorem.

In order to prove the second part, let us suppose that there exists another "weak solution" $v \in \boldsymbol{B}_{\beta,\eta}^k$. Then, $\delta := u - v \in \boldsymbol{B}_{\beta,\eta}^k$ and

$$\delta = \tilde{p} * \left([\sigma \circ u]\dot{W} \right) - \tilde{p} * \left([\sigma \circ v]\dot{W} \right) = \tilde{p} * \left([\sigma \circ u - \sigma \circ v]\dot{W} \right). \tag{3.12}$$

[The second identity follows from the very construction of our stochastic convolution, using the fact that $Z \mapsto \tilde{p} * Z\dot{W}$ is a bounded *linear* map from $\boldsymbol{B}_{\beta,\eta}^k$ to itself.] Propositions 2.1 and 2.4 together imply the following:

$$\mathcal{N}_{\beta,\eta}^k (\delta) \le z_k \sqrt{\Upsilon(\beta)} \cdot \mathcal{N}_{\beta,\eta}^k (\sigma \circ u - \sigma \circ v)$$
$$\le z_k \mathrm{Lip}_\sigma \sqrt{\Upsilon(\beta)} \cdot \mathcal{N}_{\beta,\eta}^k (\delta). \tag{3.13}$$

Thanks to (3.6), $\mathcal{N}_{\beta,\eta}^k (u - v) = \mathcal{N}_{\beta,\eta}^k (\delta) = 0$. This readily implies that u and v are modifications of one another, as well. $\square$

4   On Condition (3.6)

Let us consider the measure $\eta_m \in M(\mathbf{R})$ defined by

$$\eta_m(\mathrm{d}x) = \mathrm{e}^{-|x|/m}\,\mathrm{d}x, \tag{4.1}$$

where $m > 0$ is fixed. If $\sigma(0) = 0$, then we may take $m := \infty$, whence $\eta(\mathrm{d}x) = \mathrm{d}x$.

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Theorem 4.1. *Suppose (1.1) has a solution $u \in \cap_{m>0} \boldsymbol{B}^2_{\beta,\eta_m}$ with $u_0 = \mu$ for a nonvoid signed Borel measure μ on $\mathbf{R}$ with $|\mu|(\mathbf{R}) < \infty$. Suppose also that $\mathrm{L}_\sigma := \inf_{z \in \mathbf{R}} |\sigma(z)/z| > 0$. Then, β satisfies $\Upsilon(\beta) < \mathrm{L}_\sigma^{-2}$.*

Proof. Let $\mathcal{M}_\beta$ be the norm defined by

$$\mathcal{M}_\beta(Z) := \left(\int_0^\infty \mathrm{e}^{-\beta t} \, \mathrm{d}t \int_{-\infty}^\infty \mathrm{e}^{-|x|/m} \, \mathrm{d}x \ \|Z_t(x)\|_2^2 \right)^{1/2} . \tag{4.2}$$

Notice that M_β is similar to $\mathcal{N}_{\beta,\eta_m}$, but is missing a supremum on Z in the space variable; cf. (2.4). Moreover, $\mathcal{M}_\beta(u) \leq \mathcal{N}^2_{\beta,\eta_m}(u) < \infty$. Note that if $H, Z \in \boldsymbol{B}^2_{\beta,\eta_m}$ with one of them—say H—random and the other one deterministic, then we have $[\mathcal{M}_\beta(H + G)]^2 = [\mathcal{M}_\beta(H)]^2 + [\mathcal{M}_\beta(G)]^2$. This is a direct computation if $H, G \in \boldsymbol{W}^2$; the general case follows from approximation (we omit the details because the method appears already during the course of the proof of Proposition 2.4). It follows that

$$[\mathcal{M}_\beta(u)]^2 = [\mathcal{M}_\beta(P_\bullet\mu)]^2 + \left[\mathcal{M}_\beta \left(\tilde{p} * ([\sigma \circ u]\dot{W}) \right) \right]^2 . \tag{4.3}$$

The method of proof of Proposition 2.4, together with the simple bound, $\mathrm{e}^{-|x|/m} \geq \mathrm{e}^{-|x-y|/m} \cdot \mathrm{e}^{-|y|/m}$, shows also that

$$\mathcal{M}_\beta \left(\tilde{p} * ([\sigma \circ u]\dot{W}) \right) \geq \mathrm{L}_\sigma \mathcal{M}_\beta \left(\tilde{p} * u\dot{W} \right) . \tag{4.4}$$

But

$$\mathcal{M}_\beta(\tilde{p} * Z\dot{W}) = \left(\int_0^\infty \mathrm{e}^{-\beta t} \|p_t\|_{L^2(\mathbf{R})}^2 \, \mathrm{d}t \right)^{1/2} \cdot \mathcal{M}_\beta(Z) . \tag{4.5}$$

[Again one proves this first for nice Z's and then take limits.] Therefore,

$$\mathcal{M}_\beta \left(\tilde{p} * ([\sigma \circ u]\dot{W}) \right) \geq \mathrm{L}_\sigma \sqrt{\Upsilon(\beta)} \cdot \mathcal{M}_\beta(u) . \tag{4.6}$$

Combine this with (4.3) to find that

$$[\mathcal{M}_\beta(u)]^2 \geq [\mathcal{M}_\beta(P_\bullet\mu)]^2 + \mathrm{L}_\sigma^2 \Upsilon(\beta) [\mathcal{M}_\beta(u)]^2 . \tag{4.7}$$

Now suppose, to the contrary, that $\Upsilon(\beta) \geq \mathrm{L}_\sigma^{-2}$. Then, it follows that $\mathcal{M}_\beta(P_\bullet \mu) = 0$ regardless of the value of m; i.e., for all $m > 0$,

$$\int_0^\infty \mathrm{e}^{-\beta t}\, \mathrm{d}t \int_{-\infty}^\infty \mathrm{e}^{-|x|/m}\, \mathrm{d}x\; |(P_t \mu)(x)|^2 = 0. \tag{4.8}$$

Let $m \uparrow \infty$ and apply the monotone convergence theorem, and then the Plancherel theorem, in order to deduce that

$$
\begin{aligned}
0 &= \int_0^\infty \mathrm{e}^{-\beta t} \|P_t \mu\|_{L^2(\mathbf{R})}^2\, \mathrm{d}t \\
&= \frac{1}{2\pi} \int_0^\infty \mathrm{e}^{-\beta t}\, \mathrm{d}t \int_{-\infty}^\infty \mathrm{d}\xi\; \mathrm{e}^{-2t\mathrm{Re}\Psi(\xi)} |\hat{\mu}(\xi)|^2 \\
&= \frac{1}{2\pi} \int_{-\infty}^\infty \frac{|\hat{\mu}(\xi)|^2}{\beta + 2\mathrm{Re}\Psi(\xi)}\, \mathrm{d}\xi.
\end{aligned}
\tag{4.9}
$$

Since Ψ is never infinite, the preceding implies that $\mu \equiv 0$, which is a contradiction. It follows that $\Upsilon(\beta) < \mathrm{L}_\sigma^{-2}$. $\qquad\square$

Theorem 4.1 implies also that Condition 3.6 is sharp: Consider the case that $\mathrm{Lip}_\sigma = \mathrm{L}_\sigma$. [This is the case, for instance, for the parabolic Anderson problem where $\sigma(x) \propto x$, or when β has sharp linear growth.] Then in this case Theorem 3.1 and Theorem 4.1 together imply that (3.6) is a necessary and sufficient condition for the existence of a weak solution to (1.1) that has values in $\cap_{m \geq 1} \boldsymbol{B}_{\beta, \eta_m}^k$.

5   Examples and Remarks.

Example 5.1 (A parabolic Anderson model)**.** Let $\sigma(x) = \lambda x$. In that case, the solution u corresponds to the conditional expected density at time $t \geq 0$ of a branching Lévy process starting with distribution u_0, given white-noise random branching. The case that $\sigma(0) = 0$ and u_0 is a function with compact support is studied in [9], in which intermittency properties are derived. Here, u_0 can be a compactly supported measure (not necessarily a function). If

we let η denote the one-dimensional Lebesgue measure, then (3.5) becomes

$$\int_{-\infty}^{\infty} \frac{|\hat{\mu}(\xi)|^2}{\beta + 2\mathrm{Re}\Psi(\xi)} \, \mathrm{d}\xi < \infty. \tag{5.1}$$

For instance, if $\mu = \delta_0$, then condition (5.1) is precisely Dalang's condition (1.2), and (1.1) admits a weak solution. In this way we can now define properly the solution of the parabolic Anderson model with $u_0 = \delta_0$, which was studied in Bertini and Cancrini [1]. □

Remark 5.2 (A nonlinear stochastic wave equation)**.** It is possible to apply similar techniques to the study of the following nonlinear stochastic wave equation driven by the Laplacian:

$$\frac{\partial^2}{\partial t^2} u_t(x) = \kappa^2 (\Delta u_t)(x) + \sigma(u_t(x))\dot{W}(t,x) \qquad (t \geq 0, x \in \mathbf{R}). \tag{5.2}$$

If the initial conditions u_0 and v_0 are nice functions, then the solution to (5.2) can be written as

$$
\begin{aligned}
u_t(x) = (\Gamma_t' * u_0)(x) + (\Gamma_t * v_0)(x) \\
+ \int_{[0,t]\times\mathbf{R}} \Gamma_{t-s}(y-x)\sigma(u_s(y)) \, W(\mathrm{d}s\,\mathrm{d}y),
\end{aligned} \tag{5.3}
$$

where Γ is the fundamental solution of the 1-dimensional wave equation, namely

$$\Gamma_t(x) := \frac{1}{2}\mathbf{1}_{[-\kappa t, \kappa t]}(x) \qquad \text{for } t > 0 \text{ and } x \in \mathbf{R}, \tag{5.4}$$

and Γ_t' denotes the weak spatial derivative of Γ_t. Then, the existence and uniqueness of a weak solution to (5.2) in the case that u_0 and v_0 are (possibly signed) Borel measures on $\mathbf{R}$ can be established using the techniques of this paper, since the definition of the generalized stochastic convolution applies as well with the 1-D wave propagator Γ above. The conditions on the initial conditions have to be adapted to insure that $\Gamma_t' * u_0$ and $\Gamma_t * v_0$ both are in $\boldsymbol{B}_{\beta,\eta}^k$, but are similar to (3.5). The details on this are left to the reader, as the stochastic wave equation in dimension one has been widely studied already [4, 7, 12, 18–21].

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