An asymptotic theory for randomly-forced discrete nonlinear heat equations*

Mohammud Foondun Davar Khoshnevisan

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Abstract

We study discrete nonlinear parabolic stochastic heat equations of the form, $u_{n+1}(x) - u_n(x) = (\mathcal{L}u_n)(x) + \sigma(u_n(x))\xi_n(x)$, for $n \in \mathbf{Z}_+$ and $x \in \mathbf{Z}^d$, where $\boldsymbol{\xi} := \{\xi_n(x)\}_{n\geq 0, x\in\mathbf{Z}^d}$ denotes random forcing and $\mathcal{L}$ the generator of a random walk on $\mathbf{Z}^d$. Under mild conditions, we prove that the preceding stochastic PDE has a unique solution that grows at most exponentially in time. And that, under natural conditions, it is "weakly intermittent." Along the way, we establish a comparison principle as well as a finite-support property.

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Running Title: The discrete nonlinear heat equation.

1 Introduction

Let us consider a prototypical stochastic heat equation of the following type:

$$\left|\begin{aligned} &\frac{\partial u(t\,,x)}{\partial t} = (\mathcal{L}u)(t\,,x) + \sigma(u(t\,,x))\xi_t(x) \quad \text{for } t > 0 \text{ and } x \in \mathbf{R}, \\ &u(0\,,x) = u_0(x), \end{aligned}\right. \tag{1.1}$$

where u_0 and σ are known nonrandom functions: u_0 is bounded and measurable; $\sigma : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous; $\boldsymbol{\xi} := \{\xi_t\}_{t\geq 0}$ is an infinite-dimensional white noise; and $\mathcal{L}$ is an operator acting on the variable x. It is well known that (1.1) has a unique "mild solution" under natural conditions on $\boldsymbol{\xi}$ and $\mathcal{L}$ [14, 15, 22, 35–37, 44, 45, 51]; we can think of $\boldsymbol{\xi}$ as the "forcing term" as well as the "noise."

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Let us observe that, in (1.1), the operator $\mathcal{L}$ and the noise term compete with one another: $\mathcal{L}$ tends to flatten/smooth the solution $\boldsymbol{u}$, whereas the noise term tends to make $\boldsymbol{u}$ more irregular. This competition was studied in [21] in the case that $\sigma := 1$ and $\mathcal{L} :=$ the L^2-generator of a Lévy process.

The [parabolic] "Anderson model" is an important special case of (1.1). In that case one considers $\mathcal{L} := \kappa\partial_{xx}$ and $\sigma(z) := \nu z$ for fixed $\nu, \kappa > 0$, and interprets $u(t\,, x)$ as the average number of particles—at site x and time t—when the particles perform independent Brownian motions; every particle splits into two at rate $\xi_t(x)$—when $\xi_t(x) > 0$—and is extinguished at rate $-\xi_t(x)$—when $\xi_t(x) < 0$. See Carmona and Molchanov [10, Chapter 1] for this, together with a groundbreaking analysis of the ensuing model. The Anderson model also has important connections to stochastic analysis, statistical physics, random media, cosmology, etc. [3, 4, 6–8, 10–13, 19, 22–27, 29, 31–34, 38, 40, 41, 47, 48, 53, 54].

A majority of the sizable literature on the Anderson model is concerned with establishing a property called "intermittency" [39, 40, 43, 53, 54]. Recall that the *pth moment Liapounov exponent* $\gamma(p)$ is defined as

$$\gamma(p) := \lim_{t\to\infty} \frac{1}{t} \ln \mathrm{E}\left[u(t\,, x)^p\right], \tag{1.2}$$

provided that the limit exists. The solution $\boldsymbol{u} := \{u(t\,, x)\}_{t\geq 0, x\in\mathbf{R}^d}$ to the parabolic Anderson model is said to be *intermittent* if $\gamma(p)$ exists for all $p \geq 1$ and $p \mapsto (\gamma(p)/p)$ is strictly increasing on $[1\,, \infty)$. This mathematical definition describes a "separation of scales" phenomena, and is believed to capture many of the salient features of its physical counterpart in statistical physics and turbulence [2, 39, 43, 50, 54]. For more information see the Introductions of Bertini and Cancrini [3] and Carmona and Molchanov [10].

Recently [20] we considered (1.1) in a fully nonlinear setting with space-time white noise $\boldsymbol{\xi}$ and $\mathcal{L} :=$ the L^2-generator of a Lévy process. We showed that if σ is "asymptotically linear" and u_0 is "sufficiently large," then $p \mapsto \widetilde{\gamma}(p)/p$ is strictly increasing on $[2\,, \infty)$, where

$$\widetilde{\gamma}(p) := \limsup_{t\to\infty} \frac{1}{t} \ln \mathrm{E}\left(|u_t(x)|^p\right). \tag{1.3}$$

This gives evidence of intermittency for solutions of stochastic PDEs. Moreover, bounds on $\widetilde{\gamma}$ were given in terms of the Lipschitz constant of σ and the function

$$\widetilde{\Upsilon}(\beta) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d\xi}{\beta + 2\mathrm{Re}\,\Psi(\xi)}, \quad \text{defined for all } \beta > 0, \tag{1.4}$$

where Ψ denotes the characteristic exponent of the Lévy process generated by the $\mathcal{L}$. It is precisely this connection between $\widetilde{\Upsilon}$ and σ that allows us to describe a relationship between the smoothing effects of $\mathcal{L}$ and the roughening effect of the underlying forcing terms.

There are two physically-relevant classes of bounded initial data u_0 that arise naturally in the literature [3, 6, 40]: **(a)** Where u_0 is bounded below, away from zero; and **(b)** Where u_0 has compact support. Our earlier analysis [20] studies fairly completely Case **(a)**, but fails to say anything about Case **(b)**. We do not know much about **(b)** in fact. Our present goal is to consider instead a *discrete* setting in which we *are* able to analyze Case **(b)**.

There is a large literature on [discrete] partial difference equations of the heat type; see Agarwal [1] and its many chapter bibliographies. Except for the work by Zeldovich et al [54, §5], we have found little on fully-discrete stochastic heat equations (1.1). We will see soon that the discrete setup treated here yields many of the interesting mathematical features that one might wish for, and at low technical cost. For instance, we do not presuppose a knowledge of PDEs and/or stochastic calculus in this paper.

An outline of the paper follows: In §2 we state the main results of the paper; they are proved in §5, after we establish some auxiliary results in §3 and §4. In §6 we compute a version of the second-moment [upper] *Liapounov exponent* of the solution u to the parabolic Anderson model with temporal noise. From a physics point of view, that model is only modestly interesting; but it provides a setting in which we can rigorously verify many of the predictions of the replica method [31]. The replica method itself will not be used however.

Throughout the paper, we define

$$\|X\|_p := \{\mathrm{E}(|X|^p)\}^{1/p} \qquad \text{for all } X \in L^p(\mathrm{P}), \tag{1.5}$$

for every $p \in [1\,,\infty)$.

2 Main results

Throughout we study the following discrete version of (1.1):

$$u_{n+1}(x) - u_n(x) = (\mathcal{L}u_n)(x) + \sigma(u_n(x))\xi_n(x) \quad \text{for } n \geq 0 \text{ and } x \in \mathbf{Z}^d, \tag{2.1}$$

with [known] bounded initial function $u_0 : \mathbf{Z}^d \to \mathbf{R}$ and diffusion $\sigma : \mathbf{R} \to \mathbf{R}$. The operator $\mathcal{L}$ acts on x and is the generator of a random walk on $\mathbf{Z}^d$.

Let $\mathcal{I}$ denote the identity operator and $\mathcal{P} := \mathcal{L} + \mathcal{I}$ the transition operator for $\mathcal{L}$. Then (2.1) is equivalent to the following recursive relation:

$$u_{n+1}(x) = (\mathcal{P}u_n)(x) + \sigma(u_n(x))\xi_n(x). \tag{2.2}$$

Our first contribution is an analysis of (2.1) in the case that the ξ's are i.i.d. with common mean 0 and variance 1 [discrete white noise]. The following function $\Upsilon : (1\,,\infty) \to \mathbf{R}_+$ is the present analogue of $\widetilde{\Upsilon}$ [see (1.4)]:

$$\Upsilon(\lambda) := \frac{1}{(2\pi)^d} \int_{(-\pi,\pi)^d} \frac{\mathrm{d}\xi}{\lambda - |\phi(\xi)|^2} \qquad \text{for all } \lambda > 1, \tag{2.3}$$

where ϕ denotes the characteristic function of the increments of the walk that corresponds to $\mathcal{L}$; that is,

$$\phi(\xi) := \sum_{x \in \mathbf{Z}^d} e^{ix\cdot\xi} P_{0,x}. \tag{2.4}$$

Because Υ is continuous, strictly positive, and strictly decreasing on $(1\,,\infty)$, it has a continuous strictly increasing inverse on $(0\,,\Upsilon(1^-))$. We extend the definition of that inverse by setting

$$\Upsilon^{-1}(x) := \sup\left\{\lambda > 1 :\ \Upsilon(\lambda) > x\right\}, \tag{2.5}$$

where $\sup\varnothing := 1$. Also, let

$$\mathrm{Lip}_\sigma := \sup_{x\neq y} \frac{|\sigma(x)-\sigma(y)|}{|x-y|} \tag{2.6}$$

denote the Lipschitz constant of the function σ [Lip_σ can be infinite]. The following is a discrete counterpart of Theorems 2.1 and 2.7 of [20], and is our first main result.

Theorem 2.1. *Suppose $\boldsymbol{\xi}$ are i.i.d. with mean 0 and variance 1. If u_0 is bounded and σ is Lipschitz continuous, then* (2.1) *has an a.s.-unique solution $\boldsymbol{u}$ which satisfies the following: For all $p\in[2\,,\infty)$,*

$$\limsup_{n\to\infty} \frac1n \sup_{x\in\mathbf{Z}^d} \ln \|u_n(x)\|_p \le \frac12 \ln \Upsilon^{-1}\left((c_p \mathrm{Lip}_\sigma)^{-2}\right), \tag{2.7}$$

where $\Upsilon^{-1}(1/0) := 0$ and $c_p :=$ the optimal constant in Burkholder's inequality for discrete-parameter martingales. Conversely, if $\inf_{x\in\mathbf{Z}^d} u_0(x) > 0$ and $L_\sigma := \inf_{z\in\mathbf{R}} |\sigma(z)/z| > 0$, then

$$\inf_{x\in\mathbf{Z}^d} \limsup_{n\to\infty} \frac1n \ln\|u_n(x)\|_p \ge \frac12 \ln\Upsilon^{-1}\left(L_\sigma^{-2}\right). \tag{2.8}$$

The exact value of c_p is not known [52]. Burkholder's method itself produces $c_p \le 18pq^{1/2}$ [28, Theorem 2.10, p. 23], where $q := p/(p-1)$ denotes the conjugate to p. It is likely that better bounds are known, but we are not aware of them.

Theorem 2.1 has a continuous counterpart in [20]. Next we point out that "u_0 is bounded below" in Theorem 2.1 can some times be replaced by "u_0 has finite support." As far as we know, this does not seem to have a continuous analogue [20]. But first we recall the following standard definition:

Definition 2.2. $\mathcal{L}$ is *local* if there exists $R>0$ such that $P_{0,x}=0$ if $|x|>R$.[1]

Theorem 2.3. *Suppose $\mathcal{L}$ is local and the ξ's are i.i.d. with mean 0 and variance 1. In addition, $u_0 \not\equiv 0$ has finite support, σ is Lipschitz continuous with $\sigma(0)=0$, and $L_\sigma := \inf_{z\in\mathbf{R}}|\sigma(z)/z| > 0$. Then, for all $p\in[2\,,\infty)$, $M_n := \sup_{x\in\mathbf{Z}^d}|u_n(x)|$ satisfies*

$$\begin{aligned}\frac12\ln\Upsilon^{-1}\left(L_\sigma^{-2}\right) &\le \limsup_{n\to\infty}\frac1n\sup_{x\in\mathbf{Z}^d}\ln\|u_n(x)\|_p \\ &\le \limsup_{n\to\infty}\frac1n\ln\|M_n\|_p \le \frac12\ln\Upsilon^{-1}\left((c_p\mathrm{Lip}_\sigma)^{-2}\right).\end{aligned} \tag{2.9}$$

[1] As is sometimes customary, we identify the Fredholm operator $\mathcal{P}^n$ with its kernel, which is merely the n-step transition probability: $P^n_{x,y} = P^n_{0,y-x}$ at $(x\,,y)\in\mathbf{Z}^d\times\mathbf{Z}^d$.

We define the *upper pth-moment Liapounov exponent* $\bar{\gamma}(p)$ as follows:

$$\bar{\gamma}(p) := \limsup_{n\to\infty} \frac{1}{n} \sup_{x\in\mathbf{Z}^d} \ln \mathrm{E}\left[u_n(x)^p\right], \tag{2.10}$$

Definition 2.4. We say that $\boldsymbol{u} := \{u_n(x)\}_{n>0, x\in\mathbf{Z}^d}$ is *weakly intermittent* if $\bar{\gamma}(p) < \infty$ for all positive and finite p, and $p \mapsto (\bar{\gamma}(p)/p)$ is strictly increasing on $[2\,,\infty)$.

Corollary 2.5. *Suppose, in addition to the conditions of Theorem 2.3, that* $C_\xi := \sup_{n\geq 0}\sup_{x\in\mathbf{Z}^d}|\xi_n(x)|$ *is finite and*

$$P_{0,0} \geq C_\xi \mathrm{Lip}_\sigma. \tag{2.11}$$

Then (2.1) *with has a weakly-intermittent solution.*

We emphasize that $\bar{\gamma}(p)$ is *not* an exact discrete version of $\widetilde{\gamma}(p)$, as it is missing absolute values.

Our next result concerns the Anderson model with temporal noise. In other words we consider (2.1) with $\sigma(z) = z$, $\xi_n(x) = \xi_n$ for all $x \in \mathbf{Z}^d$, and $\boldsymbol{\xi} := \{\xi_n\}_{n=0}^\infty$ = i.i.d. random variables. The present model is motivated by Mandelbrot's analysis of random cascades in turbulence [39], and is designed to showcase a family of examples where the predictions of the replica method of Kardar [31] can be shown rigorously. We make the following assumptions:

Assumptions 2.6. Suppose:

(a) $\mathcal{L}$ is *local*;

(b) $\sup_{n\geq 0}|\xi_n| \leq P_{0,0} < 1$ [lazy, non-degenerate random walk]; and

(c) $u_0(x) \geq 0$ for all $x \in \mathbf{Z}^d$, and $0 < \sum_{z\in\mathbf{Z}^d} u_0(z) < \infty$. □

Then we offer the following.

Theorem 2.7. *Under Assumptions 2.6, the Anderson model* [(2.1) *with* $\sigma(z) = z$] *has a unique a.s.-nonnegative solution* $\boldsymbol{u}$, *and* $M_n := \sup_{x\in\mathbf{Z}^d} u_n(x)$ *satisfies*

$$\begin{aligned} &\lim_{n\to\infty}\frac{1}{n}\ln \mathrm{E}\left(M_n^p\right) = \Gamma(p) \quad \textit{for all } p \in [0\,,\infty), \textit{ and} \\ &\quad\lim_{n\to\infty}\frac{1}{n}\ln M_n = \lim_{n\to\infty}\frac{1}{n}\mathrm{E}\left(\ln M_n\right) = \Gamma'(0^+) \qquad \textit{almost surely,} \end{aligned} \tag{2.12}$$

where $\Gamma(p) := \ln \mathrm{E}[(1+\xi_1)^p]$ *for all* $p > 0$.

3 Some Auxiliary Results

Let us start with a simple existence/growth lemma. Note that σ is not assumed to be Lipschitz continuous, and the ξ's need not be random. The proof is not demanding, but the result itself is unimprovable [Remark 3.2].

Lemma 3.1. *Suppose there exist finite C_σ and $\widetilde{C}_\sigma$ such that $|\sigma(z)| \le C_\sigma|z|+\widetilde{C}_\sigma$ for all $z \in \mathbf{R}$. Suppose also that u_0 is bounded and $C_\xi := \sup_{n\ge 0}\sup_{x\in\mathbf{Z}^d}|\xi_n(x)|$ is finite. Then* (2.1) *has a unique solution $\boldsymbol{u}$ that satisfies*

$$\limsup_{n\to\infty}\frac{1}{n}\sup_{x\in\mathbf{Z}^d}\ln|u_n(x)| \le \ln(1+C_\sigma C_\xi). \tag{3.1}$$

Proof. Clearly, $\|\mathcal{P}h\|_\infty \le \|h\|_\infty$, where $\|h\|_\infty$ denotes the supremum norm of a function $h : \mathbf{Z}^d \to \mathbf{R}$. Consequently, (2.2) implies that

$$\|u_{n+1}\|_\infty \le \|u_n\|_\infty\,(1+C_\sigma C_\xi) + \widetilde{C}_\sigma C_\xi. \tag{3.2}$$

We iterate this and apply (2.2) to conclude the proof. □

Remark 3.2. Consider (2.1) with $u_0(x) \equiv 1$, $\sigma(z) = z$, and $\xi_n(x) \equiv 1$. Then, $u_n(x) = 2^n$ for all $n \ge 0$ and $x \in \mathbf{Z}^d$, and (3.1) is manifestly an identity. The results of the Introduction show that when the ξ's are mean-zero and independent, then the worst-case rate in (3.1) can be improved upon; this is another evidence of intermittency. □

The following covers the case when ξ's are random variables. This existence/growth result is proved in the same manner as Lemma 3.1; we omit the elementary proof, and also mention that the following cannot be improved upon.

Lemma 3.3. *Suppose there exist finite C_σ and $\widetilde{C}_\sigma$ such that $|\sigma(z)| \le C_\sigma|z|+\widetilde{C}_\sigma$ for all $z \in \mathbf{R}$. Suppose also that $u_0(x)$ and $\|\xi_n(x)\|_p$ are bounded uniformly in $x \in \mathbf{Z}^d$ and $n \ge 0$, for some $p \in [1\,,\infty]$. Then* (2.1) *has an a.s.-unique solution $\boldsymbol{u}$ that satisfies*

$$\limsup_{n\to\infty}\frac{1}{n}\sup_{x\in\mathbf{Z}^d}\ln\|u_n(x)\|_p \le \ln\left(1+C_\sigma K_{p,\xi}\right), \tag{3.3}$$

where $K_{p,\xi} := \sup_{n\ge 0}\sup_{x\in\mathbf{Z}^d}\|\xi_n(x)\|_p$.

3.1 A finite-support property.

Next we demonstrate that the solution to (2.1) has a *finite-support* property. A remarkable result of Mueller [41] asserts that Theorem 3.4 below does not have a naive continuum-limit analogue. The present work is closer in spirit to the compact-support theorem of Mueller and Perkins [42].

Let us consider the heat equation (2.1), and suppose that it has a unique solution $\boldsymbol{u} := \{u_n(x)\}_{n\geq 0, x\in\mathbf{Z}^d}$. We say that a function $f : \mathbf{Z}^d \to \mathbf{R}$ has *finite support* if $\{x \in \mathbf{Z}^d : f(x) \neq 0\}$ is finite. Define

$$R_n := \inf\left\{r > 0 : \ u_n(x) = 0 \text{ for all } x \in \mathbf{Z}^d \text{ such that } |x| > r\right\}, \tag{3.4}$$

and let R denote the *radius of support of* $\mathcal{P}$; that is,

$$R := \inf\left\{r > 0 : P_{0,x} = 0 \quad \text{for all } x \in \mathbf{Z}^d \text{ with } |x| > r\right\}. \tag{3.5}$$

Theorem 3.4. *Suppose $\sigma(0) = 0$ and $\mathcal{L}$ is local. If, in addition, u_0 has finite support, then so does u_n for all $n \geq 1$. In fact,*

$$\#\left\{x \in \mathbf{Z}^d : u_{n+1}(x) \neq 0\right\} \leq 2^d[(n+1)R + R_0]^d \quad \textit{for all } n \geq 0. \tag{3.6}$$

Proof. Suppose there exists $n \geq 0$ such that $u_n(x) = 0$ for all but a finite number of points $x \in \mathbf{Z}^d$. We propose to prove that u_{n+1} enjoys the same finite-support property. This clearly suffices to prove the theorem. Because $u_n(x) = 0$ for all but a finite number of x's, (2.2) tells us that for all but a finite number of points $x \in \mathbf{Z}^d$, $u_{n+1}(x) = \sum_{y\in\mathbf{Z}^d:|y-x|\leq R} P_{x,y}u_n(y)$. Thus, if u_n has finite support then so does u_{n+1}, and $R_{n+1} \leq R + R_n$. Eq. (3.6) also follows from this. $\square$

The locality of $\mathcal{L}$ cannot be dropped altogether. This general phenomenon appears earlier. For instance, Iscoe [30] showed that the super Brownian motion has a finite-support property, and Evans and Perkins [18] proved that there Iscoe's theorem does not hold if the underlying motion is nonlocal.

3.2 A comparison principle

The result of this subsection is a discrete analogue of Mueller's well-known and deep comparison principle [41]; but the proof uses very simple ideas. Throughout we assume that there exist unique solutions $\boldsymbol{v}$ and $\boldsymbol{u}$ to (2.1) with respective initial data v_0 and u_0. And assume that $\sigma : \mathbf{R} \to \mathbf{R}$ is globally Lipschitz with optimal Lipschitz constant Lip_σ.

Theorem 3.5. *Suppose $C_\xi := \sup_{n\geq 0}\sup_{x\in\mathbf{Z}^d} |\xi_n(x)|$ is finite and satisfies*

$$P_{0,0} \geq C_\xi \mathrm{Lip}_\sigma. \tag{3.7}$$

Then $u_0 \geq v_0$ implies that $u_n \geq v_n$ for all $n \geq 0$.

Proof. We propose to prove that if $u_n \geq v_n$ then $u_{n+1} \geq v_{n+1}$. Let us write $f_k := u_k - v_k$ for all $k \geq 0$. By (2.2) and (3.7),

$$\begin{aligned} f_{n+1}(x) &= (\mathcal{P}f_n)(x) + [\sigma(u_n(x)) - \sigma(v_n(x))]\,\xi_n(x) \\ &\geq (\mathcal{P}f_n)(x) - \mathrm{Lip}_\sigma \cdot |f_n(x)\xi_n(x)| \\ &\geq (\mathcal{P}f_n)(x) - P_{0,0} \cdot |f_n(x)|. \end{aligned} \tag{3.8}$$

But $(\mathcal{P}h)(x) = \sum_{y\in\mathbf{Z}^d} P_{x,y}h(y) \geq P_{0,0}\cdot h(x)$ for all $x \in \mathbf{Z}^d$, as long as $h \geq 0$. By the induction hypothesis, f_n is a nonnegative function, and hence so is f_{n+1}. This has the desired result. □

The following "positivity principle" follows readily from Theorem 3.5.

Corollary 3.6. *If $u_0 \geq 0$ in Theorem 3.5, then $u_n \geq 0$ for all $n \geq 0$.*

4 A priori estimates

In this section we develop some tools needed for the proof of Theorems 2.1 and 2.3. It might help to emphasize that we are considering the case where the random field $\boldsymbol{\xi} := \{\xi_n(x)\}_{n\geq 0, x\in\mathbf{Z}^d}$ is [discrete] *white noise.* That is, the ξ's are mutually independent, and have mean 0 and variance 1. [In fact, they will not be assumed to be identically distributed.] Note, in particular, that $K_{1,\xi} = 0$ and $K_{2,\xi} = 1$, where $K_{1,\xi}$ and $K_{2,\xi}$ were defined in Lemma 3.3.

Here and throughout, let $\mathcal{G} := \{\mathcal{G}_n\}_{n=0}^\infty$ denote the filtration generated by the infinite-dimensional "white-noise" $\{\xi_n\}_{n=0}^\infty$. Recall that a random field $\boldsymbol{f} := \{f_n(x)\}_{n\geq 0, x\in\mathbf{Z}^d}$ is $\mathcal{G}$*-predictable* if the random function f_n is measurable with respect to $\mathcal{G}_{n-1}$ for all $n \geq 1$, and f_0 is nonrandom.

Given a $\mathcal{G}$-predictable random field $\boldsymbol{f}$ and $\lambda > 1$, we define

$$\|\boldsymbol{f}\|_{\lambda,p} := \sup_{n\geq 0}\sup_{x\in\mathbf{Z}^d} \lambda^{-n}\|f_n(x)\|_p. \tag{4.1}$$

Define for all $\mathcal{G}$-predictable random fields $\boldsymbol{f}$,

$$(\mathcal{A}\boldsymbol{f})_n(x) := \sum_{j=0}^{n}\sum_{y\in\mathbf{Z}^d} P_{x,y}^{n-j}\sigma(f_j(y))\xi_j(y). \tag{4.2}$$

We begin by developing an *a priori* estimate for the operator norm of $\mathcal{A}$. This estimate is a discrete L^p-counterpart of Lemma 3.3 in [20] while the continuity estimates given by Proposition 4.4 is a discrete version of Lemma 3.4 in [20].

Proposition 4.1. *For all $\mathcal{G}$-predictable random fields $\boldsymbol{f}$ and all $\lambda > 1$,*

$$\|\mathcal{A}\boldsymbol{f}\|_{\lambda,p} \leq c_p(|\sigma(0)| + \mathrm{Lip}_\sigma\|\boldsymbol{f}\|_{\lambda,p}) \cdot \sqrt{\lambda^2\Upsilon(\lambda^2)}. \tag{4.3}$$

The proof requires a simple arithmetic result [20, Lemma 3.2]:

Lemma 4.2. $(a+b)^2 \leq (1+\epsilon)a^2 + (1+\epsilon^{-1})b^2$ *for all $a, b \in \mathbf{R}$ and $\epsilon > 0$.*

Hereforth, define

$$q_k := \sum_{z\in\mathbf{Z}^d} \left|P_{0,z}^k\right|^2 \qquad \text{for all } k \geq 0. \tag{4.4}$$

The proof of Proposition 4.1 also requires the following Fourier-analytic interpretation of the function Υ.

Lemma 4.3. $\lambda\Upsilon(\lambda) = \sum_{n=0}^{\infty} \lambda^{-n} q_n$ *for all* $\lambda > 1$.

Proof. By the Plancherel theorem [46, p. 26],

$$q_n = \frac{1}{(2\pi)^d} \int_{(-\pi,\pi)^d} |\phi(\xi)|^{2n}\,\mathrm{d}\xi. \tag{4.5}$$

Multiply the preceding by λ^{-n} and add over all $n \geq 0$ to finish. $\square$

Proof of Proposition 4.1. According to Burkholder's inequality,

$$\mathrm{E}\left(|(\mathcal{A}\boldsymbol{f})_n(x)|^p\right) \leq c_p^p \mathrm{E}\left(\left|\sum_{j=0}^{n}\sum_{y\in\mathbf{Z}^d} |P_{x,y}^{n-j}|^2 \cdot |\sigma(f_j(y))|^2\right|^{p/2}\right). \tag{4.6}$$

Since $p/2$ is a positive integer, the generalized Hölder inequality yields the following: For all $j = 0, \ldots, n$ and $y_1, \ldots, y_{p/2} \in \mathbf{Z}^d$,

$$\mathrm{E}\left(\prod_{i=1}^{p/2} |\sigma(f_j(y_i))|^2\right) \leq \prod_{i=1}^{p/2} \|\sigma(f_j(y_i))\|_p^2. \tag{4.7}$$

After a little algebra, the preceding and (4.6) together imply that

$$\|(\mathcal{A}\boldsymbol{f})_n(x)\|_p^2 \leq c_p^2 \sum_{j=0}^{n}\sum_{y\in\mathbf{Z}^d} |P_{x,y}^{n-j}|^2 \cdot \|\sigma(f_j(y))\|_p^2. \tag{4.8}$$

Because σ is Lipschitz, $|\sigma(x)| \leq |\sigma(0)| + \mathrm{Lip}_\sigma |x|$ for all $x \in \mathbf{R}$. Consequently, by Lemma 4.2 and Minkowski's inequality,

$$\begin{aligned}\|(\mathcal{A}\boldsymbol{f})_n(x)\|_p^2 \leq c_p^2(1+\epsilon)|\sigma(0)|^2 \sum_{k=0}^{n} q_k \\ + c_p^2\left(1+\epsilon^{-1}\right)\mathrm{Lip}_\sigma^2 \sum_{j=0}^{n}\sum_{y\in\mathbf{Z}^d} |P_{x,y}^{n-j}|^2 \cdot \|f_j(y)\|_p^2.\end{aligned} \tag{4.9}$$

In accord with Lemma 4.3,

$$\sum_{k=0}^{n} q_k \leq \lambda^{2n+2}\Upsilon(\lambda^2) \qquad \text{for all } n \geq 0, \tag{4.10}$$

and also

$$\sup_{y\in\mathbf{Z}^d} \|f_j(y)\|_p^2 \leq \lambda^{2j}\|\boldsymbol{f}\|_{\lambda,p}^2 \quad \text{for all } j \geq 0. \tag{4.11}$$

It follows that $\lambda^{-2n}\|(\mathcal{A}\boldsymbol{f})_n(x)\|_p^2$ is bounded above by

$$
\begin{aligned}
&c_p^2(1+\epsilon)|\sigma(0)|^2\lambda^2\Upsilon(\lambda^2)\\
&\qquad\qquad + c_p^2\left(1+\epsilon^{-1}\right)\mathrm{Lip}_\sigma^2\sum_{j=0}^{n}\sum_{y\in\mathbf{Z}^d}|P_{x,y}^{n-j}|^2\lambda^{-2(n-j)}\|\boldsymbol{f}\|_{\lambda,p}^2. \qquad (4.12)
\end{aligned}
$$

We now take supremum over all $n\geq 1$ and $x\in\mathbf{Z}^d$, and obtain

$$
\|\mathcal{A}\boldsymbol{f}\|_{\lambda,p}^2 \leq c_p^2\lambda^2\Upsilon(\lambda^2)\cdot\left\{(1+\epsilon)|\sigma(0)|^2+\left(1+\epsilon^{-1}\right)\mathrm{Lip}_\sigma^2\|\boldsymbol{f}\|_{\lambda,p}^2\right\}. \qquad (4.13)
$$

We obtain the result upon optimizing the right-hand side over $\epsilon>0$. $\square$

Next we present an *a priori* estimate of the degree to which $\mathcal{A}$ is continuous.

Proposition 4.4. *For all predictable random fields $\boldsymbol{f}$ and $\boldsymbol{g}$, and all $\lambda>1$,*

$$
\|\mathcal{A}\boldsymbol{f}-\mathcal{A}\boldsymbol{g}\|_{\lambda,p}\leq c_p\mathrm{Lip}_\sigma\|\boldsymbol{f}-\boldsymbol{g}\|_{\lambda,p}\cdot\sqrt{\lambda^2\Upsilon(\lambda^2)}. \qquad (4.14)
$$

Proof. We can, and will, assume without loss of generality that $\|\boldsymbol{f}-\boldsymbol{g}\|_{\lambda,p}<\infty$; else, there is nothing to prove. By using Burkholder's inequality and arguing as in the previous Lemma we find that

$$
\|(\mathcal{A}\boldsymbol{f})_n(x)-(\mathcal{A}\boldsymbol{g})_n(x)\|_p^2\leq c_p^2\mathrm{Lip}_\sigma^2\cdot\sum_{j=0}^{n}\sum_{y\in\mathbf{Z}^d}|P_{x,y}^{n-j}|^2\cdot\|f_j(y)-g_j(y)\|_p^2. \qquad (4.15)
$$

We can apply (4.11), but with $\boldsymbol{f}-\boldsymbol{g}$ in place of $\boldsymbol{f}$, and follow the proof of Lemma 4.3 to finish the proof. $\square$

5 Proof of Main results

Before we prove the main results we provide a version of Duhamel's principle for discrete equations.

Proposition 5.1 (Duhamel's principle)**.** *Suppose that there exists a unique solution to* (2.1)*, then for all $n\geq 0$ and $x\in\mathbf{Z}^d$,*

$$
u_{n+1}(x)=(\mathcal{P}^{n+1}u_0)(x)+\sum_{j=0}^{n}\sum_{y\in\mathbf{Z}^d}P_{x,y}^{n-j}\sigma(u_j(y))\xi_j(y) \qquad a.s. \qquad (5.1)
$$

Remark 5.2. Among other things, Proposition 5.1 implies that $\{u_n\}_{n=0}^\infty$ is an infinite-dimensional Markov chain with values in $(\mathbf{Z}^d)^{\mathbf{Z}_+}$. And that u_{n+1} is measurable with respect to $\{\xi_k(\bullet)\}_{k=0}^n$ for all $n\geq 0$. $\square$

Proof. One checks directly that (2.2) implies that $(\mathcal{P}u_n)(x)$ can be written as $(\mathcal{P}^2u_{n-1})(x)+\sum_{y\in\mathbf{Z}^d}P_{x,y}\sigma(u_{n-1}(y))\xi_{n-1}(y)$, and the proposition follows a simple induction scheme. $\square$

5.1 Remaining proofs

Proof of Theorem 2.1. We proceed in two steps: First we prove uniqueness and (2.7), and then we establish (2.8).

Step 1: Let $f_n^{(0)}(x) := u_0(x)$ for all $n \geq 0$ and $x \in \mathbf{Z}^d$. We recall the operator $\mathcal{A}$ from (4.2), and define iteratively a predictable random field $\boldsymbol{f}^{(\ell+1)}$ as follows: $f_0^{(\ell+1)}(x) := u_0(x)$ for all $x \in \mathbf{Z}^d$, and

$$f_{n+1}^{(\ell+1)}(x) := (\mathcal{P}^{n+1}u_0)(x) + \left(\mathcal{A}\boldsymbol{f}^{(\ell)}\right)_n(x), \tag{5.2}$$

for integers $n, \ell \geq 0$ and $x \in \mathbf{Z}^d$. Proposition 4.1 and induction together imply that $\|\mathcal{A}\boldsymbol{f}^{(\ell)}\|_{\lambda,p} < \infty$ for all $\lambda > 1$ and $\ell \geq 0$. And therefore, $\|\boldsymbol{f}^{(m)}\|_{\lambda,p} < \infty$ for all $m \geq 0$ and $\lambda > 1$, as well. We multiply (5.2) by λ^{-n} and use the fact that $f_0^{(m)} \equiv u_0$ to obtain

$$\|\boldsymbol{f}^{(\ell+1)} - \boldsymbol{f}^{(\ell)}\|_{\lambda,p} = \frac{1}{\lambda}\|\mathcal{A}\boldsymbol{f}^{(\ell)} - \mathcal{A}\boldsymbol{f}^{(\ell-1)}\|_{\lambda,p}. \tag{5.3}$$

Thus, Proposition 4.4 implies

$$\|\boldsymbol{f}^{(\ell+1)} - \boldsymbol{f}^{(\ell)}\|_{\lambda,p} \leq c_p \mathrm{Lip}_\sigma \sqrt{\Upsilon(\lambda^2)} \cdot \|\boldsymbol{f}^{(\ell)} - \boldsymbol{f}^{(\ell-1)}\|_{\lambda,p}. \tag{5.4}$$

This and iteration together yield

$$\|\boldsymbol{f}^{(\ell+1)} - \boldsymbol{f}^{(\ell)}\|_{\lambda,p} \leq \left(c_p \mathrm{Lip}_\sigma \sqrt{\Upsilon(\lambda^2)}\right)^{\ell} \cdot \|\boldsymbol{f}^{(1)} - \boldsymbol{f}^{(0)}\|_{\lambda,p}. \tag{5.5}$$

In order to estimate the final $(\lambda\,, p)$-norm we use (5.2) $[\ell := 0]$ and Minkowski's inequality to find that

$$\|f_{n+1}^{(1)} - f_{n+1}^{(0)}\|_p \leq 2\|u_0(x)\|_p + \|(\mathcal{A}\boldsymbol{f}^{(0)})_n\|_p. \tag{5.6}$$

We argue as before and use Proposition 4.1 to deduce that $\|\boldsymbol{f}^{(1)} - \boldsymbol{f}^{(0)}\|_{\lambda,p}$ is bounded above by $2\|u_0\|_{\lambda,p} + c_p(|\sigma(0)| + \mathrm{Lip}_\sigma\|u_0\|_{\lambda,p})\sqrt{\Upsilon(\lambda^2)}$. Thus, by (5.5),

$$\begin{aligned} &\|\boldsymbol{f}^{(\ell+1)} - \boldsymbol{f}^{(\ell)}\|_{\lambda,p} \\ &\quad\leq \left(c_p \mathrm{Lip}_\sigma \sqrt{\Upsilon(\lambda^2)}\right)^{\ell} \cdot \left\{c_p(|\sigma(0)| + \mathrm{Lip}_\sigma\|u_0\|_{\lambda,p})\sqrt{\Upsilon(\lambda^2)} + 2\|u_0\|_{\lambda,p}\right\}. \end{aligned} \tag{5.7}$$

Consequently, if $\Upsilon(\lambda^2) < (c_p\mathrm{Lip}_\sigma)^{-2}$ then $\|f^{(\ell+1)} - f^{(\ell)}\|_{\lambda,p}$ is summable in ℓ. Whence there exists a predictable $\boldsymbol{f}$ such that $\|\boldsymbol{f}^{(\ell)} - \boldsymbol{f}\|_{\lambda,p}$ tends to zero as ℓ tends to infinity, and $\boldsymbol{f}$ solves (2.1). Proposition 5.1 implies that $f_n(x) = u_n(x)$ a.s., for all $n \geq 0$ and $x \in \mathbf{Z}^d$. It follows that $\|\boldsymbol{u}\|_{\lambda,p} < \infty$ provided that $\Upsilon(\lambda^2) < (c_p\mathrm{Lip}_\sigma)^{-2}$. The first part of the theorem—that is, existence, uniqueness, and (2.7)—all follow from this finding. We now turn our attention to the second step of the proof.

Step 2: Hereforth, we assume that $\alpha := \inf_{x\in\mathbf{Z}^d} u_0(x) > 0$ and $|\sigma(z)| \geq L_\sigma|z|$ for all $z \in \mathbf{R}$. It follows readily from Proposition 5.1 that

$$\begin{aligned}\mathrm{E}\left(|u_{n+1}(x)|^2\right) &= \left|(\mathcal{P}^{n+1}u_0)(x)\right|^2 + \sum_{j=0}^{n}\sum_{y\in\mathbf{Z}^d}\left|P_{x,y}^{n-j}\right|^2\mathrm{E}\left(|\sigma(u_j(y)|^2\right)\\ &\geq \alpha^2 + L_\sigma^2\cdot\sum_{j=0}^{n}\sum_{y\in\mathbf{Z}^d}\left|P_{0,y-x}^{n-j}\right|^2\mathrm{E}\left(|u_j(y)|^2\right).\end{aligned} \tag{5.8}$$

In order to solve this we define for all $\lambda > 1$ and $z \in \mathbf{Z}^d$,

$$\mathcal{F}_\lambda(z) := \sum_{j=0}^{\infty}\lambda^{-j}|P_{0,z}^{j}|^2 \qquad \text{and} \qquad \mathcal{G}_\lambda(z) := \sum_{j=0}^{\infty}\lambda^{-j}\mathrm{E}\left(|u_j(z)|^2\right). \tag{5.9}$$

We can multiply the extreme quantities in (5.8) by $\lambda^{-(n+1)}$ and add $[n \geq 0]$ to find that

$$\mathcal{G}_\lambda(x) \geq \frac{\alpha^2\lambda}{\lambda-1} + \frac{L_\sigma^2}{\lambda}\cdot\left(\mathcal{F}_\lambda * \mathcal{G}_\lambda\right)(x). \tag{5.10}$$

This is a renewal inequality [9]; we prove that (5.10) does not have a finite solution when $\Upsilon(\lambda) \geq L_\sigma^{-2}$. If $\mathbf{1}(x) := 1$ for all $x \in \mathbf{Z}^d$, then $(\mathcal{F}_\lambda * \mathbf{1})(x) = \lambda\Upsilon(\lambda)$ for all $x \in \mathbf{Z}^d$ [Lemma 4.3]. Therefore, (5.10) yields

$$\begin{aligned}\mathcal{G}_\lambda(x) &\geq \frac{\alpha^2\lambda}{\lambda-1} + \frac{L_\sigma^2}{\lambda}\cdot\left(\frac{\alpha^2\lambda}{\lambda-1}(\mathcal{F}_\lambda * \mathbf{1})(x) + \frac{L_\sigma^2}{\lambda}\cdot(\mathcal{F}_\lambda * \mathcal{F}_\lambda * \mathcal{G}_\lambda)(x)\right)\\ &= \frac{\alpha^2\lambda}{\lambda-1}\left\{1 + \Upsilon(\lambda)L_\sigma^2\right\} + \left(\frac{L_\sigma^2}{\lambda}\right)^2\cdot(\mathcal{F}_\lambda * \mathcal{F}_\lambda * \mathcal{G}_\lambda)(x).\end{aligned} \tag{5.11}$$

This and induction together imply the following:

$$\mathcal{G}_\lambda(x) \geq \frac{\alpha^2\lambda}{\lambda-1}\sum_{n=0}^{\infty}\left(\Upsilon(\lambda)L_\sigma^2\right)^n. \tag{5.12}$$

Consequently, $\Upsilon(\lambda) \geq L_\sigma^{-2}$ implies that $\mathcal{G}_\lambda(x) = \infty$ for all $x \in \mathbf{Z}^d$. If there exists a $\lambda_0 > 1$ such that $\Upsilon(\lambda_0) > L_\sigma^{-2}$, then the preceding tells us that $\mathcal{G}_{\lambda_0} \equiv \infty$. Now suppose, in addition, that there exists $z \in \mathbf{Z}^d$ such that

$$\mathrm{E}\left(|u_n(z)|^2\right) = O(\lambda_0^n). \tag{5.13}$$

Then by the continuity of Υ we can choose a finite $\lambda > \lambda_0$ such that $\Upsilon(\lambda) \geq L_\sigma^{-2}$, whence $\mathcal{G}_\lambda \equiv \infty$. This yields a contradiction, since (5.13) implies that $\mathcal{G}_\lambda(z) \leq$ const $\times \sum_{n=0}^{\infty}(\lambda_0/\lambda)^n < \infty$. We have verified (2.8) when $p = 2$. An application of Hölder inequality proves (2.8) for all $p \geq 2$, whence the theorem. □

Proof of Theorem 2.3. Because u_0 has finite support, it is bounded. Therefore, Theorem 2.1 ensures the existence of an a.s.-unique solution $\boldsymbol{u}$ to (2.1).

Choose and fix $p \in [2\,,\infty)$, and let $L^p(\mathbf{Z}^d)$ denote the usual space of p-times summable functions $f : \mathbf{Z}^d \to \mathbf{R}$, normed via

$$\|f\|_{L^p(\mathbf{Z}^d)}^p := \sum_{x\in\mathbf{Z}^d} |f(x)|^p. \tag{5.14}$$

We also define m to be the counting measure on $\mathbf{Z}^d$ and consider the Banach space $\mathbf{B} := L^p(m \times \mathrm{P})$, all the time noting that for all random functions $g \in \mathbf{B}$,

$$\|g\|_{\mathbf{B}} = \left| \mathrm{E}\left(\sum_{x\in\mathbf{Z}^d} |g(x)|^p \right) \right|^{1/p}. \tag{5.15}$$

Evidently, $u_0 \in L^p(\mathbf{Z}^d)$; we claim that

$$\frac{1}{2} \ln \Upsilon^{-1}\left(L_\sigma^{-2}\right) \le \limsup_{n\to\infty} \frac{1}{n} \ln \|u_n\|_{\mathbf{B}} \le \frac{1}{2} \ln \Upsilon^{-1}\left((c_p \mathrm{Lip}_\sigma)^{-2}\right). \tag{5.16}$$

For every $\lambda > 1$ we define $\mathbf{B}(\lambda)$ to be the Banach space of all $\mathcal{G}$-predictable processes $\boldsymbol{f}$ with $\|\boldsymbol{f}\|_{\mathbf{B}(\lambda)} < \infty$, where

$$\|\boldsymbol{f}\|_{\mathbf{B}(\lambda)} := \sup_{n\ge 0} \lambda^{-n} \|f_n\|_{\mathbf{B}}. \tag{5.17}$$

Note that $\|\boldsymbol{f}\|_{\lambda,p} \le \|\boldsymbol{f}\|_{\mathbf{B}(\lambda)}$.

Since u_0 has finite support, we can use Theorem 3.4 to write

$$\|u_n\|_{\mathbf{B}} = O(n^{d/p}) \times \sup_{x\in\mathbf{Z}^d} \|u_n(x)\|_p. \tag{5.18}$$

Therefore, the following is valid for all $\lambda \in (0\,,\infty)$:

$$\|\boldsymbol{u}\|_{\mathbf{B}(\lambda)} \le \mathrm{const} \cdot \sup_{n\ge 0} \left(n^{d/p} \lambda^{-n} \sup_{x\in\mathbf{Z}^d} \|u_n(x)\|_p \right). \tag{5.19}$$

As a result, if we select $\lambda > \lambda_0 > \frac{1}{2}\Upsilon^{-1}((c_p\mathrm{Lip}_\sigma)^{-2}))$, then

$$\|\boldsymbol{u}\|_{\lambda,p} \le \mathrm{const} \times \|\boldsymbol{u}\|_{\mathbf{B}(\lambda_0)} < \infty, \tag{5.20}$$

thanks to the upper bound of Theorem 2.1. It follows immediately from this that $\limsup_{n\to\infty} n^{-1} \ln \|u_n\|_{\mathbf{B}} \le \lambda_0$ for all finite $\lambda_0 > \frac{1}{2}\Upsilon^{-1}((c_p\mathrm{Lip}_\sigma)^{-2})$. The second inequality in (5.16) is thus proved. Next we derive the first inequality in (5.16).

Thanks to Jensen's inequality, it suffices to consider only the case $p = 2$. According to (2.2),

$$\mathrm{E}\left(|u_{n+1}(x)|^2\right) \ge \left|(\mathcal{P}^{n+1}u_0)(x)\right|^2 + L_\sigma^2 \cdot \sum_{j=0}^{n} \sum_{y\in\mathbf{Z}^d} \left|P_{x,y}^{n-j}\right|^2 \mathrm{E}\left(|u_j(y)|^2\right). \tag{5.21}$$

Consequently,

$$\|u_{n+1}\|_{\mathbf{B}}^2 \geq \|u_0\|_{L^2(\mathbf{Z}^d)}^2 + L_\sigma^2 \cdot \sum_{j=0}^{n} q_{n-j} \|u_j\|_{\mathbf{B}}^2. \tag{5.22}$$

We multiply both sides by $\lambda^{-(n+1)}$, then sum from $n = 0$ to $n = \infty$ and finally apply Lemma 4.3, in order to obtain the following:

$$\sum_{n=1}^{\infty} \lambda^{-n} \|u_n\|_{\mathbf{B}}^2 \geq \frac{1}{\lambda - 1} \cdot \|u_0\|_{L^2(\mathbf{Z}^d)}^2 + L_\sigma^2 \Upsilon(\lambda) \cdot \sum_{k=0}^{\infty} \lambda^{-k} \|u_k\|_{\mathbf{B}}^2. \tag{5.23}$$

Because $u_0 \not\equiv 0$, we have $(\lambda - 1)^{-1} \cdot \|u_0\|_{L^2(\mathbf{Z}^d)}^2 > 0$, and this shows that $\sum_{n=1}^{\infty} \lambda^{-n} \|u_n\|_{\mathbf{B}}^2 = \infty$ whenever $L_\sigma^2 \Upsilon(\lambda) \geq 1$. In particular, it must follow that $\limsup_{n\to\infty} \rho^{-n} \|u_n\|_{\mathbf{B}}^2 = \infty$ whenever $\rho \in (1\,, \lambda]$. This implies the first inequality in (5.16).

Now we can conclude the proof from (5.16). According to Theorem 3.4,

$$\sup_{x \in \mathbf{Z}^d} \|u_n(x)\|_2 \leq \|u_n\|_{\mathbf{B}} \leq O(n^{d/2}) \times \sup_{x \in \mathbf{Z}^d} \|M_n\|_2. \tag{5.24}$$

Therefore, (5.16) implies the theorem. □

Before we prove Corollary 2.5 we state and prove an elementary convexity lemma that is due essentially to Carmona and Molchanov [6, Theorem III.1.2, p. 55].

Lemma 5.3. *Suppose $u_n(x) \geq 0$ for all $n \geq 0$ and $x \in \mathbf{Z}^d$, $\bar{\gamma}(p) < \infty$ for all $p < \infty$ and $\bar{\gamma}(2) > 0$. Then, $\boldsymbol{u}$ is weakly intermittent.*

Proof. Because $\boldsymbol{u}$ is nonnegative,

$$\bar{\gamma}(\alpha) = \limsup_{n\to\infty} \frac{1}{n} \ln \mathrm{E}\left[u_n(x)^\alpha\right] \qquad \text{for all } \alpha \geq 0. \tag{5.25}$$

Thanks to Proposition 5.1, $\mathrm{E}[u_n(x)] = (\mathcal{P}^n u_0)(x)$ is bounded above uniformly by $\sup_x u_0(x)$, which is finite. Consequently,

$$\bar{\gamma}(1) = 0 < \bar{\gamma}(2). \tag{5.26}$$

Next we claim that $\bar{\gamma}$ is convex on $\mathbf{R}_+$. Indeed, for all $a, b \geq 0$ and $\lambda \in (0\,,1)$, Hölder's inequality yields the following: For all $s \in (1\,,\infty)$ with $t := s/(s-1)$,

$$\mathrm{E}\left[u_n(x)^{\lambda a + (1-\lambda) b}\right] \leq \left\{\mathrm{E}\left[u_n(x)^{s\lambda a}\right]\right\}^{1/s} \left\{\mathrm{E}\left[u_n(x)^{t(1-\lambda)b}\right]\right\}^{1/t}. \tag{5.27}$$

Choose $s := 1/\lambda$ to deduce the convexity of $\bar{\gamma}$ from (5.25).

Now we complete the proof: By (5.26) and convexity, $\bar\gamma(p) > 0$ for all $p \geq 2$. If $p' > p \geq 2$, then we write $p = \lambda p' + (1-\lambda)$—with $\lambda := (p-1)/(p'-1)$—and apply convexity to conclude that

$$\bar\gamma(p) \leq \lambda\bar\gamma(p') + (1-\lambda)\bar\gamma(1) = \frac{p-1}{p'-1}\,\bar\gamma(p'). \tag{5.28}$$

Since (5.28) holds in particular with $p \equiv 2$, it implies that $\bar\gamma(p') > 0$. And the lemma follows from (5.28) and the inequality $(p-1)/(p'-1) < p/p'$. □

Proof of Corollary 2.5. Condition 3.7 and Theorem 3.5 imply that $u_n(x) \geq 0$, and hence (5.25) holds. Now "$\bar\gamma(2) > 0$" and "$\bar\gamma(p) < \infty$ for $p > 2$" both follow from the Theorem 2.3, and Lemma 5.3 completes the proof. □

Proof of Theorem 2.7. The assertion about the existence and uniqueness of the solution to the Anderson model (2.1) with $\sigma(z) := z$ follows from Lemma 3.1. The solution is nonnegative by Lemma 3.6. Now we prove the claims about the growth of the solution u.

It is possible to check that $U_n := \sum_{x\in\mathbf{Z}^d} u_n(x)$ can be written out explicitly as $U_n = U_0 \times \prod_{j=1}^n (1+\xi_j)$. Since $0 < U_0 < \infty$, Kolmogorov's strong law of large numbers implies that almost surely,

$$\lim_{n\to\infty} \frac{1}{n} \ln U_n = \lim_{n\to\infty} \frac{1}{n} \mathrm{E}\left[\ln U_n\right] = \mathrm{E}\left[\ln(1+\xi_1)\right] = \Gamma'(0^+). \tag{5.29}$$

Also, $\lim_{n\to\infty} n^{-1} \ln \mathrm{E}\,(U_n^p) = \Gamma(p)$ for all $p \geq 0$. Because $M_n \leq U_n$, we have

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{n} \ln M_n \leq \Gamma'(0^+) \qquad \text{a.s.,} \quad \limsup_{n\to\infty} \frac{1}{n} \mathrm{E}\,(\ln M_n) \leq \Gamma'(0^+), \text{ and} \\ &\limsup_{n\to\infty} \frac{1}{n} \ln \mathrm{E}\,(M_n^p) \leq \Gamma(p) \quad \text{for all } p \in [0\,,\infty). \end{aligned} \tag{5.30}$$

Next we strive to establish the complementary inequalities to these.

In order to derive the second, and final, half of the theorem we choose and fix some $x_0 \in \mathbf{Z}^d$ such that $u_0(x_0) > 0$. Let $\boldsymbol{v} := \{v_n(x)\}_{n\geq 0, x\in\mathbf{Z}^d}$ solve (2.1) with $\sigma(z) = z$, subject to $v_0(x) = u_0(x_0)$ if $x = x_0$ and $v_0(x) = 0$ otherwise. The existence and uniqueness of v follows from Lemma 3.1. By Corollary 3.6,

$$0 \leq v_n(x) \leq u_n(x) \qquad \text{for all } n \geq 0 \text{ and } x \in \mathbf{Z}^d. \tag{5.31}$$

Let $V_n := \sum_{x\in\mathbf{Z}^d} v_n(x)$. Then,

$$\lim_{n\to\infty} \frac{1}{n} \ln V_n = \lim_{n\to\infty} \frac{1}{n} \mathrm{E}\left[\ln V_n\right] = \mathrm{E}\left[\ln(1+\xi_1)\right] = \Gamma'(0^+). \tag{5.32}$$

Also, $\lim_{n\to\infty} n^{-1} \ln \mathrm{E}\,(V_n^p) = \Gamma(p)$ for all $p \in [0\,,\infty)$. Recall R from (3.5). Because $v_0 = 0$ off of $\{x_0\}$, (3.6) implies that $V_n \leq 2^d\{nR+1\}^d \times \sup_{x\in\mathbf{Z}^d} v_n(x)$.

Owing to Theorem 3.5, $\sup_{x\in\mathbf{Z}^d} v_n(x) \le M_n$. Therefore,

$$\begin{gathered}\liminf_{n\to\infty}\frac{1}{n}\ln M_n \ge \Gamma'(0^+), \quad \liminf_{n\to\infty}\frac{1}{n}\mathrm{E}\left(\ln M_n\right) \ge \Gamma'(0^+), \quad \text{and}\\ \liminf_{n\to\infty}\frac{1}{n}\ln \mathrm{E}\left(M_n^p\right) \ge \Gamma(p) \qquad \text{for all } p\in[0\,,\infty).\end{gathered} \tag{5.33}$$

Together with (5.30), these bounds prove Theorem 2.7. □

6 An example

Let us consider (2.1) in the special case that: (i) ξ's are independent mean-zero variance-one random variables; (ii) $\sigma(z) = \nu z$ for a fixed $\nu > 0$; (iii) u_0 has finite support; and (iv) $\mathcal{L}$ is the generator of a simple symmetric random walk on $\mathbf{Z}$. That is,

$$(\mathcal{L}h)(x) = \frac{h(x+1)+h(x-1)-2h(x)}{2}, \tag{6.1}$$

for every function $h:\mathbf{Z}\to\mathbf{R}$ and all $x\in\mathbf{Z}$. The operator $2\mathcal{L}$ is called the *graph Laplacian* on $\mathbf{Z}$, and the resulting form,

$$u_{n+1}(x) - u_n(x) = (\mathcal{L}u_n)(x) + \nu u_n(x)\xi_n(x), \tag{6.2}$$

of (2.1) is an *Anderson model* of a parabolic type [6, 40]. Theorems 2.1 and 2.3 together imply that the upper Liapounov exponent of the solution to (2.1) is $\ln\Upsilon^{-1}(\nu^{-2})$ in this case. We compute the quantity $\Upsilon^{-1}(\nu^{-2})$ next. The following might suggest that one cannot hope to compute upper Liapounov exponents explicitly in general.

Proposition 6.1. *If $\nu > 0$, then*

$$\Upsilon^{-1}\left(\nu^{-2}\right) = \inf\left\{\lambda > 1 :\ {}_1F_1\left(\frac{1}{2}\,; 1\,; \frac{1}{\lambda}\right) < \frac{\lambda}{\nu^2}\right\}. \tag{6.3}$$

Proof. Recall the q_k's from (4.4). According to Plancherel's theorem and symmetry,

$$q_n = \frac{1}{\pi}\int_0^\pi \left(\frac{1+\cos(2\xi)}{2}\right)^n \mathrm{d}\xi. \tag{6.4}$$

We may apply the half-angle formula for cosines, and then Wallis's formula (Davis [16, **6.1.49**, p. 258]), in order to find that if $n \ge 1$ then

$$q_n = \frac{(2n-1)!!}{(2n)!!}, \tag{6.5}$$

where "!!" denotes the double factorial. Therefore, ${}_1F_1(1/2\,;1\,;\bullet)$ is the generating function of the sequence $\{q_n\}_{n=0}^{\infty}$; confer with Slater [49, **13.1.2**, p. 504]. This and Lemma 4.3 together prove that

$$\lambda\Upsilon(\lambda) = {}_1F_1\left(\frac{1}{2}\,;1\,;\frac{1}{\lambda}\right), \tag{6.6}$$

and the lemma follows since Υ is a continuous and strictly decreasing function on $(0\,,\infty)$. □

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Mohammud Foondun & Davar Khoshnevisan
Department of Mathematics, University of Utah, Salt Lake City, UT 84112-0090
Emails: `mohammud@math.utah.edu` & `davar@math.utah.edu`
URLs: `http://www.math.utah.edu/~mohammud` & `http://www.math.utah.edu/~davar`