

Slices of Brownian Sheet: New Results, and Open Problems

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Abstract. We can view Brownian sheet as a sequence of interacting Brownian motions or *slices*. Here we present a number of results about the slices of the sheet. A common feature of our results is that they exhibit phase transition. In addition, a number of open problems are presented.

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1. Introduction

Let $B := \{B(s, t)\}_{s, t \geq 0}$ denote a two-parameter Brownian sheet in $\mathbf{R}^d$. That is, B is a centered Gaussian process with covariance matrix,

$$\text{Cov}(B_i(s, t), B_j(u, v)) = \min(s, u) \cdot \min(t, v) \cdot \delta_{i,j}. \quad (1.1)$$

We can assume without loss of generality that B is continuous. Moreover, it is convenient to think of B as the distribution function of a d -dimensional white noise $\hat{B}$ on $\mathbf{R}_+^2$; i.e., we may think of $B(s, t)$ as

$$B(s, t) = \hat{B}([0, s] \times [0, t]). \quad (1.2)$$

These properties were discovered first in Čentsov [3].

Choose and fix some number $s > 0$. The *slice of B along s* is the stochastic process $\{B(s, t)\}_{t \geq 0}$. It is easy to see that if s is non-random then the slice of B along s is a scaled Brownian motion. More precisely, $t \mapsto s^{-1/2}B(s, t)$ is standard d -dimensional Brownian motion. It is not too difficult to see that if s is random, then the slice along s need not be a Brownian motion. For instance, the slice along a non-random s hits points if and only if $d = 1$. But there are random values of s such that the slice along s hits zero up to dimension $d = 3$; see (1.3) below. Nonetheless, one may expect the slice along s to look like Brownian motion in some sense, even for some random values of s . [For example, all slices share the Brownian property that they are continuous paths.]

A common question in infinite-dimensional stochastic analysis is to ask if there are slices that behave differently from d -dimensional Brownian motion in a prescribed manner. There is a large literature on this subject; see the survey paper [14]. In this paper we present some new examples where there is, generally, a “cut-off phenomenon” or “phase transition.”

Our first example is related to the zero-set of the Brownian sheet. Orey and Pruitt [25] have proven that $B^{-1}\{0\}$ is non-trivial if and only if the spatial dimension d is three or less. That is,

$$P\{B(s, t) = 0 \text{ for some } s, t > 0\} > 0 \iff d \leq 3. \quad (1.3)$$

See also Fukushima [11] and Penrose [26]. Khoshnevisan [16] has derived the following refinement: For all non-random, compact sets $E, F \subset (0, \infty)$,

$$P\{B^{-1}\{0\} \cap (E \times F) \neq \emptyset\} > 0 \iff \text{Cap}_{d/2}(E \times F) > 0, \quad (1.4)$$

where Cap_β denotes “ β -dimensional Riesz capacity.” [These capacities are recalled in the appendix.] The Orey–Pruitt theorem (1.3) follows immediately from (1.4) and Taylor’s theorem [Appendix A.1].

Now consider the projection $\mathcal{Z}_d$ of $B^{-1}\{0\}$ onto the x -axis. That is,

$$\mathcal{Z}_d := \{s \geq 0 : B(s, t) = 0 \text{ for some } t > 0\}. \quad (1.5)$$

Thus, $s \in \mathcal{Z}_d$ if and only if the slice of B along s hits zero. Of course, zero is always in $\mathcal{Z}_d$, and the latter is a.s. closed. Our first result characterizes the polar sets of $\mathcal{Z}_d$.

Theorem 1.1. *For all non-random, compact sets $F \subset (0, \infty)$,*

$$\mathbb{P}\{\mathcal{Z}_d \cap F \neq \emptyset\} > 0 \iff \text{Cap}_{(d-2)/2}(F) > 0. \quad (1.6)$$

Theorem 1.1 and Taylor's theorem [Appendix A.1] together provide us with a new proof of the Orey–Pruitt theorem (1.3). Furthermore, we can apply a codimension argument [15, Theorem 4.7.1, p. 436] to find that

$$\dim_{\text{H}} \mathcal{Z}_d = 1 \wedge \left(2 - \frac{d}{2}\right)^+ \quad \text{a.s.}, \quad (1.7)$$

where $\dim_{\text{H}}$ denotes Hausdorff dimension [Appendix A.3]. Consequently, when $d \in \{2, 3\}$, the [Hausdorff] dimension of $\mathcal{Z}_d$ is equal to $2 - (d/2)$. Oddly enough, this is precisely the dimension of $B^{-1}\{0\}$ as well; see Rosen [29, 30]. But $\mathcal{Z}_d$ is the projection of $B^{-1}\{0\}$ onto the x -axis. Therefore, one might guess that $B^{-1}\{0\}$ and $\mathcal{Z}_d$ have the same dimension because all slices of B have the property that their zero-sets have zero dimension. If B were a generic function of two variables, then such a result would be false, as there are simple counter-examples. Nevertheless, the “homogeneity” of the slices of B guarantees that our intuition is correct in this case.

Theorem 1.2. *If $d \in \{2, 3\}$, then the following holds outside a single $\mathbb{P}$ -null set:*

$$\dim_{\text{H}}(B^{-1}\{0\} \cap (\{s\} \times (0, \infty))) = 0 \quad \text{for all } s > 0. \quad (1.8)$$

Remarks 1.3. 1. Equation (1.8) is not valid when $d = 1$. In that case, Penrose [26] proved that $\dim_{\text{H}}(B^{-1}\{0\} \cap (\{s\} \times (0, \infty))) = 1/2$ for all $s > 0$. In particular, Penrose's theorem implies that $\mathcal{Z}_1 = \mathbf{R}_+$ a.s.; the latter follows also from an earlier theorem of Shigekawa [31].

2. Almost surely, $\mathcal{Z}_d = \{0\}$ when $d \geq 4$; see (1.3). This and the previous remark together show that “ $d \in \{2, 3\}$ ” covers the only interesting dimensions.
3. The fact that Brownian motion misses singletons in $\mathbf{R}^d$ for $d \geq 2$ implies that the Lebesgue measure of $\mathcal{Z}_d$ is a.s. zero when $d \in \{2, 3\}$.
4. It is not hard to see that the probability in Theorem 1.1 is zero or one. Used in conjunction with Theorem 1.1, this observation demonstrates that $\mathcal{Z}_d$ is a.s. everywhere-dense when $d \leq 3$.

Next, we consider the random set,

$$\mathcal{D}_d := \{s \geq 0 : B(s, t_1) = B(s, t_2) \text{ for some } t_2 > t_1 > 0\}. \quad (1.9)$$

We can note that $s \in \mathcal{D}_d$ if and only if the slice of B along s has a double point.

Lyons [20] has proven that $\mathcal{D}_d$ is non-trivial if and only if $d \leq 5$. That is,

$$\mathbf{P} \{ \mathcal{D}_d \neq \{0\} \} > 0 \iff d \leq 5. \quad (1.10)$$

See also Mountford [23]. Lyons's theorem (1.10) is an improvement to an earlier theorem of Fukushima [11] which asserts the necessity of the condition " $d \leq 6$." Our next result characterizes the polar sets of $\mathcal{D}_d$.

Theorem 1.4. *For all non-random, compact sets $F \subset (0, \infty)$,*

$$\mathbf{P} \{ \mathcal{D}_d \cap F \neq \emptyset \} > 0 \iff \text{Cap}_{(d-4)/2}(F) > 0. \quad (1.11)$$

Lyons's theorem (1.10) follows at once from this and Taylor's theorem. In addition, a codimension argument reveals that almost surely,

$$\dim_{\mathbf{H}} \mathcal{D}_d = 1 \wedge \left(3 - \frac{d}{2} \right)^+. \quad (1.12)$$

This was derived earlier by Mountford [23] who used different methods.

Remark 1.5. Penrose [26, 27] has shown that $\mathcal{D}_d = \mathbf{R}_+^d$ a.s. when $d \leq 3$. Also recall Lyons' theorem (1.10). Thus, Theorem 1.4 has content only when $d \in \{4, 5\}$.

In summary, our Theorems 1.1 and 1.4 state that certain unusual slices of the sheet can be found in the "target set" F if and only if F is sufficiently large in the sense of capacity. Next we introduce a property which is related to more delicate features of the set F . Before doing so, let us set $d \geq 3$ and define

$$\mathcal{R}(s) := \inf \left\{ \alpha > 0 : \liminf_{t \rightarrow \infty} \frac{(\log t)^{1/\alpha}}{t^{1/2}} |B(s, t)| < \infty \right\} \quad \text{for all } s > 0. \quad (1.13)$$

Thus, $\mathcal{R}(s)$ is the critical escape-rate—at the logarithmic level—for the slice of B along s . Because $t \mapsto s^{-1/2} B(s, t)$ is standard Brownian motion for all fixed $s > 0$, the integral test of Dvoretzky and Erdős [8] implies that

$$\mathbf{P} \{ \mathcal{R}(s) = d - 2 \} = 1 \quad \text{for all } s > 0. \quad (1.14)$$

That is, the typical slice of B escapes at log-rate $(d-2)$. This leads to the question, "When are all slices of B transient"? Stated succinctly, the answer is: "If and only if $d \geq 5$." See Fukushima [11] for the sufficiency of the condition " $d \geq 5$," and Kôno [18] for the necessity. Further information can be found in Dalang and Khoshnevisan [4]. Next we try to shed further light on the rate of convergence of the transient slices of B . Our characterization is in terms of packing dimension $\dim_{\mathbf{p}}$, which is recalled in Appendix B.2.

Theorem 1.6. *Choose and fix $d \geq 3$, and a non-random compact set $F \subset (0, \infty)$. Then with probability one:*

1. $\mathcal{R}(s) \geq d - 2 - 2 \dim_{\mathbf{p}} F$ for all $s \in F$.
2. If $\dim_{\mathbf{p}} F < (d - 2)/2$, then $\mathcal{R}(s) = d - 2 - 2 \dim_{\mathbf{p}} F$ for some $s \in F$.

Remark 1.7. The condition that $\dim_{\mathbf{p}} F < (d - 2)/2$ is always met when $d \geq 5$.

The organization of this paper is as follows: After introducing some basic real-variable computations in Section 2, we prove Theorem 1.1 in Section 3. Our argument is entirely harmonic-analytic and does not require any probability theory; this proof rests on a projection theorem for capacities which may be of independent interest. Theorems 1.4 and 1.2 are respectively proved in Sections 4 and 6. Section 5 contains a variant of Theorem 1.4, and Section 7 contains the proof of Theorem 1.6 and much more. There is also a final Section 8 wherein we record some open problems.

Throughout, any n -vector x is written, coordinatewise, as $x = (x_1, \dots, x_n)$. Moreover, $|x|$ will always denote the ℓ^1 -norm of $x \in \mathbf{R}^n$; i.e.,

$$|x| := |x_1| + \dots + |x_n|. \quad (1.15)$$

Generic constants that do not depend on anything interesting are denoted by $c, c_1, c_2, \dots$; they are *always* assumed to be positive and finite, and their values may change between, as well as within, lines.

Let A denote a Borel set in $\mathbf{R}^n$. The collection of all Borel probability measures on A is always denoted by $\mathcal{P}(A)$.

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2. Preliminary Real-Variable Estimates

Our analysis depends on the properties of three classes of functions. We develop the requisite estimates here in this section. Aspects of these lemmas overlap with Lemmas 1.2 and 2.5 of Dalang and Khoshnevisan [4].

Here and throughout, we define for all $\epsilon > 0$ and $x \in \mathbf{R}$,

$$\begin{aligned} f_\epsilon(x) &:= \left(\frac{\epsilon}{|x|^{1/2}} \wedge 1 \right)^d, \\ F_\epsilon(x) &:= \int_0^1 f_\epsilon(y + |x|) dy, \\ G_\epsilon(x) &:= \int_0^1 F_\epsilon(y + |x|) dy. \end{aligned} \quad (2.1)$$

Our first technical lemma attaches a “meaning” to f_ϵ .

Lemma 2.1. *Let $\mathbf{g}$ denote a d -vector of i.i.d. standard-normal variables. Then there exist a constant c such that for all $\sigma, \epsilon > 0$,*

$$cf_\epsilon(\sigma^2) \leq \mathbf{P} \{ \sigma |\mathbf{g}| \leq \epsilon \} \leq f_\epsilon(\sigma^2). \quad (2.2)$$

Proof. This is truly an elementary result. However, we include a proof to acquaint the reader with some of the methods that we use later on.

Let $M := \max_{1 \leq i \leq d} |\mathbf{g}_i|$, and note that $|\mathbf{g}| \geq M$. Therefore,

$$\mathbb{P}\{\sigma|\mathbf{g}| \leq \epsilon\} \leq \left(\int_{-\epsilon/\sigma}^{\epsilon/\sigma} \frac{e^{-u^2/2}}{(2\pi)^{1/2}} du \right)^d \leq \left(\frac{\epsilon}{\sigma} \right)^d, \quad (2.3)$$

because $(2/\pi)^{1/2} \exp(-u^2/2) \leq 1$. The upper bound of the lemma follows because $\mathbb{P}\{\sigma|\mathbf{g}| \leq \epsilon\}$ is also at most one. To derive the lower bound we use the inequality $|\mathbf{g}| \leq Md$ to find that when $\epsilon \leq \sigma$,

$$\begin{aligned} \mathbb{P}\{\sigma|\mathbf{g}| \leq \epsilon\} &\geq \left(\int_{-\epsilon/(\sigma d)}^{\epsilon/(\sigma d)} \frac{e^{-u^2/2}}{(2\pi)^{1/2}} du \right)^d \geq \left(\frac{2}{\pi d^2} \right)^{d/2} e^{-1/(2d^2)} \left(\frac{\epsilon}{\sigma} \right)^d \\ &= \left(\frac{2}{\pi d^2} \right)^{d/2} e^{-1/(2d^2)} f_\epsilon(\sigma^2) := c_1 f_\epsilon(\sigma^2). \end{aligned} \quad (2.4)$$

The same reasoning shows that when $\epsilon > \sigma$,

$$\begin{aligned} \mathbb{P}\{\sigma|\mathbf{g}| \leq \epsilon\} &\geq \left(\int_{-1}^1 \frac{e^{-u^2/2}}{(2\pi)^{1/2}} du \right)^d = \left(\int_{-1}^1 \frac{e^{-u^2/2}}{(2\pi)^{1/2}} du \right)^d f_\epsilon(\sigma^2) \\ &:= c_2 f_\epsilon(\sigma^2). \end{aligned} \quad (2.5)$$

The lemma follows with $c := \min(c_1, c_2)$. $\square$

Next we find bounds for F_ϵ in terms of the function $U_{(d-2)/2}$ that is defined in (A.3).

Lemma 2.2. *There exists $c > 1$ such that for all $0 \leq y \leq 2$ and $\epsilon > 0$,*

$$F_\epsilon(y) \leq c\epsilon^d U_{(d-2)/2}(y). \quad (2.6)$$

In addition, for all $y \geq \epsilon^2$,

$$F_\epsilon(y) \geq \frac{\epsilon^d}{c} U_{(d-2)/2}(y). \quad (2.7)$$

Proof. Evidently,

$$F_\epsilon(y) = \int_0^1 f_\epsilon(x+y) dx \leq \epsilon^d \int_0^1 \frac{dx}{(x+y)^{d/2}} = \epsilon^d \int_y^{1+y} \frac{dx}{x^{d/2}}, \quad (2.8)$$

and this is an equality when $y \geq \epsilon^2$. The remainder of the proof is a direct computation. $\square$

As regards the functions G_ϵ , we first note that

$$G_\epsilon(x) = \iint_{[0,1]^2} f_\epsilon(x + |y|) dy. \quad (2.9)$$

The following captures a more useful property of G_ϵ .

Lemma 2.3. *There exists $c > 1$ such that for all $0 < x \leq 2$ and $\epsilon > 0$,*

$$G_\epsilon(x) \leq c\epsilon^d U_{(d-4)/2}(x). \quad (2.10)$$

If, in addition, $x \geq \epsilon^2$ then

$$G_\epsilon(x) \geq \frac{\epsilon^d}{c} U_{(d-4)/2}(x). \quad (2.11)$$

Lemma 2.3 follows from Lemma 2.2 and one or two elementary and direct computations.

We conclude this section with a final technical lemma.

Lemma 2.4. *For all $x, \epsilon > 0$,*

$$G_\epsilon(x) \geq \frac{1}{2} \int_0^2 F_\epsilon(x+y) dy. \quad (2.12)$$

Proof. We change variables to find that

$$\int_0^2 F_\epsilon(x+y) dy = \frac{1}{2} \int_0^1 F_\epsilon\left(x + \frac{y}{2}\right) dy \geq \frac{1}{2} \int_0^1 F_\epsilon(x+y) dy, \quad (2.13)$$

by monotonicity. This proves the lemma. $\square$

3. Proof of Theorem 1.1

In light of (1.4) it suffices to prove that

$$\text{Cap}_{d/2}([0, 1] \times F) > 0 \quad \text{iff} \quad \text{Cap}_{(d/2)-1}(F) > 0. \quad (3.1)$$

The following harmonic-analytic fact does the job, and a little more; it must be well known, but we could not find it in a suitable form in the literature.

Recall that a function $f : \mathbf{R}^n \rightarrow [0, \infty]$ is of *strict positive type* if: (i) f is locally integrable away from $0 \in \mathbf{R}^n$; and (ii) the Fourier transform of f is *strictly* positive. Corresponding to such a function f we can define a function $\Pi_m f$ [equivalently, the operator Π_m] as follows:

$$(\Pi_m f)(x) := \int_{[0,1]^m} f(x+y) dy \quad \text{for all } x \in \mathbf{R}^{n-m}. \quad (3.2)$$

It is easy to see that

$$(\Pi_m f)(x) := \iint_{[0,1]^m \times [0,1]^m} f(x+y-z) dy dz \quad \text{for all } x \in \mathbf{R}^{n-m}. \quad (3.3)$$

This is a direct computation when $m = 1$; the general case is proved by induction. Then, we have

Theorem 3.1 (Projection theorem for capacities). *Let $n > 1$ be an integer, and suppose that $f : \mathbf{R}^n \rightarrow [0, \infty]$ is of strict positive type and continuous on $\mathbf{R}^n \setminus \{0\}$. Then, for all integers $1 \leq m < n$ and compact sets $F \subset \mathbf{R}^{n-m}$,*

$$\text{Cap}_f([0, 1]^m \times F) = \text{Cap}_{\Pi_m f}(F). \quad (3.4)$$

The proof is divided into two parts. The first part is easier, and will be dispensed with first.

Proof of Theorem 3.1 (The Upper Bound). Let λ_m denote the Lebesgue measure on $[0, 1]^m$, normalized to have mass one. If $\mu \in \mathcal{P}(F)$, then evidently,

$$I_{\Pi_m f}(\mu) = I_f(\lambda_m \times \mu) \geq \inf_{\nu \in \mathcal{P}([0, 1]^m \times F)} I_f(\nu). \quad (3.5)$$

The equality follows from (3.3) and the theorem of Fubini–Tonelli. But it is clear that $\lambda_m \times \mu \in \mathcal{P}([0, 1]^m \times F)$, whence $\text{Cap}_{\Pi_m f}(F) \leq \text{Cap}_f([0, 1]^m \times F)$. This completes our proof. $\square$

We need some preliminary developments for the lower bound. For this portion, we identify the hypercube $[0, 1]^m$ with the m -dimensional torus $\mathbf{T}^m$ in the usual way. In particular, note that $\mathbf{T}^m$ is compact in the resulting quotient topology. Any probability measure μ on $[0, 1]^m \times F$ can be identified with a probability measure on $\mathbf{T}^m \times F$ in the usual way. We continue to write the latter measure as μ as well. Throughout the remainder of this section, $f : \mathbf{R}^n \rightarrow [0, \infty]$ is a fixed function of strict positive type that is also continuous on $\mathbf{R}^n \setminus \{0\}$.

Lemma 3.2. *Suppose $\mathbf{T}^m \times F$ has positive f -capacity. Then, there exists a probability measure $\mathbf{e}_{\mathbf{T}^m \times F}$ —the “equilibrium measure”—on $\mathbf{T}^m \times F$ such that*

$$I_f(\mathbf{e}_{\mathbf{T}^m \times F}) = [\text{Cap}_f(\mathbf{T}^m \times F)]^{-1} < \infty. \quad (3.6)$$

Proof. For all $\epsilon > 0$ we can find $\mu_\epsilon \in \mathcal{P}(\mathbf{T}^m \times F)$ such that

$$I_f(\mu_\epsilon) \leq \frac{1 + \epsilon}{\text{Cap}_f(\mathbf{T}^m \times F)}. \quad (3.7)$$

All μ_ϵ ’s are probability measures on the same compact set $\mathbf{T}^m \times F$. Choose an arbitrary weak limit $\mu_0 \in \mathcal{P}(\mathbf{T}^m \times F)$ of the sequence $\{\mu_\epsilon\}_{\epsilon > 0}$, as $\epsilon \rightarrow 0$. It follows from Fatou’s lemma that

$$\begin{aligned} \liminf_{\epsilon \rightarrow 0} I_f(\mu_\epsilon) &\geq \liminf_{\eta \rightarrow 0} \liminf_{\epsilon \rightarrow 0} \iint_{\{|x-y| \geq \eta\}} f(x-y) \mu_\epsilon(dx) \mu_\epsilon(dy) \\ &\geq \liminf_{\eta \rightarrow 0} \iint_{\{|x-y| \geq \eta\}} f(x-y) \mu_0(dx) \mu_0(dy) \\ &= I_f(\mu_0). \end{aligned} \quad (3.8)$$

Thanks to (3.7), $I_f(\mu_0)$ is at most equal to the reciprocal of the f -capacity of $\mathbf{T}^m \times F$. On the other hand, the said capacity is bounded above by $I_f(\sigma)$ for all $\sigma \in \mathcal{P}(\mathbf{T}^m \times F)$, whence follows the lemma. $\square$

The following establishes the uniqueness of the equilibrium measure.

Lemma 3.3. *Suppose $\mathbf{T}^m \times F$ has positive f -capacity χ . If $I_f(\mu) = I_f(\nu) = 1/\chi$ for some $\mu, \nu \in \mathcal{P}(\mathbf{T}^m \times F)$, then $\mu = \nu = \mathbf{e}_{\mathbf{T}^m \times F}$.*

Proof. We denote by $\mathcal{F}$ the Fourier transform on any and every (locally compact) abelian group G ; $\mathcal{F}$ is normalized as follows: For all group characters ξ , and all $h \in L^1(G)$,

$$(\mathcal{F}h)(\xi) = \int_G (x, \xi) h(x) dx, \quad (3.9)$$

where (x, ξ) is the usual duality relation between $x \in G$ and the character ξ , and “ dx ” denotes Haar measure (normalized to be one if G is compact; counting measure if G is discrete; and mixed in the obvious way, when appropriate). Because f is of positive type and continuous away from the origin,

$$I_f(\mu) = \frac{1}{(2\pi)^n} \int_{\mathbf{T}^m \times \mathbf{R}^{n-m}} (\mathcal{F}f)(\xi) |(\mathcal{F}\mu)(\xi)|^2 d\xi; \quad (3.10)$$

see Kahane [12, Eq. (5), p. 134].

Using (3.10) (say) we can extend the definition of $I_f(\kappa)$ to all signed measures κ that have finite absolute mass. We note that $I_f(\kappa)$ is real and non-negative, but could feasibly be infinite; $I_f(\kappa)$ is strictly positive if κ is not identically equal to the zero measure. The latter follows from the strict positivity of f .

Let ρ and σ denote two signed measures that have finite absolute mass. Then, we can define, formally,

$$I_f(\sigma, \rho) := \iint \left[\frac{f(x-y) + f(y-x)}{2} \right] \sigma(dx) \rho(dy). \quad (3.11)$$

This is well-defined if $I_f(|\sigma|, |\rho|) < \infty$, for instance. Evidently, $I_f(\sigma, \rho) = I_f(\rho, \sigma)$ and $I_f(\sigma, \sigma) = I_f(\sigma)$. Finally, by the Cauchy-Schwarz inequality,

$$|I_f(\sigma, \rho)| \leq I_f(\sigma) I_f(\rho). \quad (3.12)$$

Now suppose to the contrary that the μ and ν of the statement of the lemma are distinct. Then, by (3.10),

$$0 < I_f\left(\frac{\mu - \nu}{2}\right) = \frac{I_f(\mu) + I_f(\nu) - 2I_f(\mu, \nu)}{4} = \frac{\chi^{-1} - I_f(\mu, \nu)}{2}, \quad (3.13)$$

where, we recall, $\chi^{-1} = I_f(\mathbf{e}_{\mathbf{T}^m \times F})$ denotes the reciprocal of the f -capacity of $\mathbf{T}^m \times F$. Consequently, $I_f(\mu, \nu)$ is strictly less than $I_f(\mathbf{e}_{\mathbf{T}^m \times F})$. From this we can deduce that

$$\begin{aligned} I_f\left(\frac{\mu + \nu}{2}\right) &= \frac{I_f(\mu) + I_f(\nu) + 2I_f(\mu, \nu)}{4} = \frac{\chi^{-1} + I_f(\mu, \nu)}{2} \\ &< I_f(\mathbf{e}_{\mathbf{T}^m \times F}) \leq I_f\left(\frac{\mu + \nu}{2}\right). \end{aligned} \quad (3.14)$$

And this is a contradiction. Therefore, $\mu = \nu$; also μ is equal to $\mathbf{e}_{\mathbf{T}^m \times F}$ because of the already-proved uniqueness together with Lemma 3.2. $\square$

Proof of Theorem 3.1 (The Lower Bound). It remains to prove that

$$\text{Cap}_{\Pi_m f}(F) \geq \text{Cap}_f([0, 1]^m \times F). \quad (3.15)$$

We will prove the seemingly-weaker statement that

$$\text{Cap}_{\Pi_m f}(F) \geq \text{Cap}_f(\mathbf{T}^m \times F). \quad (3.16)$$

This is seemingly weaker because $\text{Cap}_f(\mathbf{T}^m \times F) = \text{Cap}_f([0, 1]^m \times F)$. But, in fact, our proof will reveal that for all $q > 1$,

$$\text{Cap}_{\Pi_m f}(F) \geq q^{-m} \text{Cap}_f([0, q]^m \times F). \quad (3.17)$$

the right-hand side is at least $q^{-m} \text{Cap}_f([0, 1]^m \times F)$. Therefore, we can let $q \downarrow 1$ to derive (3.15), and therefore the theorem.

Having our ultimate goal (3.16) in mind, we can assume without loss of generality that

$$\text{Cap}_f(\mathbf{T}^m \times F) > 0, \quad (3.18)$$

so that $\mathbf{e}_{\mathbf{T}^m \times F}$ exists and is the unique minimizer in the definition of $\text{Cap}_f(\mathbf{T}^m \times F)$ (Lemmas 3.2 and 3.3).

Let us write any $z \in \mathbf{T}^m \times \mathbf{R}^{n-m}$ as $z = (z', z'')$, where $z' \in \mathbf{T}^m$ and $z'' \in \mathbf{R}^{n-m}$.

For all $a, b \in \mathbf{T}^m \times \mathbf{R}^{n-m}$ define $\tau_a(b) = a + b$. We emphasize that the first m coordinates of $\tau_a(b)$ are formed by addition in $\mathbf{T}^m$ [i.e., component-wise addition mod 1 in $[0, 1]^m$], whereas the next $n - m$ coordinates of $\tau_a(b)$ are formed by addition in $\mathbf{R}^{n-m}$. In particular, $\tau_a(\mathbf{T}^m \times F) = \mathbf{T}^m \times (a'' + F)$.

For all $a \in \mathbf{T}^m \times \mathbf{R}^{n-m}$, $\mathbf{e}_{\mathbf{T}^m \times F} \circ \tau_a^{-1}$ is a probability measure on $\tau_a(\mathbf{T}^m \times F)$. Moreover, it is easy to see that $\mathbf{e}_{\mathbf{T}^m \times F}$ and $\mathbf{e}_{\mathbf{T}^m \times F} \circ \tau_a^{-1}$ have the same f -energy. Therefore, whenever $a'' = 0$, $\mathbf{e}_{\mathbf{T}^m \times F} \circ \tau_a^{-1}$ is a probability measure on $\mathbf{T}^m \times F$ that minimizes the f -capacity of $\mathbf{T}^m \times F$. The uniqueness of $\mathbf{e}_{\mathbf{T}^m \times F}$ proves that

$$\mathbf{e}_{\mathbf{T}^m \times F} = \mathbf{e}_{\mathbf{T}^m \times F} \circ \tau_a^{-1} \quad \text{whenever } a'' = 0. \quad (3.19)$$

See Lemma 3.3. Now let X be a random variable with values in $\mathbf{T}^m \times F$ such that the distribution of X is $\mathbf{e}_{\mathbf{T}^m \times F}$. The preceding display implies that for all $a' \in \mathbf{T}^m$, the distribution of $(X' + a', X'')$ is the same as that of (X', X'') . The uniqueness of normalized Haar measure λ_m then implies that X' is distributed as λ_m . In fact, for all Borel sets $A \subset \mathbf{T}^m$ and $B \subset \mathbf{R}^{n-m}$,

$$\begin{aligned} \mathbf{e}_{\mathbf{T}^m \times F}(A \times B) &= \mathbf{P}\{X' \in A, X'' \in B\} \\ &= \int_{\mathbf{T}^m} \mathbf{P}\{X' \in a' + A, X'' \in B\} da' \\ &= \mathbf{E}[\lambda_m(A - X') ; X'' \in B] \\ &= \lambda_m(A) \mathbf{P}\{X'' \in B\} := \lambda_m(A) \mu(B). \end{aligned} \quad (3.20)$$

Now we compute directly to find that

$$\text{Cap}_f(\mathbf{T}^m \times F) = \frac{1}{I_f(\lambda_m \times \mu)} = \frac{1}{I_{\Pi_m f}(\mu)} \leq \frac{1}{\inf_{\sigma \in \mathcal{P}(F)} I_{\Pi_m f}(\sigma)}. \quad (3.21)$$

This proves (3.16), and therefore the theorem. $\square$

Finally we are ready to present the following:

Proof of Theorem 1.1. The function U_α is of strict positive type for all $0 < \alpha < d$. The easiest way to see this is to merely recall the following well-known fact from harmonic analysis: In the sense of distributions, $\mathcal{F}U_\alpha = c_{d,\alpha}U_{d-\alpha}$ for a positive and finite constant $c_{d,\alpha}$ [32, Lemma 1, p. 117]. We note also that U_α is continuous away from the origin. Thus, we can combine (1.4) with Theorem 3.1 to find that

$$\mathbb{P}\{\mathcal{Z}_d \cap F \neq \emptyset\} > 0 \iff \text{Cap}_{\Pi_1 U_{d/2}}(F) > 0. \quad (3.22)$$

But for all $x \geq \epsilon^2 > 0$,

$$(\Pi_1 U_{d/2})(x) = \int_0^1 \frac{dy}{|x+y|^{d/2}} = \frac{F_\epsilon(x)}{\epsilon^d}. \quad (3.23)$$

Therefore, in accord with Lemmas 2.2 and 2.4,

$$c_1 U_{(d-2)/2}(x) \leq (\Pi_1 U_{d/2})(x) \leq c_2 U_{(d-2)/2}(x), \quad (3.24)$$

for all $\epsilon > 0$ and $x \geq 2\epsilon^2$. Because (c_1, c_2) does not depend on ϵ , the displayed bounds are valid for all $x > 0$, whence it follows that

$$\frac{1}{c_2} \text{Cap}_{(d-2)/2}(F) \leq \text{Cap}_{\Pi_1 U_{d/2}}(F) \leq \frac{1}{c_1} \text{Cap}_{(d-2)/2}(F). \quad (3.25)$$

This and (3.22) together prove the theorem. $\square$

4. Proof of Theorem 1.4

Let $B^{(1)}$ and $B^{(2)}$ be two independent Brownian sheets in $\mathbf{R}^d$, and define for all $\mu \in \mathcal{P}(\mathbf{R}_+)$,

$$J_\epsilon(\mu) := \frac{1}{\epsilon^d} \iint_{[1,2]^2} \int \mathbf{1}_{\mathbf{A}(\epsilon; s, t)} \mu(ds) dt, \quad (4.1)$$

where $\mathbf{A}(\epsilon; a, b)$ is the event

$$\mathbf{A}(\epsilon; a, b) := \left\{ |B^{(2)}(a, b_2) - B^{(1)}(a, b_1)| \leq \epsilon \right\}, \quad (4.2)$$

for all $1 \leq a, b_1, b_2 \leq 2$ and $\epsilon > 0$.

Lemma 4.1. *We have*

$$\inf_{0 < \epsilon < 1} \inf_{\mu \in \mathcal{P}([1,2])} \mathbb{E}[J_\epsilon(\mu)] > 0. \quad (4.3)$$

Proof. The distribution of $B^{(2)}(s, t_2) - B^{(1)}(s, t_1)$ has a density function that is bounded below, uniformly for all $1 \leq s, t_1, t_2 \leq 2$. $\square$

Next we present a bound for the second moment of $J_\epsilon(\mu)$. For technical reasons, we first alter $J_\epsilon(\mu)$ slightly. Henceforth, we define

$$\hat{J}_\epsilon(\mu) := \frac{1}{\epsilon^d} \iint \int_{[1,3]^2} \mathbf{1}_{\mathbf{A}(\epsilon; s, t)} \mu(ds) dt. \quad (4.4)$$

Lemma 4.2. *There exists a constant c such that for all Borel probability measures μ on $\mathbf{R}_+$ and all $0 < \epsilon < 1$,*

$$\mathbb{E} \left[\left(\hat{J}_\epsilon(\mu) \right)^2 \right] \leq \frac{c I_{G_\epsilon}(\mu)}{\epsilon^d} \leq c I_{(d-4)/2}(\mu). \quad (4.5)$$

Proof. For all $\epsilon > 0$, $1 \leq s, u \leq 2$, and $t, v \in [1, 2] \times [3, 4]$ define

$$P_\epsilon(s, u; t, v) := \mathbb{P}(\mathbf{A}(\epsilon; s, t) \cap \mathbf{A}(\epsilon; u, v)). \quad (4.6)$$

We claim that there exists a constant c_1 —independent of (s, u, t, v, ϵ) —such that

$$P_\epsilon(s, u; t, v) \leq c_1 \epsilon^d f_\epsilon(|s - u| + |t - v|). \quad (4.7)$$

Lemmas 2.3 and 2.4 of Dalang and Khoshnevisan [4] contain closely-related, but non-identical, results.

Let us assume (4.7) for the time being and prove the theorem. We will establish (4.7) subsequently.

Owing to (4.7) and the Fubini–Tonelli theorem,

$$\begin{aligned} \mathbb{E} \left[\left(\hat{J}_\epsilon(\mu) \right)^2 \right] &\leq \frac{c_1}{\epsilon^d} \iint \int_{[1,3]^2 \times [[1,3]]^2} f_\epsilon(|s - u| + |t - v|) dt dv \mu(ds) \mu(du) \\ &\leq \frac{c}{\epsilon^d} \iint G_\epsilon(s - u) \mu(ds) \mu(du) \\ &= \frac{c I_{G_\epsilon}(\mu)}{\epsilon^d}. \end{aligned} \quad (4.8)$$

See (2.9). This is the first inequality of the lemma. The second follows from the first and Lemma 2.3. Now we proceed to derive (4.7).

By symmetry, it suffices to estimate $P_\epsilon(s, u; t, v)$ in the case that $s \leq u$. Now we carry out the estimates in two separate cases.

Case 1. First we consider the case $t_1 \leq v_1$ and $t_2 \leq v_2$. Define $\hat{B}^{(i)}$ to be the white noise that corresponds to the sheet $B^{(i)}$ ($i = 1, 2$). Then, consider

$$\begin{aligned} H_1^{(1)} &:= \hat{B}^{(1)}([0, s] \times [0, t_1]), & H_2^{(1)} &:= \hat{B}^{(1)}([0, s] \times [t_1, v_1]), \\ H_3^{(1)} &:= \hat{B}^{(1)}([s, u] \times [0, v_1]), \\ H_1^{(2)} &:= \hat{B}^{(2)}([0, s] \times [0, t_2]), & H_2^{(2)} &:= \hat{B}^{(2)}([0, s] \times [t_2, v_2]), \\ H_3^{(2)} &:= \hat{B}^{(2)}([s, u] \times [0, v_2]). \end{aligned} \quad (4.9)$$

Then, the H 's are all totally independent Gaussian random vectors. Moreover, we can find independent d -vectors $\{\mathbf{g}_j^{(i)}\}_{1 \leq i \leq 2, 1 \leq j \leq 3}$ of i.i.d. standard-normals such that

$$\begin{aligned} H_1^{(1)} &= (st_1)^{1/2} \mathbf{g}_1^{(1)}, \quad H_2^{(1)} = (s(v_1 - t_1))^{1/2} \mathbf{g}_2^{(1)}, \\ H_3^{(1)} &= (v_1(u - s))^{1/2} \mathbf{g}_3^{(1)}, \\ H_1^{(2)} &= (st_2)^{1/2} \mathbf{g}_1^{(2)}, \quad H_2^{(2)} = (s(v_2 - t_2))^{1/2} \mathbf{g}_2^{(2)}, \\ H_3^{(2)} &= (v_2(u - s))^{1/2} \mathbf{g}_3^{(2)}. \end{aligned} \quad (4.10)$$

In addition,

$$\begin{aligned} P_\epsilon(s, u; t, v) &= \mathbb{P} \left\{ \begin{aligned} &|H_1^{(2)} - H_1^{(1)}| \leq \epsilon \\ &|H_1^{(2)} + H_2^{(2)} + H_3^{(2)} - H_1^{(1)} - H_2^{(1)} - H_3^{(1)}| \leq \epsilon \end{aligned} \right\} \\ &\leq \mathbb{P} \left\{ |H_1^{(2)} - H_1^{(1)}| \leq \epsilon \right\} \\ &\quad \times \mathbb{P} \left\{ |H_2^{(2)} + H_3^{(2)} - H_2^{(1)} - H_3^{(1)}| \leq 2\epsilon \right\}. \end{aligned} \quad (4.11)$$

The first term on the right is equal to the following:

$$\mathbb{P} \left\{ (s(t_1 + t_2))^{1/2} |\mathbf{g}| \leq \epsilon \right\} \leq c_2 \epsilon^d, \quad (4.12)$$

where $c_2 > 0$ does not depend on (s, t, u, v, ϵ) ; see Lemma 2.1. Also, the second term is equal to the following:

$$\begin{aligned} &\mathbb{P} \left\{ (s(v_2 - t_2) + v_2(u - s) + s(v_1 - t_1) + v_1(u - s))^{1/2} |\mathbf{g}| \leq 2\epsilon \right\} \\ &\leq \mathbb{P} \left\{ (|v - t| + (u - s))^{1/2} |\mathbf{g}| \leq 2\epsilon \right\} \\ &\leq c_3 f_\epsilon(|u - s| + |t - v|), \end{aligned} \quad (4.13)$$

and $c_3 > 0$ does not depend on (s, t, u, v, ϵ) . We obtain (4.7) by combining (4.12) and (4.13). This completes the proof of Case 1.

Case 2. Now we consider the case that $t_2 \geq v_2$ and $t_1 \leq v_1$. We can replace the $H_i^{(j)}$'s of Case 1 with the following:

$$\begin{aligned} H_1^{(1)} &:= \hat{B}^{(1)}([0, s] \times [0, t_1]), \quad H_2^{(1)} := \hat{B}^{(1)}([0, s] \times [t_1, v_1]), \\ H_3^{(1)} &:= \hat{B}^{(1)}([s, u] \times [0, v_1]), \\ H_1^{(2)} &:= \hat{B}^{(2)}([0, s] \times [0, v_2]), \quad H_2^{(2)} := \hat{B}^{(2)}([0, s] \times [v_2, t_2]), \\ H_3^{(2)} &:= \hat{B}^{(2)}([s, u] \times [0, v_2]). \end{aligned} \quad (4.14)$$

It follows then that

$$P_\epsilon(s, u; t, v) = \mathbb{P} \left\{ \begin{aligned} &|H_1^{(2)} + H_2^{(2)} - H_1^{(1)}| \leq \epsilon \\ &|H_1^{(2)} + H_3^{(2)} - H_1^{(1)} - H_2^{(1)} - H_3^{(1)}| \leq \epsilon \end{aligned} \right\}. \quad (4.15)$$

One can check covariances and see that the density function of $H_1^{(2)} - H_1^{(1)}$ is bounded above by a constant $c_1 > 0$ that does not depend on (s, t, u, v, ϵ) . Therefore,

$$\begin{aligned} P_\epsilon(s, u; t, v) &\leq c_1 \int_{\mathbf{R}^d} \mathbf{P} \left\{ \begin{array}{l} |H_1^{(2)} + z| \leq \epsilon \\ |H_3^{(2)} - H_2^{(1)} - H_3^{(1)} + z| \leq \epsilon \end{array} \right\} dz \\ &= c_1 \int_{\{|w| \leq \epsilon\}} \mathbf{P} \left\{ |H_3^{(2)} - H_1^{(2)} + H_2^{(1)} - H_3^{(1)} + w| \leq \epsilon \right\} dw \\ &\leq c_1 (2\epsilon)^d \mathbf{P} \left\{ |H_3^{(2)} - H_1^{(2)} + H_2^{(1)} - H_3^{(1)}| \leq 2\epsilon \right\}. \end{aligned} \quad (4.16)$$

The component-wise variance of this particular combination of $H_j^{(i)}$'s is equal to $(u-s)(v_1+v_2) + s(v_1-t_1+v_2-t_2) \geq (u-s) + |t-v|$. Whence follows (4.7) in the present case.

Symmetry considerations, together with Cases 1 and 2, prove that (4.7) holds for all possible configurations of (s, u, t, v) . This completes our proof. $\square$

For all $i \in \{1, 2\}$ and $s, t \geq 0$, we define $\mathcal{F}_{s,t}^{(i)}$ to be the σ -algebra generated by $\{B^{(i)}(u, v)\}_{0 \leq u \leq s, 0 \leq v \leq t}$; as usual, we can assume that $\mathcal{F}^{(i)}$'s are complete and right-continuous in the partial order “ $\prec$ ” described as follows: For all $s, t, u, v \geq 0$, $(s, t) \prec (u, v)$ iff $s \leq u$ and $t \leq v$. Based on $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$, we define

$$\mathcal{F}_{s;t,v} := \mathcal{F}_{s,t}^{(1)} \vee \mathcal{F}_{s,v}^{(2)} \quad \text{for all } s, t, v \geq 0. \quad (4.17)$$

The following proves that Cairoli's maximal L^2 -inequality holds with respect to the family of $\mathcal{F}_{s;t,v}$'s.

Lemma 4.3. *Choose and fix a number $p > 1$. Then for all almost surely non-negative random variables $Y \in \mathcal{L}^p := L^p(\Omega, \bigvee_{s,t,v \geq 0} \mathcal{F}_{s;t,v}, \mathbf{P})$,*

$$\left\| \sup_{s,t,v \in \mathbf{Q}_+} \mathbf{E}[Y \mid \mathcal{F}_{s;t,v}] \right\|_{\mathcal{L}^p} \leq \left(\frac{p}{p-1} \right)^3 \|Y\|_{\mathcal{L}^p}. \quad (4.18)$$

Proof. We propose to prove that for all $s, s', t, t', v, v' \geq 0$, and all bounded random variables Y that are $\mathcal{F}_{s';t',v'}$ -measurable,

$$\mathbf{E}[Y \mid \mathcal{F}_{s;t,v}] = \mathbf{E}[Y \mid \mathcal{F}_{s \wedge s'; t \wedge t', v \wedge v'}] \quad \text{a.s.} \quad (4.19)$$

This proves that the three-parameter filtration $\{\mathcal{F}_{s;t,v}\}_{s,t,v \in \mathbf{Q}_+}$ is *commuting* in the sense of Khoshnevisan [15, p. 35]. Corollary 3.5.1 of the same reference [15, p. 37] would then finish our proof.

By a density argument, it suffices to demonstrate (4.19) in the case that $Y = Y_1 Y_2$, where Y_1 and Y_2 are bounded, and measurable with respect to $\mathcal{F}_{s',t'}^{(1)}$ and $\mathcal{F}_{s',v'}^{(2)}$, respectively. But in this case, independence implies that almost surely,

$$\mathbf{E}[Y \mid \mathcal{F}_{s;t,v}] = \mathbf{E}[Y_1 \mid \mathcal{F}_{s,t}^{(1)}] \mathbf{E}[Y_2 \mid \mathcal{F}_{s,v}^{(2)}]. \quad (4.20)$$

By the Cairoli–Walsh commutation theorem [15, Theorem 2.4.1, p. 237], $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$ are each two-parameter, commuting filtrations. Theorem 3.4.1 of Khoshnevisan [15, p. 36] implies that almost surely,

$$\begin{aligned} \mathbb{E}[Y_1 \mid \mathcal{F}_{s,t}^{(1)}] &= \mathbb{E}[Y_1 \mid \mathcal{F}_{s \wedge s', t \wedge t'}^{(1)}], \\ \mathbb{E}[Y_2 \mid \mathcal{F}_{s,v}^{(2)}] &= \mathbb{E}[Y_2 \mid \mathcal{F}_{s \wedge s', v \wedge v'}^{(2)}]. \end{aligned} \quad (4.21)$$

Plug this into (4.20) to obtain (4.19) in the case that Y has the special form $Y_1 Y_2$, as described above. The general form of (4.19) follows from the mentioned special case and density. $\square$

Lemma 4.4. *Choose and fix a number $p > 1$. Then for all almost surely non-negative random variables $Y \in \mathcal{L}^p := L^p(\Omega, \vee_{s,t,v \geq 0} \mathcal{F}_{s;t,v}, \mathbb{P})$, we can find a continuous modification of the three-parameter process $\{\mathbb{E}[Y \mid \mathcal{F}_{s;t,v}]\}_{s,t,v \geq 0}$. Consequently,*

$$\left\| \sup_{s,t,v \geq 0} \mathbb{E}[Y \mid \mathcal{F}_{s;t,v}] \right\|_{\mathcal{L}^p} \leq \left(\frac{p}{p-1} \right)^3 \|Y\|_{\mathcal{L}^p}. \quad (4.22)$$

Proof. First suppose $Y = Y_1 Y_2$ where $Y_i \in \mathcal{L}^p(\Omega, \vee_{s,t \geq 0} \mathcal{F}_{s;t}^{(i)}, \mathbb{P})$. In this case, (4.20) holds by independence. Thanks to Wong and Zakai [34], each of the two conditional expectations on the right-hand side of (4.20) has a representation in terms of continuous, two-parameter and one-parameter stochastic integrals. This proves the continuity of $(s, t, v) \mapsto \mathbb{E}[Y \mid \mathcal{F}_{s;t,v}]$ in the case where Y has the mentioned special form. In the general case, we can find $Y^1, Y^2, \dots$ such that: (i) Each Y^i has the mentioned special form; and (ii) $\|Y^n - Y\|_{\mathcal{L}^p} \leq 2^{-n}$. We can write, for all integers $n \geq 1$,

$$|\mathbb{E}[Y^{n+1} \mid \mathcal{F}_{s;t,v}] - \mathbb{E}[Y^n \mid \mathcal{F}_{s;t,v}]| \leq \sum_{k=n}^{\infty} |\mathbb{E}[Y^{k+1} - Y^k \mid \mathcal{F}_{s;t,v}]|. \quad (4.23)$$

Take supremum over $s, t, v \in \mathbf{Q}_+$ and apply Lemma 4.3 to find that

$$\begin{aligned} & \sum_{n=1}^{\infty} \left\| \sup_{s,t,v \in \mathbf{Q}_+} |\mathbb{E}[Y^{n+1} \mid \mathcal{F}_{s;t,v}] - \mathbb{E}[Y^n \mid \mathcal{F}_{s;t,v}]| \right\|_{\mathcal{L}^p} \\ & \leq c \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \|Y^{k+1} - Y^k\|_{\mathcal{L}^p} < \infty. \end{aligned} \quad (4.24)$$

Because each $\mathbb{E}[Y^n \mid \mathcal{F}_{s;t,v}]$ is continuous in (s, t, v) , $\mathbb{E}[Y \mid \mathcal{F}_{s;t,v}]$ has a continuous modification. The ensuing maximal inequality follows from continuity and Lemma 4.3. $\square$

Lemma 4.5. *There exists a constant c such that the following holds outside a single null set: For all $0 < \epsilon < 1$, $1 \leq a, b_1, b_2 \leq 2$, and $\mu \in \mathcal{P}(\mathbf{R}_+)$,*

$$\mathbb{E} \left[\hat{J}_\epsilon(\mu) \mid \mathcal{F}_{a; b_1, b_2} \right] \geq \frac{c}{\epsilon^d} \int_{F \cap [a, 2]} G_\epsilon(s - a) \mu(ds) \cdot \mathbf{1}_{\mathbf{A}(\epsilon/2; a, b)}. \quad (4.25)$$

Remark 4.6. As the proof will show, we may have to redefine the left-hand side of (4.25) on a null-set to make things work seamlessly. The details are standard, elementary probability theory and will go without further mention.

Proof. Throughout this proof we write $\mathcal{E} := \mathcal{E}_{a; b; \epsilon}(\mu) := \mathbb{E}[\hat{J}_\epsilon(\mu) \mid \mathcal{F}_{a; b_1, b_2}]$. Evidently,

$$\mathcal{E} \geq \frac{1}{\epsilon^d} \int_{b_1}^3 \int_{b_2}^3 \int_{F \cap [a, 2]} \mathbb{P}(\mathbf{A}(\epsilon; s, t) \mid \mathcal{F}_{a; b_1, b_2}) \mu(ds) dt_2 dt_1. \quad (4.26)$$

A white-noise decomposition implies the following: For all $s \geq a$, $t_1 \geq b_1$, and $t_2 \geq b_2$,

$$\begin{aligned} B^{(1)}(s, t_1) &= B^{(1)}(a, b_1) + b_1^{1/2} W_1^1(s - a) + a^{1/2} W_2^1(t_1 - b_1) \\ &\quad + V^1(s - a, t_1 - b_1), \\ B^{(2)}(s, t_2) &= B^{(2)}(a, b_2) + b_2^{1/2} W_1^2(s - a) + a^{1/2} W_2^2(t_2 - b_2) \\ &\quad + V^2(s - a, t_2 - b_2). \end{aligned} \quad (4.27)$$

Here: the W_j^i 's are standard, linear Brownian motions; the V^i 's are Brownian sheets; and the collection $\{W_j^i, V^i, B^i(a, b_i)\}_{i,j=1}^2$ is totally independent. By appealing to this decomposition in conjunction with (4.26) we can infer that the following is a lower bound for $\mathcal{E}$, almost surely on the event $\mathbf{A}(\epsilon/2; a, b)$:

$$\begin{aligned} &\frac{1}{\epsilon^d} \int_{b_1}^3 \int_{b_2}^3 \int_{F \cap [a, 2]} \mu(ds) dt_2 dt_1 \\ &\times \mathbb{P} \left\{ \left| \begin{aligned} &b_2^{1/2} W_1^2(s - a) + a^{1/2} W_2^2(t_2 - b_2) + V^2(s - a, t_2 - b_2) \\ &- b_1^{1/2} W_1^1(s - a) - a^{1/2} W_2^1(t_1 - b_1) - V^1(s - a, t_1 - b_1) \end{aligned} \right| \leq \frac{\epsilon}{2} \right\} \\ &= \frac{1}{\epsilon^d} \int_{b_1}^3 \int_{b_2}^3 \int_{F \cap [a, 2]} \mathbb{P} \left\{ \sigma |\mathbf{g}| \leq \frac{\epsilon}{2} \right\} \mu(ds) dt_2 dt_1. \end{aligned} \quad (4.28)$$

Here, $\mathbf{g}$ is a d -vector of i.i.d. standard-normals, and σ^2 is equal to the quantity $b_2(s - a) + a(t_2 - b_2) + (s - a)(t_2 - b_2) + b_1(s - a) + a(t_1 - b_1) + (s - a)(t_1 - b_1)$. The range of possible values of a and b is respectively $[1, 2]$ and $[1, 2]^2$. This means that we can find a constant $c > 0$ —independent of (a, b, s, t) —such that $\sigma^2 \leq c\{|s - a| + |t - b|\}$. Apply this bound to the previous display; then appeal to Lemma 2.1 to find that (4.25) holds a.s., but the null-set could feasibly depend on (a, b, ϵ) .

To ensure that the null-set can be chosen independently from (a, b, ϵ) , we first note that the integral on the right-hand side of (4.25) is: (i) Continuous in $\epsilon > 0$; (ii) independent of $b \in [1, 2]^2$; and (iii) lower semi-continuous in $a \in [1, 2]$. Similarly, $(a, b, \epsilon) \mapsto \mathbf{1}_{\mathbf{A}(\epsilon; a, b)}$ is left-continuous in $\epsilon > 0$ and lower semi-continuous in $(a, b) \in [1, 2]^3$. Therefore, it suffices to prove that the left-hand side of (4.25) is a.s. continuous in $(a, b) \in [1, 2]^3$, and left-continuous in $\epsilon > 0$. The left-continuity assertion about $\epsilon > 0$ is evident; continuity in (a, b) follows if we could prove that for all bounded random variables Y , $(a, b) \mapsto \mathbb{E}[Y \mid \mathcal{F}_{a; b_1, b_2}]$ has an a.s.-continuous modification. But this follows from Lemma 4.4. $\square$

Next we state and prove a quantitative capacity estimate.

Proposition 4.7. *Consider the collection of times of double-points:*

$$D(\omega) := \left\{ 1 \leq s \leq 2 : \inf_{t \in [1, 2]^2} |B^{(2)}(s, t_2) - B^{(1)}(s, t_1)|(\omega) = 0 \right\}. \quad (4.29)$$

Then there exists a constant $c > 1$ such that for all compact, non-random sets $F \subseteq [1, 2]$,

$$\frac{1}{c} \text{Cap}_{(d-4)/2}(F) \leq \mathbb{P}\{D \cap F \neq \emptyset\} \leq c \text{Cap}_{(d-4)/2}(F). \quad (4.30)$$

Proof. Define the closed random sets,

$$D_\epsilon(\omega) := \left\{ 1 \leq s \leq 2 : \inf_{t \in [1, 2]^2} |B^{(2)}(s, t_2) - B^{(1)}(s, t_1)|(\omega) \leq \epsilon \right\}. \quad (4.31)$$

Also, choose and fix a probability measure $\mu \in \mathcal{P}(F)$. It is manifest that D_ϵ intersects F almost surely on the event $\{J_\epsilon(\mu) > 0\}$. Therefore, we can apply the Paley–Zygmund inequality to find that

$$\mathbb{P}\{D_\epsilon \cap F \neq \emptyset\} \geq \frac{(\mathbb{E}[J_\epsilon(\mu)])^2}{\mathbb{E}\left[(J_\epsilon(\mu))^2\right]} \geq \frac{(\mathbb{E}[J_\epsilon(\mu)])^2}{\mathbb{E}\left[\left(\hat{J}_\epsilon(\mu)\right)^2\right]}. \quad (4.32)$$

Let $\epsilon \downarrow 0$ and appeal to compactness to find that

$$\mathbb{P}\{D \cap F \neq \emptyset\} \geq \frac{\liminf_{\epsilon \rightarrow 0} (\mathbb{E}[J_\epsilon(\mu)])^2}{c I_{(d-4)/2}(\mu)}. \quad (4.33)$$

[We have used the second bound of Lemma 4.2.] According to Lemma 4.1, the numerator is bounded below by a strictly positive number that does not depend on μ . Therefore, the lower bound of our proposition follows from optimizing over all $\mu \in \mathcal{P}(F)$.

In order to derive the upper bound we can assume, without any loss in generality, that $\mathbb{P}\{D_\epsilon \cap F \neq \emptyset\} > 0$; for otherwise there is nothing to prove.

For all $0 < \epsilon < 1$ define

$$\tau_\epsilon := \inf \left\{ s \in F : \inf_{t \in [1, 2]^2} |B^{(2)}(s, t_2) - B^{(1)}(s, t_1)| \leq \epsilon \right\}. \quad (4.34)$$

As usual, $\inf \emptyset := \infty$. It is easy to see that τ_ϵ is a stopping time with respect to the one-parameter filtration $\{\mathcal{H}_s\}_{s \geq 0}$, where

$$\mathcal{H}_s := \bigvee_{t, v \geq 0} \mathcal{F}_{s; t, v} \quad \text{for all } s \geq 0. \quad (4.35)$$

We note also that there exist $[0, \infty]$ -valued random variables τ'_ϵ and τ''_ϵ such that: (i) $\tau'_\epsilon \vee \tau''_\epsilon = \infty$ iff $\tau_\epsilon = \infty$; and (ii) almost surely on $\{\tau_\epsilon < \infty\}$,

$$\left| B^{(2)}(\tau_\epsilon, \tau'_\epsilon) - B^{(1)}(\tau_\epsilon, \tau''_\epsilon) \right| \leq \epsilon. \quad (4.36)$$

Define

$$p_\epsilon := \mathbb{P}\{\tau_\epsilon < \infty\}, \quad \text{and} \quad \nu_\epsilon(\bullet) := \mathbb{P}\{\tau_\epsilon \in \bullet \mid \tau_\epsilon < \infty\}. \quad (4.37)$$

We can note that

$$\inf_{\epsilon > 0} p_\epsilon \geq \mathbb{P}\{D \cap F \neq \emptyset\}, \quad (4.38)$$

and this is strictly positive by our earlier assumption. Consequently, ν_ϵ is well defined as a classical conditional probability, and $\nu_\epsilon \in \mathcal{P}(F)$. Now consider the process $\{M^\epsilon\}_{0 < \epsilon < 1}$ defined as follows:

$$M_{a; b_1, b_2}^\epsilon := \mathbb{E} \left[\hat{J}_\epsilon(\nu_\epsilon) \mid \mathcal{F}_{a; b_1, b_2} \right]. \quad (4.39)$$

Thanks to Lemmas 4.4 and 4.5,

$$\begin{aligned} & \mathbb{E} \left[\sup_{a, b_1, b_2 \in \mathbf{R}_+^3} (M_{a; b_1, b_2}^\epsilon)^2 \right] \\ & \geq \mathbb{E} \left[\left(M_{\tau_\epsilon; \tau'_\epsilon, \tau''_\epsilon}^\epsilon \right)^2 \right] \\ & \geq \frac{cp_\epsilon}{\epsilon^{2d}} \mathbb{E} \left[\left(\int_{F \cap [\tau_\epsilon, 2]} G_\epsilon(s - \tau_\epsilon) \nu_\epsilon(ds) \right)^2 \mid \tau_\epsilon < \infty \right] \\ & \geq \frac{cp_\epsilon}{\epsilon^{2d}} \left(\mathbb{E} \left[\int_{F \cap [\tau_\epsilon, 2]} G_\epsilon(s - \tau_\epsilon) \nu_\epsilon(ds) \mid \tau_\epsilon < \infty \right] \right)^2. \end{aligned} \quad (4.40)$$

The last line is a consequence of the Cauchy–Schwarz inequality. We can bound the squared term on the right-hand side as follows:

$$\begin{aligned} & \mathbb{E} \left[\int_{F \cap [\tau_\epsilon, 2]} G_\epsilon(s - \tau_\epsilon) \nu_\epsilon(ds) \mid \tau_\epsilon < \infty \right] \\ & = \iint_{\{s \in F \cap [u, 2]\}} G_\epsilon(s - u) \nu_\epsilon(ds) \nu_\epsilon(du) \\ & \geq \frac{1}{2} \iint G_\epsilon(s - u) \nu_\epsilon(ds) \nu_\epsilon(du) = \frac{1}{2} I_{G_\epsilon}(\nu_\epsilon). \end{aligned} \quad (4.41)$$

Plug this in (4.40), and appeal to Lemmas 4.2 and 4.3, to find that

$$\begin{aligned} \frac{cp_\epsilon}{4\epsilon^{2d}} (I_{G_\epsilon}(\nu_\epsilon))^2 &\leq \mathbb{E} \left[\sup_{a,b_1,b_2 \in \mathbf{Q}_+} (M_{a;b_1,b_2}^\epsilon)^2 \right] \\ &\leq 2^6 \mathbb{E} \left[\left(\hat{J}_\epsilon(\nu_\epsilon) \right)^2 \right] \leq \frac{c}{\epsilon^d} I_{G_\epsilon}(\nu_\epsilon). \end{aligned} \quad (4.42)$$

Solve this, using (4.38), to find that

$$\mathbb{P}\{D \cap F \neq \emptyset\} \leq \frac{c}{I_{G_\epsilon}(\nu_\epsilon)}. \quad (4.43)$$

Choose and fix a number $\eta > 0$. In accord with Lemma 2.3,

$$I_{G_\epsilon}(\nu_\epsilon) \geq \iint_{\{|s-u| \geq \eta\}} U_{(d-4)/2}(s-u) \nu_\epsilon(ds) \nu_\epsilon(du), \quad (4.44)$$

for all $0 < \epsilon < \eta^{1/2}$. Recall that $\{\nu_\epsilon\}_{\epsilon>0}$ is a net of probability measures on F . Because F is compact, Prohorov's theorem ensures that there exists a subsequential weak limit $\nu_0 \in \mathcal{P}(F)$ of $\{\nu_\epsilon\}_{\epsilon>0}$, as $\epsilon \rightarrow 0$. Therefore, we can apply Fatou's lemma to find that

$$\begin{aligned} \liminf_{\epsilon \rightarrow 0} I_{G_\epsilon}(\nu_\epsilon) &\geq \lim_{\eta \rightarrow 0} \iint_{\{|s-u| \geq \eta\}} U_{(d-4)/2}(s-u) \nu_0(ds) \nu_0(du) \\ &= I_{(d-4)/2}(\nu_0). \end{aligned} \quad (4.45)$$

Together with (4.43), the preceding implies that $\mathbb{P}\{D \cap F \neq \emptyset\}$ is at most some constant divided by $I_{(d-4)/2}(\nu_0)$. This, in turn, is bounded by a constant multiple of $\text{Cap}_{(d-4)/2}(F)$. The proposition follows. $\square$

Proof of Theorem 1.4. Let I and J be disjoint, closed intervals in $(0, \infty)$ with the added property that $x < y$ for all $x \in I$ and $y \in J$. Define

$$\mathcal{D}_d(I, J) := \{s > 0 : B(s, t_1) = B(s, t_2) \text{ for some } t_1 \in I \text{ and } t_2 \in J\}. \quad (4.46)$$

We intend to prove that

$$\mathbb{P}\{\mathcal{D}_d(I, J) \cap F \neq \emptyset\} > 0 \iff \text{Cap}_{(d-4)/2}(F) > 0. \quad (4.47)$$

Evidently, this implies Theorem 1.4. Without loss of much generality, we may assume that $I = [\frac{1}{2}, \frac{3}{2}]$, $J = [\frac{7}{2}, \frac{9}{2}]$, and $F \subseteq [1, 2]$. Now consider the random fields,

$$\begin{aligned} B^{(2)}(s, t) &:= B(s, \tfrac{5}{2} + t) - B(s, \tfrac{5}{2}) \\ B^{(1)}(s, t) &:= B(s, \tfrac{5}{2} - t) - B(s, \tfrac{5}{2}), \end{aligned} \quad (4.48)$$

for $0 \leq s, t \leq 5/2$. Then two covariance computations reveal that the random fields $\{B^{(1)}(s, \frac{5}{2} - t) - B(s, \frac{5}{2})\}_{1 \leq s, t \leq 2}$ and $\{B^{(2)}(s, \frac{5}{2} + t) - B(s, \frac{5}{2})\}_{1 \leq s, t \leq 2}$ are independent Brownian sheets. On the other hand, the following are easily seen to be equivalent: (i) There exists $(s, t_1, t_2) \in [1, 2]^3$ such that $B^{(1)}(s, t_1) = B^{(2)}(s, t_2)$;

and (ii) There exists $(s, t_1, t_2) \in [1, 2] \times I \times J$ such that $B(s, t_1) = B(s, t_2)$. Therefore, (4.47) follows from Proposition 4.7. This completes our proof. $\square$

5. More on Double-Points

Consider the random sets

$$\begin{aligned}\hat{\mathcal{D}}_d &:= \{(s, t_1, t_2) \in \mathbf{R}_+^3 : B(s, t_1) = B(s, t_2)\}, \\ \bar{\mathcal{D}}_d &:= \{(s, t_1) \in \mathbf{R}_+^2 : B(s, t_1) = B(s, t_2) \text{ for some } t_2 > 0\}.\end{aligned}\tag{5.1}$$

The methods of this paper are not sufficiently delicate to characterize the polar sets of $\hat{\mathcal{D}}_d$ and $\mathcal{D}_d$. I hasten to add that I believe such a characterization is within reach of the existing technology [16]. Nonetheless it is not too difficult to prove the following by appealing solely to the techniques developed here.

Theorem 5.1. *For all non-random compact sets $E \subset (0, \infty)^2$ and $G \subset (0, \infty)^3$,*

$$\begin{aligned}\text{Cap}_{d/2}(G) > 0 &\implies \mathbf{P}\{\hat{\mathcal{D}}_d \cap G \neq \emptyset\} > 0 \implies \mathcal{H}_{d/2}(G) > 0, \\ \text{Cap}_{(d-2)/2}(E) > 0 &\implies \mathbf{P}\{\bar{\mathcal{D}}_d \cap E \neq \emptyset\} > 0 \implies \mathcal{H}_{(d-2)/2}(E) > 0.\end{aligned}\tag{5.2}$$

where $\mathcal{H}_\alpha$ denotes the α -dimensional Hausdorff measure [Appendix A.3].

Proof. Let $B^{(1)}$ and $B^{(2)}$ be two independent, two-parameter Brownian sheets on $\mathbf{R}^d$. It suffices to prove that there exists a constant $c > 1$ such that for all non-random compact sets $E \subseteq [1, 2]^2$ and $G \subseteq [1, 2]^3$,

$$\begin{aligned}c^{-1}\text{Cap}_{d/2}(G) &\leq \mathbf{P}\{\hat{\mathcal{T}}_d \cap G \neq \emptyset\} \leq c\mathcal{H}_{d/2}(G), \\ c^{-1}\text{Cap}_{(d-2)/2}(E) &\leq \mathbf{P}\{\bar{\mathcal{T}}_d \cap E \neq \emptyset\} \leq c\mathcal{H}_{(d-2)/2}(E),\end{aligned}\tag{5.3}$$

where

$$\begin{aligned}\hat{\mathcal{T}}_d &:= \{(s, t_1, t_2) \in [1, 2]^3 : B^{(2)}(s, t_2) = B^{(1)}(s, t_1)\}, \\ \bar{\mathcal{T}}_d &:= \{(s, t_1) \in [1, 2]^2 : B^{(2)}(s, t_2) = B^{(1)}(s, t_1) \text{ for some } t_2 > 0\}.\end{aligned}\tag{5.4}$$

[This sort of reasoning has been employed in the proof of Theorem 1.1 already; we will not repeat the argument here.] We begin by deriving the first bound in (5.3).

Recall (4.2). Choose and fix $\mu \in \mathcal{P}(G)$, and define for all $\epsilon > 0$,

$$\mathcal{J}_\epsilon(\mu) := \frac{1}{\epsilon^d} \iiint \mathbf{1}_{\mathbf{A}(\epsilon; s, t)} \mu(ds dt_1 dt_2).\tag{5.5}$$

The proof of Lemma 4.1 shows that

$$\inf_{0 < \epsilon < 1} \inf_{\mu \in \mathcal{P}([1, 2]^3)} \mathbf{E}[\mathcal{J}_\epsilon(\mu)] > 0.\tag{5.6}$$

Similarly, we can apply (4.7) to find that

$$\begin{aligned} \mathbb{E} \left[(\mathcal{J}_\epsilon(\mu))^2 \right] &\leq \frac{c}{\epsilon^d} \iiint f_\epsilon(|s-u|+|t-v|) \mu(ds dt_1 dt_2) \mu(du dv_1 dv_2) \\ &\leq c I_{d/2}(\mu). \end{aligned} \quad (5.7)$$

We have used the obvious inequality, $f_\epsilon(x) \leq \epsilon^d |x|^{-d/2}$. The lower bound in (5.3) follows from the previous two moment-bounds, and the Paley-Zygmund-inequality; we omit the details.

For the proof of the upper bound it is convenient to introduce some notation. Define

$$\begin{aligned} \Delta(s; t) &:= B^{(2)}(s, t_2) - B^{(1)}(s, t_1) \quad \text{for all } s, t_1, t_2 \geq 0, \\ \mathcal{U}(x; \epsilon) &:= [x_1, x_1 + \epsilon] \times [x_2, x_2 + \epsilon] \times [x_3, x_3 + \epsilon] \quad \text{for all } x \in \mathbf{R}^3, \epsilon > 0. \end{aligned} \quad (5.8)$$

Then,

$$\mathbb{P} \left\{ \hat{\mathcal{T}}_d \cap \mathcal{U}(x; \epsilon) \neq \emptyset \right\} \leq \mathbb{P} \left\{ |\Delta(x)| \leq \Theta(x; \epsilon) \right\}, \quad (5.9)$$

where $\Theta(x; \epsilon) := \sup_{y \in \mathcal{U}(x; \epsilon)} |\Delta(y) - \Delta(x)|$. The density function of $\Delta(x)$ is bounded above, uniformly for all $x \in [1, 2]^3$. Furthermore, $\Delta(x)$ is independent of $\Theta(x; \epsilon)$. Therefore, there exists a constant c such that uniformly for all $0 < \epsilon < 1$ and $x \in [1, 2]^3$,

$$\mathbb{P} \left\{ \hat{\mathcal{T}}_d \cap \mathcal{U}(x; \epsilon) \neq \emptyset \right\} \leq c \mathbb{E} \left[(\Theta(x; \epsilon))^d \right] \leq c \epsilon^{d/2}. \quad (5.10)$$

The final inequality holds because: (i) Brownian-sheet scaling dictates that $\Theta(x; \epsilon)$ has the same law as $\epsilon^{d/2} \Theta(x; 1)$; and (ii) $\Theta(x; 1)$ has moments of all order, with bounds that do not depend on $x \in [1, 2]^3$ [25, Lemma 1.2].

To prove the upper bound we can assume that $\mathcal{H}_{d/2}(G) < \infty$. In this case we can find $x_1, x_2, \dots \in [1, 2]^3$ and $r_1, r_2, \dots \in (0, 1)$ such that $G \subseteq \cup_{i=1}^\infty \mathcal{U}(x_i; r_i)$ and $\sum_{i=1}^\infty r_i^{d/2} \leq 2\mathcal{H}_{d/2}(G)$. Thus, by (5.10),

$$\begin{aligned} \mathbb{P} \left\{ \hat{\mathcal{T}}_d \cap G \neq \emptyset \right\} &\leq \sum_{i \geq 1} \mathbb{P} \left\{ \hat{\mathcal{T}}_d \cap \mathcal{U}(x_i; r_i) \neq \emptyset \right\} \\ &\leq c \sum_{i \geq 1} r_i^{d/2} \leq 2c \mathcal{H}_{d/2}(G). \end{aligned} \quad (5.11)$$

This completes our proof of the first bound in (5.3).

In order to prove the lower bound for $\bar{\mathcal{T}}_d$ note that $\bar{\mathcal{T}}_d$ intersects E if and only if $\hat{\mathcal{T}}_d$ intersects $[0, 1] \times E$. In (3.1) we proved that if E is a one-dimensional, compact set, then $\text{Cap}_{d/2}([0, 1] \times E) = \text{Cap}_{(d-2)/2}(E)$. A similar proof shows the same fact holds in any dimension, whence follows the desired lower bound for the probability that $\bar{\mathcal{T}}_d$ intersects E .

To conclude, it suffices to prove that

$$\mathcal{H}_{d/2}([0, 1] \times E) > 0 \implies \mathcal{H}_{(d-2)/2}(E) > 0. \quad (5.12)$$

But this follows readily from Frostman's lemma [Appendix A.3]. Indeed, the positivity of $\mathcal{H}_{d/2}([0, 1] \times E)$ is equivalent to the existence of $\mu \in \mathcal{P}([0, 1] \times E)$ and

a constant c such that the μ -measure of all balls [in $\mathbf{R}^3$] of radius $r > 0$ is at most $cr^{d/2}$. Define $\bar{\mu}(C) := \mu([0, 1] \times C)$ for all Borel sets $C \subseteq \mathbf{R}^2$. Evidently, $\bar{\mu} \in \mathcal{P}(E)$, and a covering argument, together with the Frostman property of μ , imply that $\bar{\mu}$ of all two-dimensional balls of radius $r > 0$ is at most $cr^{(d/2)-1}$. Another application of the Frostman lemma finishes the proof. $\square$

6. Proof of Theorem 1.2

Define for all $s > 0$, every $\omega \in \Omega$, and all Borel sets $I \subseteq \mathbf{R}_+$,

$$T_d^I(s)(\omega) := \{t \in I : B(s, t)(\omega) = 0\}. \quad (6.1)$$

Equivalently, $T_d^I(s) = B^{-1}\{0\} \cap (\{s\} \times (0, \infty)) \cap I$. It suffices to prove that for all closed intervals $I \subset (0, \infty)$,

$$\dim_{\mathbf{H}} T_d^I(s) = 0 \quad \text{for all } s > 0 \quad \text{a.s.} \quad (6.2)$$

[N.B.: The order of the quantifiers!]. This, in turn, proves that

$$\dim_{\mathbf{H}} T_d^{\mathbf{R}^+}(s) = \sup_I \dim_{\mathbf{H}} T_d^I(s) = 0 \quad \text{for all } s > 0, \quad (6.3)$$

where the supremum is taken over all closed intervals $I \subset (0, \infty)$ with rational end-points. Theorem 1.2 follows suit. Without loss of much generality, we prove (1.8) for $I := [1, 2]$; the more general case follows from this after a change of notation. To simplify the exposition, we write

$$T_d(s) := T_d^{[1, 2]}(s). \quad (6.4)$$

Consider the following events:

$$\mathbf{G}_k(n) := \left\{ \sup_{\substack{1 \leq s, t \leq 2 \\ s \leq u \leq s+(1/k) \\ t \leq v \leq t+(1/k)}} |B(u, v) - B(s, t)| \leq n \left(\frac{\log k}{k} \right)^{1/2} \right\}, \quad (6.5)$$

where $k, n \geq 3$ are integers. We will use the following folklore lemma. A generalization is spelled out explicitly in Lacey [19, Eq. (3.8)].

Lemma 6.1. *For all $\gamma > 0$ there exists $n_0 = n_0(\gamma)$ such that for all $n, k \geq n_0$,*

$$\mathbf{P}(\mathbf{G}_k(n)) \geq 1 - n_0 k^{-\gamma}. \quad (6.6)$$

Next we mention a second folklore result.

Lemma 6.2. *Let $\{W(t)\}_{t \geq 0}$ denote a standard Brownian motion in $\mathbf{R}^d$. Then, there exists a constant c such that for all integers $m \geq 1$ and $1 \leq r_1 \leq r_2 \leq \dots \leq r_m \leq 2$,*

$$\mathbf{P} \left\{ \max_{1 \leq i \leq m} |W(r_i)| \leq \epsilon \right\} \leq c \epsilon^d \prod_{2 \leq i \leq m} \left(\frac{\epsilon}{(r_i - r_{i-1})^{1/2}} \wedge 1 \right)^d. \quad (6.7)$$

Proof. If $|W(r_i)| \leq \epsilon$ for all $i \leq m$ then $|W(r_1)| \leq \epsilon$, and $|W(r_i) - W(r_{i-1})| \leq 2\epsilon$ for all $2 \leq i \leq m$. Therefore,

$$\mathbb{P} \left\{ \max_{1 \leq i \leq m} |W(r_i)| \leq \epsilon \right\} \leq \mathbb{P} \{|W(r_1)| \leq \epsilon\} \prod_{2 \leq i \leq m} \mathbb{P} \{|W(r_i - r_{i-1})| \leq 2\epsilon\}. \quad (6.8)$$

A direct computation yields the lemma from this. $\square$

Now define

$$I_{i,j}(k) := \left[1 + \frac{i}{k}, 1 + \frac{(i+1)}{k} \right] \times \left[1 + \frac{j}{k}, 1 + \frac{(j+1)}{k} \right], \quad (6.9)$$

where i and j can each run through $\{0, \dots, k-1\}$, and $k \geq 1$ is an integer. We say that $I_{i,j}(k)$ is *good* if $I_{i,j}(k) \cap B^{-1}\{0\} \neq \emptyset$. With this in mind, we define

$$N_{i,k} := \sum_{0 \leq j \leq k-1} \mathbf{1}_{\{I_{i,j}(k) \text{ is good}\}} \quad (6.10)$$

Lemma 6.3. *Suppose $d \in \{2, 3\}$. Then, for all $\gamma > 0$ there exists $\alpha = \alpha(d, \gamma) > 1$ large enough that*

$$\max_{0 \leq i \leq k-1} \mathbb{P} \left\{ N_{i,k} \geq \alpha (\log k)^{(8-d)/2} \right\} = O(k^{-\gamma}), \quad (6.11)$$

as k tends to infinity.

Proof. On $\mathbf{G}_k(n)$ we have the set-wise inclusion,

$$\{I_{i,j}(k) \text{ is good}\} \subseteq \left\{ \left| B \left(1 + \frac{i}{k}, 1 + \frac{j}{k} \right) \right| \leq n \left(\frac{\log k}{k} \right)^{1/2} \right\}. \quad (6.12)$$

Therefore, for all integer $p \geq 1$,

$$\begin{aligned} & \mathbb{E} \left[N_{i,k}^p ; \mathbf{G}_k(n) \right] \\ & \leq \sum_{0 \leq j_1, \dots, j_p \leq k-1} \mathbb{P} \left\{ \max_{1 \leq \ell \leq p} \left| B \left(1 + \frac{i}{k}, 1 + \frac{j_\ell}{k} \right) \right| \leq n \left(\frac{\log k}{k} \right)^{1/2} \right\} \\ & = \sum_{0 \leq j_1, \dots, j_p \leq k-1} \mathbb{P} \left\{ \max_{1 \leq \ell \leq p} \left| \left(1 + \frac{i}{k} \right)^{1/2} W \left(1 + \frac{j_\ell}{k} \right) \right| \leq n \left(\frac{\log k}{k} \right)^{1/2} \right\} \\ & \leq p! \sum_{0 \leq j_1 \leq \dots \leq j_p \leq k-1} \mathbb{P} \left\{ \max_{1 \leq \ell \leq p} \left| W \left(1 + \frac{j_\ell}{k} \right) \right| \leq n \left(\frac{\log k}{k} \right)^{1/2} \right\}, \end{aligned} \quad (6.13)$$

where W denotes a standard d -dimensional Brownian motion. Because the latter quantity does not depend on the value of i , Lemma 6.2 shows that

$$\begin{aligned} & \max_{0 \leq i \leq k-1} \mathbb{E} \left[N_{i,k}^p ; \mathbf{G}_k(n) \right] \\ & \leq cp! n^{pd} \left(\frac{\log k}{k} \right)^{d/2} \sum_{0 \leq j_1 \leq \dots \leq j_p \leq k-1} \prod_{2 \leq \ell \leq p} \left(\frac{\log k}{j_\ell - j_{\ell-1}} \right)^{d/2}, \end{aligned} \quad (6.14)$$

for all k large, where we are interpreting $1/0$ as one.

Now first consider the case $d = 3$. We recall our (somewhat unusual) convention about $1/0$, and note that

$$\sum_{0 \leq j_1 \leq \dots \leq j_p \leq k-1} \prod_{2 \leq \ell \leq p} (j_\ell - j_{\ell-1})^{-3/2} \leq k \left(\sum_{l \geq 0} \frac{1}{l^{3/2}} \right)^{p-1}. \quad (6.15)$$

Therefore, when $d = 3$ we can find a constant c_1 —independent of (p, k) —such that

$$\max_{0 \leq i \leq k-1} \mathbb{E} \left[N_{i,k}^p ; \mathbf{G}_k(n) \right] \leq p! \frac{(c_1 \log k)^{3p/2}}{k^{1/2}} \leq p! (c_1 \log k)^{3p/2}. \quad (6.16)$$

By enlarging c_1 , if need be, we find that this inequality is valid for all $k \geq 1$. This proves readily that

$$\max_{0 \leq i \leq k-1} \mathbb{E} \left[\exp \left(\frac{N_{i,k}}{2(c_1 \log k)^{3/2}} \right) ; \mathbf{G}_k(n) \right] \leq \sum_{p \geq 0} 2^{-p} = 2. \quad (6.17)$$

Therefore, Chebyshev's inequality implies that for all $i, k, p \geq 1$ and $a > 0$,

$$\max_{0 \leq i \leq k-1} \mathbb{P} \left\{ N_{i,k} \geq 2\gamma c_1^{3/2} (\log k)^{5/2} ; \mathbf{G}_k(n) \right\} \leq 2k^{-\gamma}. \quad (6.18)$$

Note that c_1 may depend on n . But we can choose n large enough—once and for all—such that the probability of the complement of $\mathbf{G}_k(n)$ is at most $nk^{-\gamma}$ (Lemma 6.1). This proves the lemma in the case that $d = 3$.

The case $d = 2$ is proved similarly, except (6.15) is replaced by

$$\sum_{0 \leq j_1 \leq \dots \leq j_p \leq k-1} \prod_{2 \leq \ell \leq p} (j_\ell - j_{\ell-1})^{-1} \leq k \left(\sum_{0 \leq l \leq k} \frac{1}{l} \right)^{p-1} \leq k(c_2 \log k)^{p-1}, \quad (6.19)$$

where c_2 does not depend on (k, p) , and [as before] $1/0 := 1$. Equation (6.16), when $d = 2$, becomes:

$$\max_{0 \leq i \leq k-1} \mathbb{E} \left[N_{i,k}^p ; \mathbf{G}_k(n) \right] \leq p! (c_2 \log k)^p. \quad (6.20)$$

This forms the $d = 2$ version of (6.17):

$$\max_{0 \leq i \leq k-1} \mathbb{E} \left[\exp \left(\frac{N_{i,k}}{2c_2 \log k} \right) ; \mathbf{G}_k(n) \right] \leq 2. \quad (6.21)$$

Thus, (6.18), when $d = 2$, becomes

$$\max_{0 \leq i \leq k-1} \mathbb{P} \left\{ N_{i,k} \geq 2\gamma c_2 (\log k)^2 ; \mathbf{G}_k(n) \right\} \leq 2k^{-\gamma}. \quad (6.22)$$

The result follows from this and Lemma 6.1 after we choose and fix a sufficiently large n . $\square$

Estimating $N_{i,k}$ is now a simple matter, as the following shows.

Lemma 6.4. *If $d \in \{2, 3\}$, then with probability one,*

$$\max_{0 \leq i \leq k-1} N_{i,k} = O\left((\log k)^{(8-d)/2}\right) \quad (k \rightarrow \infty). \quad (6.23)$$

Proof. By Lemma 6.3, there exists $\alpha > 0$ so large that for all $k \geq 1$ and $0 \leq i \leq k-1$, $P\{N_{i,k} \geq \alpha(\log k)^{(8-d)/2}\} \leq \alpha k^{-3}$. Consequently,

$$P\left\{\max_{0 \leq i \leq k-1} N_{i,k} \geq \alpha(\log k)^{(8-d)/2}\right\} \leq \alpha k^{-2}. \quad (6.24)$$

The lemma follows from this and the Borel–Cantelli lemma. $\square$

We are ready to prove Theorem 1.2. As was mentioned earlier, it suffices to prove (6.2), and this follows from our next result.

Proposition 6.5. *Fix $d \in \{2, 3\}$ and define the measure-function*

$$\Phi(x) := [\log_+(1/x)]^{-(8-d)/2}. \quad (6.25)$$

Then, $\sup_{1 \leq s \leq 2} \mathcal{H}_\Phi(T_d(s)) < \infty$ a.s.

The reason is provided by the following elementary lemma whose proof is omitted.

Lemma 6.6. *Suppose φ is a measure function such that $\liminf_{x \downarrow 0} x^{-\alpha} \varphi(x) = \infty$ for some $\alpha > 0$. Then, for all Borel sets $A \subset \mathbf{R}^n$,*

$$\mathcal{H}_\varphi(A) < \infty \implies \mathcal{H}_\alpha(A) < \infty \implies \dim_{\mathbf{H}} A \leq \alpha. \quad (6.26)$$

Now we prove Proposition 6.5.

Proof of Proposition 6.5. We can construct a generous cover of $T_d(s)$ as follows: For all irrational $s \in [i/k, (i+1)/k]$, we cover $T_d(s)$ intervals of the form

$$\left[1 + \frac{j}{k}, 1 + \frac{(j+1)}{k}\right], \quad (6.27)$$

where j can be any integer in $\{0, \dots, k-1\}$ as long as $I_{i,j}(k)$ is good. Therefore, for any measure-function φ ,

$$\sup_{\substack{1 \leq s \leq 2: \\ s \text{ is irrational}}} \mathcal{H}_\varphi^{(1/k)}(T_d(s)) \leq \varphi(1/k) \max_{0 \leq i \leq k-1} N_{i,k}. \quad (6.28)$$

Now we choose the measure-function $\varphi(x) := \Phi(x)$ and let $k \rightarrow \infty$ to find that $\mathcal{H}_\Phi(T_d(s))$ is finite, uniformly over all irrational $s \in [1, 2]$. The case of rational s 's is simpler to analyse. Indeed, $T_d(s) = \emptyset$ a.s. for all rational $s \in [1, 2]$. This is because d -dimensional Brownian motion ($d \in \{2, 3\}$) does not hit zero. $\square$

Remark 6.7. The form of Lemma 6.4 changes dramatically when $d = 1$. Indeed, one can adjust the proof of Lemma 6.4 to find that a.s.,

$$\max_{0 \leq i \leq k-1} N_{i,k} = O\left(k^{1/2}(\log k)^{3/2}\right) \quad (k \rightarrow \infty). \quad (6.29)$$

This yields fairly readily that the upper Minkowski dimension [written as $\dim_{\mathbf{M}}$] of $T_1(s)$ is at most $1/2$ simultaneously for all $s > 0$. Let $\dim_{\mathbf{p}}$ denote the packing dimension, and recall (B.7). Then, the preceding and the theorem of Penrose [26] together prove that almost surely,

$$\dim_{\mathbf{H}} T_1(s) = \dim_{\mathbf{p}} T_1(s) = \dim_{\mathbf{M}} T_1(s) = \frac{1}{2} \quad \text{for all } s > 0. \quad (6.30)$$

7. On Rates of Escape

Throughout this section, we choose and fix a non-decreasing and measurable function $\psi : (0, \infty) \rightarrow (0, \infty)$ such that $\lim_{t \rightarrow \infty} \psi(t) = \infty$. Define, for all Borel-measurable sets $F \subset \mathbf{R}$,

$$\Upsilon_F(\psi) := \int_1^\infty \left[\frac{K_F(1/\psi(x))}{(\psi(x))^{(d-2)/2}} \wedge 1 \right] \frac{dx}{x}. \quad (7.1)$$

Theorem 7.1. *If $d \geq 3$, then for all non-random, compact sets $F \subset (0, \infty)$, the following holds with probability one:*

$$\liminf_{t \rightarrow \infty} \inf_{s \in F} \left(\frac{\psi(t)}{t} \right)^{1/2} |B(s, t)| = \begin{cases} 0 & \text{if } \Upsilon_F(\psi) = \infty, \\ \infty & \text{otherwise.} \end{cases} \quad (7.2)$$

Remark 7.2. Although the infimum over all $s \in E$ is generally an uncountable one, measurability issues do not arise. Our proof actually shows that the event in (7.2) is a subset of a null set. Thus, we are assuming tacitly that the underlying probability space is complete. This convention applies to the next theorem as well.

Definition 7.3. Let $F \subset (0, \infty)$ be non-random and compact, and $\psi : (0, \infty) \rightarrow (0, \infty)$ measurable and non-decreasing. Then we say that $(F, \psi) \in \text{FIN}_{\text{loc}}$ if there exists a denumerable decomposition $F = \cup_{n \geq 1} F_n$ of F in terms of closed intervals $F_1, F_2, \dots$ —all with rational end-points—such that $\Upsilon_{F_n}(\psi) < \infty$ for all $n \geq 1$.

This brings us to the main theorem of this section. Its proof is a little delicate because we have to get three different estimates, each of which is valid only on a certain scale. This proof is motivated by the earlier work of the author with David Levin and Pedro Méndez [17].

Theorem 7.4. *If $d \geq 3$, then for all non-random, compact sets $F \subset (0, \infty)$, the following holds with probability one:*

$$\inf_{s \in F} \liminf_{t \rightarrow \infty} \left(\frac{\psi(t)}{t} \right)^{1/2} |B(s, t)| = \begin{cases} 0 & \text{if } (F, \psi) \notin \text{FIN}_{\text{loc}}, \\ \infty & \text{otherwise.} \end{cases} \quad (7.3)$$

The key estimate, implicitly referred to earlier, is the following.

Theorem 7.5. *If $d \geq 3$ then there exists a constant c such that for all non-random compact sets $F \subseteq [1, 2]$ and $0 < \epsilon < 1$,*

$$\frac{1}{c} [\epsilon^{d-2} K_F(\epsilon^2) \wedge 1] \leq \mathbb{P} \left\{ \inf_{s \in F} \inf_{1 \leq t \leq 2} |B(s, t)| \leq \epsilon \right\} \leq c [\epsilon^{d-2} K_F(\epsilon^2) \wedge 1]. \quad (7.4)$$

Let us mention also the the next result without proof; it follows upon combining Theorems 4.1 and 4.2 of our collaborative effort with Robert Dalang [4], together with Brownian scaling:

Lemma 7.6. *If $d \geq 3$, then there exists c such that for all $1 \leq a < b \leq 2$, $0 < \epsilon < 1$, and $n \geq 1$ such that $(b - a) \geq c\epsilon^2$,*

$$\frac{1}{c} (b - a)^{(d-2)/2} \leq \mathbb{P} \left\{ \inf_{\substack{a \leq s \leq b \\ 1 \leq t \leq 2}} |B(s, t)| \leq \epsilon \right\} \leq c (b - a)^{(d-2)/2}. \quad (7.5)$$

Remark 7.7. Dalang and Khoshnevisan [4] state this explicitly for $d \in \{3, 4\}$. However, the key estimates are their Lemmas 2.1 and 2.6, and they require only that $d > 2$.

Proof of Theorem 7.5 (The Upper Bound). Fix $n \geq 1$. Define $I_j := [j/n, (j + 1)/n)$, and let $\chi_j = 1$ if $I_j \cap F \neq \emptyset$ and $\chi_j = 0$ otherwise. Then in accord with Lemma 7.6,

$$\begin{aligned} & \mathbb{P} \left\{ \inf_{s \in F} \inf_{1 \leq t \leq 2} |B(s, t)| \leq \frac{1}{(cn)^{1/2}} \right\} \\ & \leq \sum_{n \leq j \leq 2n-1} \mathbb{P} \left\{ \inf_{s \in I_j} \inf_{1 \leq t \leq 2} |B(s, t)| \leq \frac{1}{(cn)^{1/2}} \right\} \chi_j \\ & \leq cn^{-(d-2)/2} M_n(F). \end{aligned} \quad (7.6)$$

This, in turn, is bounded above by $cn^{-(d-2)/2} K_F(1/n)$; see (B.2). The lemma follows in the case that $\epsilon = (cn)^{-1/2}$. The general case follows from a *monotonicity argument*, which we rehash (once) for the sake of completeness.

Suppose $(c(n+1))^{-1/2} \leq \epsilon \leq (cn)^{-1/2}$. Then,

$$\begin{aligned} \mathbb{P} \left\{ \inf_{s \in F} \inf_{1 \leq t \leq 2} |B(s, t)| \leq \epsilon \right\} & \leq \mathbb{P} \left\{ \inf_{s \in F} \inf_{1 \leq t \leq 2} |B(s, t)| \leq \frac{1}{(cn)^{1/2}} \right\} \\ & \leq cn^{-(d-2)/2} K_F(1/n) \\ & \leq c\epsilon^{d-2} K_F(c\epsilon^2). \end{aligned} \quad (7.7)$$

Equation (B.3) implies that $K_F(c\epsilon^2) = O(K_F(\epsilon^2))$ as $\epsilon \rightarrow 0$, and finishes our proof of the upper bound. $\square$

Before proving the lower bound, let us mention a worthwhile heuristic argument. If, in Lemma 7.6, the condition “ $(b - a) \geq c\epsilon^2$ ” is replaced by $(b - a) \ll \epsilon^2$, then the bounds both change to ϵ^{d-2} . This is the probability that a single Brownian motion hits $\mathcal{B}(0; \epsilon)$ some time during $[1, 2]$; compare with Lemma C.1. This suggests that the “correlation length” among the slices is of order ϵ^2 . That is, slices that are within ϵ^2 of one another behave much the same; those that are further apart than ϵ^2 are nearly independent. We use our next result in order to actually prove the latter heuristic.

Proposition 7.8. *If $d \geq 3$ then there exists a constant c such that for all $1 \leq s, u \leq 2$ and $0 < \epsilon < 1$, if $|u - s| \geq \epsilon^2$ then*

$$\mathbb{P} \left\{ \inf_{1 \leq t \leq 2} |B(s, t)| \leq \epsilon, \inf_{1 \leq v \leq 2} |B(u, v)| \leq \epsilon \right\} \leq c\epsilon^{d-2}|u - s|^{(d-2)/2}. \quad (7.8)$$

Proof. Without loss of generality we may choose and fix $2 \geq u > s \geq 1$. Now the processes $\{B(s, t)\}_{t \geq 0}$ and $\{B(u, v)\}_{v \geq 0}$ can be decomposed as follows:

$$B(s, t) = s^{1/2}Z(t), \quad B(u, v) = s^{1/2}Z(v) + (u - s)^{1/2}W(v), \quad (7.9)$$

where W and Z are independent d -dimensional Brownian motions. Thus, we are interested in estimating the quantity p_ϵ , where

$$\begin{aligned} p_\epsilon &:= \mathbb{P} \left\{ \inf_{1 \leq t \leq 2} |Z(t)| \leq \frac{\epsilon}{s^{1/2}}, \inf_{1 \leq v \leq 2} \left| Z(v) + \left(\frac{u - s}{s} \right)^{1/2} W(v) \right| \leq \frac{\epsilon}{s^{1/2}} \right\} \\ &\leq \mathbb{P} \left\{ \inf_{1 \leq t \leq 2} |Z(t)| \leq \epsilon, \inf_{1 \leq v \leq 2} |Z(v) + (u - s)^{1/2}W(v)| \leq \epsilon \right\}. \end{aligned} \quad (7.10)$$

The proposition follows from Lemma C.2 in Appendix C below. $\square$

Proof of Theorem 7.5 (The Lower Bound). We make a discretization argument, once more. Let $n := K_F(\epsilon^2)$ and find maximal Kolmogorov points $s_1 < \dots < s_n$ —all in F —such that $s_{i+1} - s_i \geq \epsilon^2$ for all $1 \leq i < n$. Define

$$J_\epsilon(n) := \sum_{1 \leq i \leq n} \mathbf{1}_{\{|B(s_i, t)| \leq \epsilon \text{ for some } t \in [1, 2]\}}. \quad (7.11)$$

According to Lemma C.1,

$$\frac{1}{c}n\epsilon^{d-2} \leq \mathbb{E}[J_\epsilon(n)] \leq cn\epsilon^{d-2}. \quad (7.12)$$

On the other hand, the condition $|s_j - s_i| \geq \epsilon^2$ and Proposition 7.8 together ensure that

$$\mathbb{E} \left[(J_\epsilon(n))^2 \right] \leq \mathbb{E}[J_\epsilon(n)] + c(\mathbb{E}[J_\epsilon(n)])^2. \quad (7.13)$$

Now to prove the lower bound we first assume that $n\epsilon^{d-2} \leq 1$. The previous display implies then that $\mathbb{E}[(J_\epsilon(n))^2] \leq c\mathbb{E}[J_\epsilon(n)]$. Combine this inequality with

(7.12) and the Paley–Zygmund inequality to find that

$$\mathbb{P} \left\{ \inf_{s \in F} \inf_{1 \leq t \leq 2} |B(s, t)| \leq \epsilon \right\} \geq \mathbb{P} \{J_\epsilon(n) > 0\} \geq \frac{(\mathbb{E}[J_\epsilon(n)])^2}{\mathbb{E}[(J_\epsilon(n))^2]} \geq cn\epsilon^{d-2}. \quad (7.14)$$

On the other hand, if $n\epsilon^{d-2} \geq 1$, then the left-hand side is bounded away from zero, by a similar bound. This is the desired result. $\square$

Lemma 7.9. *Let $d \geq 3$, and $f : [1, 2] \rightarrow \mathbf{R}^d$ be a fixed, non-random, measurable function. Then there exists a constant c such that for all integers $1 \leq k \leq n$*

$$\begin{aligned} & \mathbb{P} \left\{ \inf_{\substack{1 \leq s \leq k/n \\ 1 \leq t \leq 2}} |B(s, t) - f(s)| \leq \frac{1}{n^{1/2}} \right\} \\ & \leq c \left(kn^{-(d-2)/2} + \sum_{n \leq i \leq n+k-1} (\Omega_{i,n}(f))^{d-2} \right), \end{aligned} \quad (7.15)$$

where for all continuous functions h ,

$$\Omega_{i,n}(h) := \sup_{i/n \leq t \leq (i+1)/n} |h(t) - h(i/n)|. \quad (7.16)$$

Proof. Lemma 7.9 holds for similar reasons as does Proposition 7.8, but is simpler to prove. Indeed, the probability in question is at most

$$\sum_{n \leq i \leq n+k-1} \mathbb{P} \left\{ \inf_{i/n \leq s \leq (i+1)/n} |B(s, t) - f(s)| \leq \frac{1}{n^{1/2}} \right\}. \quad (7.17)$$

This, in turn, is less than or equal to

$$\sum_{n \leq i \leq n+k-1} \mathbb{P} \left\{ \inf_{1 \leq t \leq 2} |B(\frac{i}{n}, t)| \leq \frac{1}{n^{1/2}} + \sup_{1 \leq t \leq 2} \Omega_{i,n}(B(\bullet, t)) + \Omega_{i,n}(f) \right\}. \quad (7.18)$$

By the Markov property, $B((i/n), \bullet)$ is a d -dimensional Brownian motion that is independent of $\sup_{1 \leq t \leq 2} \Omega_{i,n}(B(\bullet, t))$. Standard modulus-of-continuity bounds show that the $L^{d-2}(\mathbb{P})$ -norm of $\sup_{1 \leq t \leq 2} \Omega_{i,n}(B(\bullet, t))$ is at most a constant times $n^{-(d-2)/2}$; the details will be explained momentarily. Since $(i/n) \geq 1$, these observations, in conjunction with Lemma C.1 [Appendix C] imply the lemma. It remains to prove that there exists a c such that for all $n \geq 1$,

$$\max_{n \leq i \leq 2n} \mathbb{E} \left[\sup_{1 \leq t \leq 2} (\Omega_{i,n}(B(\bullet, t)))^{d-2} \right] \leq cn^{-(d-2)/2}. \quad (7.19)$$

Choose and fix $n \geq 1$, $n \leq i \leq 2n$, and $v \in [i/n, (i+1)/n]$. Then the process $t \mapsto B(v, t) - B(i/n, t)$ is manifestly a martingale with respect to the filtration generated by the infinite-dimensional process $t \mapsto B(\bullet, t)$. Consequently, $T \mapsto \sup_{1 \leq t \leq T} (\Omega_{i,n}(B(\bullet, t)))^{d-2}$ is a sub-martingale, and (7.19) follows from Doob's inequality and Brownian-sheet scaling. This completes our proof. $\square$

Lemma 7.9, together with a monotonicity argument, implies the following.

Lemma 7.10. *Let $d \geq 3$, and $f : [1, 2] \rightarrow \mathbf{R}^d$ be a fixed, non-random, measurable function. Then there exists a constant c such that for all $1 \leq a \leq 2$ and $0 < \epsilon < 1$,*

$$\begin{aligned} & \mathbb{P} \left\{ \inf_{a \leq s \leq a+\epsilon^2} \inf_{1 \leq t \leq 3} |B(s, t) - f(s)| \leq \epsilon \right\} \\ & \leq c \left(\epsilon^{d-2} + \sup_{a \leq u \leq a+\epsilon^2} |f(u) - f(a)|^{d-2} \right), \end{aligned} \quad (7.20)$$

Proof of Theorem 7.1. First, assume that $\Upsilon(\psi) < \infty$; this is the *first half*.

Define for all $n = 0, 1, 2, \dots$,

$$\begin{aligned} \psi_n &:= \psi(2^n), \\ \mathbf{A}_n &:= \left\{ \inf_{s \in F} \inf_{2^n \leq t \leq 2^{n+1}} |B(s, t)| \leq (2^n / \psi_n)^{1/2} \right\}. \end{aligned} \quad (7.21)$$

We combine Theorem 7.5 with the Brownian-sheet scaling to deduce the following:

$$\frac{1}{c} \left[\psi_n^{-(d-2)/2} K_F(1/\psi_n) \wedge 1 \right] \leq \mathbb{P}(\mathbf{A}_n) \leq c \left[\psi_n^{-(d-2)/2} K_F(1/\psi_n) \wedge 1 \right]. \quad (7.22)$$

After doing some algebra we find that because $\Upsilon_F(\psi)$ is finite, then so is the quantity $\sum_{n \geq 1} \mathbb{P}(\mathbf{A}_n)$. By the Borel–Cantelli lemma,

$$\liminf_{n \rightarrow \infty} \left(\frac{\psi_n}{2^n} \right)^{1/2} \inf_{s \in F} \inf_{2^n \leq t \leq 2^{n+1}} |B(s, t)| \geq 1 \quad \text{a.s.} \quad (7.23)$$

If $2^n \leq t \leq 2^{n+1}$ then $(\psi_n/2^n)^{1/2} \leq (2\psi(t)/t)^{1/2}$. It follows that almost surely,

$$\liminf_{t \rightarrow \infty} \left(\frac{\psi(t)}{t} \right)^{1/2} \inf_{s \in F} |B(s, t)| \geq \frac{1}{2^{1/2}}. \quad (7.24)$$

But if $\Upsilon_F(\psi)$ is finite then so is $\Upsilon_F(r\psi)$, for any $r > 0$; see (B.3). Therefore, we can apply the preceding to $r\psi$ in place of ψ , and then let $r \rightarrow 0$ to find that

$$\Upsilon_F(\psi) < \infty \implies \liminf_{t \rightarrow \infty} \left(\frac{\psi(t)}{t} \right)^{1/2} \inf_{s \in F} |B(s, t)| = \infty \quad \text{a.s.} \quad (7.25)$$

This concludes the proof of the first half.

For the *second half* we assume that $\Upsilon_F(\psi) = \infty$. The preceding analysis proves that $\sum_{n \geq 1} \mathbb{P}(\mathbf{A}_n) = \infty$. According to the Borel–Cantelli lemma, it suffices to prove that

$$\limsup_{N \rightarrow \infty} \frac{\sum_{1 \leq n < m \leq N} \mathbb{P}(\mathbf{A}_n \cap \mathbf{A}_m)}{\left(\sum_{1 \leq n \leq N} \mathbb{P}(\mathbf{A}_n) \right)^2} < \infty. \quad (7.26)$$

Define for all integers $n \geq 1$, and all $s, t \geq 0$,

$$\begin{aligned} \mathcal{A}_n &:= \text{the } \sigma\text{-algebra generated by } \{B(\bullet, v)\}_{0 \leq v \leq 2^n}, \\ \Delta_n(s, t) &:= B(s, t + 2^n) - B(s, 2^n). \end{aligned} \quad (7.27)$$

The Markov properties of the Brownian sheet imply that whenever $m > n \geq 1$:

(i) Δ_m is a Brownian sheet that is independent of $\mathcal{A}_n$; and (ii) $\mathbf{A}_n \in \mathcal{A}_n$. Thus,

we apply these properties in conjunction with Brownian-sheet scaling to find that a.s., $P(\mathbf{A}_m | \mathcal{A}_n)$ is equal to

$$\begin{aligned} & P \left(\inf_{s \in F} \inf_{2^m - 2^n \leq t \leq 2^{m+1} - 2^n} |\Delta_n(s, t) - B(s, 2^n)| \leq \left(\frac{2^m}{\psi_m} \right)^{1/2} \middle| \mathcal{A}_n \right) \\ &= P \left(\inf_{1 \leq t \leq (2^{m+1} - 2^n)/\alpha} \left| \Delta_n(s, t) - \frac{B(s, 2^n)}{\alpha^{1/2}} \right| \leq \left(\frac{2^m}{\alpha \psi_m} \right)^{1/2} \middle| \mathcal{A}_n \right), \end{aligned} \quad (7.28)$$

where $\alpha := 2^m - 2^n$. Because $m \geq n + 1$, $(2^{m+1} - 2^n)/\alpha \leq 3$ and $2^m/\alpha \leq 2$. Therefore, almost surely,

$$P(\mathbf{A}_m | \mathcal{A}_n) \leq P \left(\inf_{s \in F} \inf_{1 \leq t \leq 3} \left| \Delta_n(s, t) - \frac{B(s, 2^n)}{\alpha^{1/2}} \right| \leq \left(\frac{2}{\psi_m} \right)^{1/2} \middle| \mathcal{A}_n \right). \quad (7.29)$$

We can cover E with at most $K := M_{[2/\psi_m]}(F)$ intervals of the form $I_i := [i/\ell, (i+1)/\ell]$, where $\ell := [\psi_m/2]$. Having done this, a simple bound, together with Lemma 7.10 yield the following: With probability one, $P(\mathbf{A}_m | \mathcal{A}_n)$ is bounded above by

$$\begin{aligned} & \sum_{1 \leq i \leq K} P \left(\inf_{s \in I_i} \inf_{1 \leq t \leq 3} \left| \Delta_n(s, t) - \frac{B(s, 2^n)}{\alpha^{1/2}} \right| \leq \left(\frac{2}{\psi_m} \right)^{1/2} \middle| \mathcal{A}_n \right) \\ & \leq cK \left(\psi_m^{-(d-2)/2} + \Omega \right), \end{aligned} \quad (7.30)$$

where

$$\begin{aligned} \Omega &:= \alpha^{-(d-2)/2} \max_{1 \leq i \leq K} E \left[\sup_{s \in I_i} |B(s, 2^n) - B(i/\ell, 2^n)|^{d-2} \right] \\ &= \alpha^{-(d-2)/2} 2^{n(d-2)/2} E \left[\sup_{0 \leq s \leq 1/\ell} |B(s, 1)|^{d-2} \right] \\ &= c\alpha^{-(d-2)/2} 2^{n(d-2)/2} \ell^{-(d-2)/2}. \end{aligned} \quad (7.31)$$

Therefore, the bound $2^n/\alpha \leq 1$ implies that $\Omega \leq c\ell^{-(d-2)/2} \leq c\psi_m^{-(d-2)/2}$. On the other hand, by (B.2) and (B.3), $K \leq K_F(1/\psi_m)$. Therefore, the preceding paragraph and (7.22) together imply that $P(\mathbf{A}_m | \mathcal{A}_n) \leq cP(\mathbf{A}_m)$ a.s., where c does not depend on (n, m, ω) . Therefrom, we conclude that $P(\mathbf{A}_m | \mathbf{A}_n) \leq cP(\mathbf{A}_m)$, whence (7.26). $\square$

We are ready to prove Theorem 7.4.

Proof of Theorem 7.4. Suppose, first, that $(F, \psi) \in \text{FIN}_{loc}$. According to Theorem 7.1, we can write $F = \cup_{n \geq 1} F_n$ a.s., where the F_n 's are closed intervals with rational end-points, such that

$$\inf_{s \in F_n} \liminf_{t \rightarrow \infty} \left(\frac{\psi(t)}{t} \right)^{1/2} |B(s, t)| = \infty \quad \text{for all } n \geq 1. \quad (7.32)$$

This proves that a.s.,

$$\inf_{s \in F} \liminf_{t \rightarrow \infty} \left(\frac{\psi(t)}{t} \right)^{1/2} |B(s, t)| = \infty, \quad (7.33)$$

and this is half of the assertion of the theorem.

Conversely, suppose $(F, \psi) \notin \text{FIN}_{loc}$. Then, given any decomposition $F = \cup_{n \geq 1} F_n$ in terms of closed, rational intervals $F_1, F_2, \dots$,

$$\liminf_{t \rightarrow \infty} \inf_{s \in F_n} \left(\frac{\psi(t)}{t} \right)^{1/2} |B(s, t)| = 0 \quad \text{for all } n \geq 1. \quad (7.34)$$

Define for all $k, n \geq 1$,

$$O_{k,n} := \left\{ s > 0 : \inf_{t \geq k} \left[\left(\frac{\psi(t)}{t} \right)^{1/2} |B(s, t)| \right] < \frac{1}{n} \right\}. \quad (7.35)$$

Then (7.34) implies that every $O_{k,n}$ is relatively open and everywhere dense in F a.s. By the Baire category theorem, $\cap_{k,n \geq 1} O_{k,n}$ has the same properties, and this proves the theorem. $\square$

With Theorem 7.4 under way, we can finally derive Theorem 1.6 of the Introduction, and conclude this section.

Proof of Theorem 1.6. Throughout, define for all $\alpha > 0$,

$$\psi_\alpha(x) := [\log_+(x)]^{2/\alpha} \quad \text{for all } x > 0. \quad (7.36)$$

Note that for any ψ , as given by Theorem 7.4, and for all $\nu > 0$,

$$\Upsilon_F(\psi) < \infty \quad \text{iff} \quad \int_1^\infty \left[\frac{K_F(1/\psi(x))}{(\psi(x))^{(d-2)/2}} \wedge \nu \right] \frac{dx}{x} < \infty. \quad (7.37)$$

Therefore,

$$\begin{aligned} & \text{if } K_F(\epsilon) = O\left(\epsilon^{-(d-2)/2}\right) \quad (\epsilon \rightarrow 0) \quad \text{then} \\ \Upsilon_F(\psi) < \infty & \iff \int_1^\infty \frac{K_F(1/\psi(x))}{x(\psi(x))^{(d-2)/2}} dx < \infty. \end{aligned} \quad (7.38)$$

Suppose $d \geq 4$. Then $K_F(\epsilon) \leq c\epsilon^{-1}$, and so by (7.38) and a little calculus,

$$\Upsilon_F(\psi_\alpha) < \infty \iff \int_1^\infty \frac{K_F(1/s)}{s^{(d-\alpha)/2}} ds < \infty. \quad (7.39)$$

According to this and (B.7), if $\alpha > d - 2 - 2\dim_M F$ is strictly positive, then $\Upsilon_F(\psi_\alpha) < \infty$. Theorem 7.1 then implies that, in this case,

$$\liminf_{t \rightarrow \infty} \inf_{s \in F} \frac{(\log t)^{1/\alpha}}{t^{1/2}} |B(s, t)| = 0 \quad \text{a.s.} \quad (7.40)$$

Similarly, if $0 < \alpha < d - 2 - 2 \dim_{\mathbf{M}} F$, then

$$\liminf_{t \rightarrow \infty} \inf_{s \in F} \frac{(\log t)^{1/\alpha}}{t^{1/2}} |B(s, t)| = \infty \quad \text{a.s.} \quad (7.41)$$

Write $F = \cup_{n \geq 1} F_n$ and “regularize” to find that:

1. If $\alpha > d - 2 - 2 \dim_{\mathbf{P}} F$ is strictly positive, then

$$\inf_{s \in F} \liminf_{t \rightarrow \infty} \frac{(\log t)^{1/\alpha}}{t^{1/2}} |B(s, t)| = 0 \quad \text{a.s.} \quad (7.42)$$

2. If $0 < \alpha < d - 2 - 2 \dim_{\mathbf{P}} F$ then

$$\inf_{s \in F} \liminf_{t \rightarrow \infty} \frac{(\log t)^{1/\alpha}}{t^{1/2}} |B(s, t)| = \infty \quad \text{a.s.} \quad (7.43)$$

The theorem follows in the case that $d \geq 4$.

When $d = 3$, the condition $\dim_{\mathbf{M}} F < 1/2$ guarantees that $K_F(\epsilon) = O(\epsilon^{-1/2})$. Now follow through the proof of the case $d \geq 4$ to finish. $\square$

8. Open Problems

8.1. Slices and Zeros

Theorem 1.2 is a metric statement. Is there a topological counterpart? The following is one way to state this formally.

Open Problem 1. *Suppose $d \in \{2, 3\}$. Is it true that outside a single null set, $B^{-1}\{0\} \cap (\{s\} \times (0, \infty))$ is a finite set for all $s > 0$?*

I conjecture that the answer is “no.” In fact, it is even possible that there exists a non-trivial measure function ϕ such that: (i) $\lim_{r \rightarrow 0} \phi(r) = \infty$; and (ii) $\mathcal{H}_\phi$ -measure of $B^{-1}\{0\} \cap (\{s\} \times (0, \infty))$ is positive for some $s > 0$.

8.2. Smallness of Double-Points for Slices

Theorem 5.1 and a codimension argument together imply that with probability one,

$$\begin{aligned} \dim_{\mathbf{H}} \hat{\mathcal{D}}_d &= \left(3 - \frac{d}{2}\right)_+, \quad \text{and} \\ \dim_{\mathbf{H}} \bar{\mathcal{D}}_d &= 2 \wedge \left(3 - \frac{d}{2}\right)_+. \end{aligned} \quad (8.1)$$

This might suggest that, therefore, none of the slices accrue any of the dimension.

Open Problem 2. *Define, for all $s \geq 0$,*

$$\mathcal{Y}_d(s) := \{(t_1, t_2) \in \mathbf{R}_+^2 : B(s, t_1) = B(s, t_2)\}. \quad (8.2)$$

Then is it the case that if $d \in \{4, 5\}$ then, outside a single null-set, $\dim_{\mathbf{H}} \mathcal{Y}_d(s) = 0$ for all $s \geq 0$?

I conjecture that the answer is “yes.” Answering this might rely on studying closely the methods of the literature on “local non-determinism.” See, in particular, Berman [2], Pitt [28], and the recent deep work of Xiao [35]. On the other hand, I believe it should be not too hard to prove that the answer to the corresponding problem for $d \leq 3$ is “no,” due to the existence of continuous intersection local times [27]. [I have not written out a complete proof in the $d \leq 3$ case, mainly because I do not have a proof, or disproof, in the case that $d \in \{4, 5\}$. This is the more interesting case because there are no intersection local times.]

Open Problem 1 has the following analogue for double-points.

Open Problem 3. *Let $d \in \{4, 5\}$. Then is it true that outside a single null set, $\mathcal{Y}_d(s)$ is a finite set for all $s > 0$?*

The answer to this question is likely to be “no.” In fact, as was conjectured for Open Problem 1, here too there might exist slices that have positive $\mathcal{H}_\phi$ -measure in some gauge ϕ . If so, then there are in fact values of s for which $\mathcal{Y}_d(s)$ is uncountable.

8.3. Marstrand’s Theorem for Projections

Marstrand [21] proved that almost every lower-dimensional orthogonal projection of a Borel set A has the same Hausdorff dimension as A . Theorem 1.1 proves that a given projection (say, onto the x -axis) of the zero-set of Brownian sheet has the same “Marstrand property.” I believe that the proof can be adjusted to show that, in fact, any non-random orthogonal projection of $B^{-1}\{0\}$ has the same Hausdorff dimension as $B^{-1}\{0\}$ itself.

Open Problem 4. *Is there a (random) orthogonal projection such that the said projection of $B^{-1}\{0\}$ has a different Hausdorff dimension than $2 - (d/2)$?*

I believe that the answer is “no.” However, I have no proof nor counter-proof. Similar questions can be asked about double-points. I will leave them to the interested reader.

8.4. Non-Linear SPDEs

Consider d independent, two-dimensional white noises, $\dot{B}_1, \dots, \dot{B}_d$, together with the following system of d non-interacting stochastic PDEs with additive noise: For a fixed $T > 0$,

$$\begin{aligned} \frac{\partial^2 u^i}{\partial t \partial x}(t, x) &= \hat{B}_i(t, x) + b^i(u(t, x)), \\ u^i(0, x) &= u_0(x) \quad \text{all } -\infty < x < \infty, \\ \frac{\partial u^i}{\partial t}(0, x) &= u_1(x) \quad \text{all } -\infty < x < \infty, \end{aligned} \tag{8.3}$$

where $1 \leq i \leq N$, and u_0 and u_1 are non-random and smooth, as well as bounded (say). Then, as long as $b := (b^1, \dots, b^d)$ is bounded and Borel-measurable the law of the process $u := (u^1, \dots, u^d)$ is mutually absolutely continuous with respect to

the law of the two-parameter, d -dimensional Brownian sheet B . See Proposition 1.6 of Nualart and Pardoux [24]. Therefore, the theorems of the preceding sections apply to the process u equally well.

Open Problem 5. *Suppose $\sigma : \mathbf{R}^d \rightarrow \mathbf{R}^d$ is a strongly elliptic, bounded, C^∞ function. Is it the case that the results of the previous sections apply to the solution of $(\partial^2 u^i / \partial t \partial x) = b^i(u) + \sigma^i(u) \cdot \hat{B}$ with reasonable boundary conditions?*

There is some evidence that the answer is “yes.” See Dalang and Nualart [7] where a closely-related problem is solved.

Finally, we end with an open-ended question about parabolic SPDEs, about which we know far less at this point. We will state things about the additive linear case only. This case seems to be sufficiently difficult to analyse at this point in time.

Open Problem 6. *Consider the following system of linear parabolic SPDE:*

$$\frac{\partial u^i}{\partial t}(t, x) = \frac{\partial^2 u^i}{\partial x^2}(t, x) + \hat{B}_i(t, x), \quad (8.4)$$

with reasonable boundary conditions. Is there an analysis of the “slices” of u along different values of t that is analogous to the results of the present paper?

Some results along these lines will appear in forthcoming work with Robert Dalang and Eulalia Nualart [5, 6].

Appendix A. Capacity and Dimension

For the sake of completeness, we begin with a brief review of Hausdorff measures. Further information can be found in Kahane [12, Chapter 10], Khoshnevisan [15, Appendices C and D], and Mattila [22, Chapter 4].

A.1. Capacity

Recall that $\mathcal{P}(F)$ denotes the collection of all probability measures on the Borel set F , and $|x|$ is the ℓ^1 -norm of the vector x .

Let $f : \mathbf{R}^n \rightarrow [0, \infty]$ be Borel measurable. Then for all $\mu \in \mathcal{P}(\mathbf{R}^n)$, the f -energy of μ is defined by

$$I_f(\mu) := \iint f(x - y) \mu(dx) \mu(dy). \quad (A.1)$$

If $F \subset \mathbf{R}^n$ is Borel-measurable, then its f -capacity can be defined by

$$\text{Cap}_f(F) := \left[\inf_{\mu \in \mathcal{P}(F)} I_f(\mu) \right]^{-1}, \quad (A.2)$$

where $\inf \emptyset := \infty$ and $1/\infty := 0$.

Let $\beta \in \mathbf{R}$ and $x \in \mathbf{R} \setminus \{0\}$ define

$$U_\beta(x) := \begin{cases} 1, & \text{if } \beta < 0, \\ \log_+(1/|x|), & \text{if } \beta = 0, \\ |x|^{-\beta}, & \text{if } \beta > 0. \end{cases} \quad (\text{A.3})$$

Also, we define U_β at zero by continuously extending U_β to a $[0, \infty]$ -valued function on all of $\mathbf{R}$. Then we write $I_\beta(\mu)$ in place of $I_{U_\beta}(\mu)$, and $\text{Cap}_\beta(F)$ in place of $\text{Cap}_{U_\beta}(F)$; $I_\beta(\mu)$ is the *Riesz* [or Bessel–Riesz] capacity of μ , and Cap_β is [Bessel–] *Riesz capacity* of F .

The following is a central property of capacities [15, p. 523].

Taylor’s Theorem (Taylor [33]). *If $F \subset \mathbf{R}^n$ is compact then $\text{Cap}_n(F) = 0$. Consequently, for all $\beta \geq n$, $\text{Cap}_\beta(F)$ is zero also.*

A.2. Hausdorff Measures

A Borel-measurable function $\varphi : \mathbf{R}_+ \rightarrow [0, \infty]$ is said to be a *measure function* if: (i) φ is non-decreasing near zero; and (ii) $\varphi(2x) = O(\varphi(x))$ as $x \rightarrow 0$. Next, we choose and fix a measure function φ and a Borel set A in $\mathbf{R}^n$. For all $r > 0$ we define

$$\mathcal{H}_\varphi^{(r)}(A) := \inf \left\{ \sum_{j \geq 1} \varphi(\delta_j) : A \subseteq \bigcup_{j \geq 1} \mathcal{B}(x^{(j)}; \delta_j), \sup_{j \geq 1} \delta_j \leq r, x^{(l)} \in \mathbf{R}^n \right\}, \quad (\text{A.4})$$

where $\mathcal{B}(x; r) := \{y \in \mathbf{R}^n : |x - y| \leq r\}$ is the ℓ^1 -ball of radius $r > 0$ about $x \in \mathbf{R}^n$. The *Hausdorff φ -measure* $\mathcal{H}_\varphi(A)$ of A can then be defined as the non-increasing limit,

$$\mathcal{H}_\varphi(A) := \lim_{r \downarrow 0} \mathcal{H}_\varphi^{(r)}(A). \quad (\text{A.5})$$

This defines a Borel [outer-] measure on Borel subsets of $\mathbf{R}^n$.

A.3. Hausdorff Dimension

An important special case of $\mathcal{H}_\varphi$ arises when we consider $\varphi(x) = x^\alpha$. In this case we may write $\mathcal{H}_\alpha$ instead; this is the α -dimensional *Hausdorff measure*. The *Hausdorff dimension* of A is

$$\dim_{\text{H}} A := \sup \{\alpha > 0 : \mathcal{H}_\alpha(A) > 0\} = \inf \{\alpha > 0 : \mathcal{H}_\alpha(A) < \infty\}. \quad (\text{A.6})$$

Hausdorff dimension has the following regularity property: If $A_1, A_2, \dots$ are Borel sets, then

$$\dim_{\text{H}} \left(\bigcup_{i \geq 1} A_i \right) = \sup_{i \geq 1} \dim_{\text{H}} A_i. \quad (\text{A.7})$$

In general, this fails if the union is replaced by an uncountable one. For instance, consider the example $\mathbf{R} = \bigcup_{x \in \mathbf{R}} \{x\}$. The following is another key fact:

Frostman’s Lemma (Frostman [10]). *Let A be a compact subset of $\mathbf{R}^n$. Then $\mathcal{H}_\alpha(A) > 0$ if and only if we can find a constant c and a $\mu \in \mathcal{P}(A)$ such that $\mu(\mathcal{B}(x; r)) \leq cr^\alpha$ for all $r > 0$ and $x \in \mathbf{R}^n$.*

See also Theorem 1 of Kahane [12, p. 130], Theorem 2.1.1 of Khoshnevisan [15, p. 517], and Theorem 8.8 of Mattila [22, p. 112].

Appendix B. Entropy and Packing

The material of this appendix can be found, in expanded form and with a detailed bibliography, in Khoshnevisan et al [17]. Throughout, $F \subset \mathbf{R}$ is a Borel-measurable set.

B.1. Minkowski Content and Kolmogorov Capacitance

There are various ways to describe the size of the set F . We have seen already the role of capacity, Hausdorff measures, and Hausdorff dimension. Alternatively, we can consider the rate of growth of the *Minkowski content* of F ; this is the function $\mathbf{N} \ni n \mapsto M_n(F)$ defined as follows:

$$M_n(F) := \# \left\{ i \in \mathbf{Z} : F \cap \left[\frac{i}{n}, \frac{i+1}{n} \right) \neq \emptyset \right\}. \quad (\text{B.1})$$

Also, we can consider the *Kolmogorov entropy* (known also as “capacitance” or “packing number”) of F ; this is the function $(0, \infty) \ni \epsilon \mapsto K_F(\epsilon)$, where $K_F(\epsilon)$ is equal to the maximum number K for which there exists $x_1, \dots, x_K \in F$ such that $\min_{i \neq j} |x_i - x_j| \geq \epsilon$. Any such sequence $\{x_i\}_{1 \leq i \leq K_F(\epsilon)}$ is referred to as a *Kolmogorov sequence*.

While $M_n(F)$ is easier to work with, $K_F(\epsilon)$ has the nice property that $K_F(\epsilon) \geq K_F(\delta) \geq 1$ whenever $0 < \epsilon < \delta$. There are two other properties that deserve mention. The first is that [17, Proposition 2.7]

$$K_F(1/n) \leq M_n(F) \leq 3K_F(1/n) \quad \text{for all } n \geq 1. \quad (\text{B.2})$$

The second property is the following [17, eq. (2.8)]:

$$K_F(\epsilon) \leq 6K_F(2\epsilon) \quad \text{for all } \epsilon > 0. \quad (\text{B.3})$$

B.2. Minkowski and Packing Dimension

The (upper) *Minkowski dimension* of F is the number

$$\dim_{\text{M}} F := \limsup_{n \rightarrow \infty} \frac{\log M_n(F)}{\log n}. \quad (\text{B.4})$$

This is known also as the (upper) “box dimension” of F , and gauges the size of F . There is a related *lower Minkowski dimension*; it is defined by

$$\underline{\dim}_{\text{M}} F := \liminf_{n \rightarrow \infty} \frac{\log M_n(F)}{\log n}. \quad (\text{B.5})$$

A handicap of the gauge $\dim_{\text{M}}$ is that it assigns the value one to the rationals in $[0, 1]$; whereas we often wish to think of $\mathbf{Q} \cap [0, 1]$ as a “zero-dimensional” set. In such cases, a different notion of dimension can be used.

The (upper) *packing dimension* of F is the “regularization” of $\dim_{\mathbf{M}} F$ in the following sense:

$$\dim_{\mathbf{P}} F := \sup \left\{ \dim_{\mathbf{M}} F_k; F = \bigcup_{i \geq 1} F_i, F_i \text{'s are closed and bounded} \right\}. \quad (\text{B.6})$$

Then it is not hard to see that $\dim_{\mathbf{P}}(\mathbf{Q} \cap [0, 1]) = 0$, as desired. Furthermore, we have the relation,

$$\underline{\dim}_{\mathbf{M}} F \leq \dim_{\mathbf{H}} F \leq \dim_{\mathbf{P}} F \leq \dim_{\mathbf{M}} F. \quad (\text{B.7})$$

These are often equalities; e.g., when F is a self-similar fractal. However, there are counter-examples for which either one, or both, of these inequalities can be strict. Furthermore, one has [17, Proposition 2.9] the following integral representations:

$$\begin{aligned} \dim_{\mathbf{M}} F &= \inf \left\{ q \in \mathbf{R} : \int_1^\infty K_F(1/s) \frac{ds}{s^{1+q}} < \infty \right\}, \\ \dim_{\mathbf{P}} F &= \inf \left\{ q \in \mathbf{R} : \begin{array}{l} \exists F_1, F_2, \dots \text{ closed and bounded} \\ \text{such that } F = \bigcup_{i \geq 1} F_i, \text{ and} \\ \int_1^\infty s^{-1-q} K_{F_n}(1/s) ds < \infty \text{ for all } n \geq 1 \end{array} \right\}. \end{aligned} \quad (\text{B.8})$$

Appendix C. Some Hitting Estimates for Brownian Motion

Throughout this section, X and Y denote two independent, standard Brownian motions in $\mathbf{R}^d$, where $d \geq 3$. We will need the following technical lemmas about Brownian motion. The first lemma is contained in Propositions 1.4.1 and 1.4.3 of Khoshnevisan [15, pp. 353 and 355].

Lemma C.1. *For all $r \in (0, 1)$,*

$$\sup_{a \in \mathbf{R}^d} \mathbf{P} \left\{ \inf_{1 \leq t \leq 3/2} |a + X(t)| \leq r \right\} \leq cr^{d-2} \leq c\mathbf{P} \left\{ \inf_{1 \leq t \leq 2} |X(t)| \leq r \right\}. \quad (\text{C.1})$$

We will also need the following variant.

Lemma C.2. *There exists a constant c such that for all $0 < r < \rho < 1$,*

$$\mathbf{P} \left(\inf_{1 \leq t \leq 2} |\rho Y(t) + X(t)| \leq r \mid \inf_{1 \leq s \leq 2} |X(s)| \leq r \right) \leq c\rho^{d-2}. \quad (\text{C.2})$$

Remark C.3. The condition “ $0 < \epsilon < \rho < 1$ ” can be replaced with “ $0 < \epsilon \leq \alpha\rho$ ” for any fixed finite $\alpha > 0$. However, this lemma fails to hold for values of $\rho = o(\epsilon)$ as can be seen by simply letting ρ tend to zero in the left-hand side of (C.2): The left-hand side converges to one while the right-hand side converges to zero.

Proof. Define $T := \inf\{1 \leq t \leq 2 : |X(s)| \leq r\}$, where $\inf \emptyset := \infty$, as usual. Then,

$$\begin{aligned} P_1 &:= \mathbb{P} \left(\inf_{T \leq t \leq 2} |\rho Y(t) + X(t)| \leq r \mid T < \infty \right) \\ &= \mathbb{P} \left(\inf_{0 \leq s \leq 2-T} |\rho Y(T+s) + X(T+s)| \leq r \mid T < \infty \right) \\ &\leq \mathbb{P} \left(\inf_{0 \leq s \leq 2-T} |\rho Y(T+s) + \hat{X}(s)| \leq 2r \mid T < \infty \right), \end{aligned} \quad (\text{C.3})$$

where $\hat{X}(s) := X(T+s) - X(T)$ for all $s \geq 0$. By the strong Markov property of X ,

$$P_1 \leq \sup_{1 \leq t \leq 2} \mathbb{P} \left\{ \inf_{0 \leq s \leq 1} |\rho Y(t+s) + X(s)| \leq 2r \right\}. \quad (\text{C.4})$$

In order to estimate this quantity, let us fix an arbitrary $t \in [1, 2]$, and define

$$\begin{aligned} S &:= \inf\{0 \leq s \leq 1 : |\rho Y(t+s) + X(s)| \leq 2r\}, \\ Z &:= \int_0^2 \mathbf{1}_{\{|\rho Y(t+s) + X(s)| \leq 3r\}} ds. \end{aligned} \quad (\text{C.5})$$

Then,

$$\begin{aligned} \mathbb{E}[Z \mid S < \infty] &\geq \mathbb{E} \left[\int_S^2 \mathbf{1}_{\{|\rho Y(t+s) + X(s)| \leq 3r\}} ds \mid S < \infty \right] \\ &\geq \mathbb{E} \left[\int_0^{2-S} \mathbf{1}_{\{|\rho \mathcal{Y}(t+s) + \mathcal{X}(s)| \leq r\}} ds \mid S < \infty \right], \end{aligned} \quad (\text{C.6})$$

where $\mathcal{Y}(u) := Y(u+S) - Y(S)$ and $\mathcal{X}(u) := X(u+S) - X(S)$ for all $u \geq 0$. The process $u \mapsto \rho Y(t+u) + X(u)$ is a Lévy process, and S is a stopping time with respect to the latter process. Therefore, by the strong Markov property,

$$\begin{aligned} \mathbb{E}[Z \mid S < \infty] &\geq \int_0^1 \mathbb{P} \{ |\rho \mathcal{Y}(t+s) + \mathcal{X}(s)| \leq r \} ds \\ &= \int_0^1 \mathbb{P} \left\{ (\rho^2(t+s) + s)^{1/2} |\mathbf{g}| \leq \epsilon \right\} ds \\ &\geq \int_0^1 \mathbb{P} \left\{ (\rho^2 t + s)^{1/2} |\mathbf{g}| \leq \epsilon \right\} ds, \end{aligned} \quad (\text{C.7})$$

where $\mathbf{g}$ is a d -vector of i.i.d. standard-normal variables. Recall (2.1). Thanks to Lemmas 2.1 and 2.2,

$$\inf_{1 \leq t \leq 2} \mathbb{E}[Z \mid S < \infty] \geq c \int_0^1 f_\epsilon(\rho^2 + s) ds = c F_\epsilon(\rho^2) \geq c \epsilon^d \rho^{-(d-2)}. \quad (\text{C.8})$$

We have appealed to the condition $\rho > \epsilon$ here. Another application of Lemma 2.1 yields the following:

$$\sup_{1 \leq t \leq 2} \mathbb{E}[Z \mid S < \infty] \leq \frac{\mathbb{E}[Z]}{\mathbb{P}\{S < \infty\}} \leq \frac{c \epsilon^d}{\mathbb{P}\{S < \infty\}}. \quad (\text{C.9})$$

Recall (C.4) to find that the preceding two displays together imply that $P_1 \leq c\rho^{d-2}$. Thus, it suffices to prove that

$$P_2 := P\left(\inf_{1 \leq t \leq T} |\rho Y(t) + X(t)| \leq r \mid T < \infty\right) \leq c\rho^{d-2}. \quad (\text{C.10})$$

The estimate on P_2 is derived by using the method used to bound P_1 ; but we apply the latter method to the time-inverted Brownian motion $\{tX(1/t)\}_{t>0}$ in place of X . We omit the numerous, messy details. $\square$

References

- [1] Adler, Robert J. (1978). The uniform dimension of the level sets of a Brownian sheet. *Ann. Probab.* **6**(3), 509–515 [Corrigendum in: *Ann. Probab.* **8**(5), 1001–1002 (1980)]
- [2] Berman, Simeon M. (1973/1974). Local nondeterminism and local times of Gaussian processes. *Indiana Univ. Math. J.* **23**, 69–94
- [3] Čentsov (1965). Wiener random fields depending on several parameters, *Dokl. Akad. Nauk SSSR (N.S.)* **106**, 607–609 (In Russian)
- [4] Dalang, Robert C. and Davar Khoshnevisan (2004). Recurrent lines in two-parameter isotropic stable Lévy sheets, *Stoch. Proc. Appl.* **114**(4), 81–107
- [5] Dalang, Robert C., Davar Khoshnevisan, and Eulalia Nualart (2005). Hitting probabilities for parabolic SPDEs with additive noise, In preparation
- [6] Dalang, Robert C., Davar Khoshnevisan, and Eulalia Nualart (2005). Hitting probabilities for parabolic SPDEs with multiplicative noise, In preparation
- [7] Dalang, Robert C. and Eulalia Nualart (2004). Potential theory for hyperbolic SPDEs, *Ann. Probab.* **32**(3a), 2099–2148
- [8] Dvoretzky, A., and P. Erdős (1951). Some problems on random walk in space, *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, 1950, pp. 353–367, University of California Press, Berkeley and Los Angeles, 1951
- [9] Falconer, K. J. and J. D. Howroyd (1997). Packing dimensions of projections and dimension profiles, *Math. Proc. Cambridge Philos. Soc.* **121**(2), 269–286
- [10] Frostman, Otto (1935). Potentiel d’équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions, *Meddel. Lunds. Univ. Mat. Sem.* **3**, 1–118
- [11] Fukushima, Masatoshi (1984). Basic properties of Brownian motion and a capacity on the Wiener space, *J. Math. Soc. Japan* **36**(1), 161–176
- [12] Kahane, Jean-Pierre (1985). *Some Random Series of Functions*, Cambridge University Press, Cambridge.
- [13] Kaufman, Robert (1968). On Hausdorff dimension of projections, *Mathematika* **15**, 153–155
- [14] Khoshnevisan, Davar (2004). Brownian sheet and quasi-sure analysis, Asymptotic methods in stochastics, 25–47, Fields Inst. Commun., 44, Amer. Math. Soc., Providence, RI, 2004.
- [15] Khoshnevisan, Davar (2002). *Multiparameter Processes*, Springer, New York

- [16] Khoshnevisan, Davar (1999). Brownian sheet images and Bessel–Riesz capacity, *Trans. Amer. Math. Soc.*, **351**(7), 2607–2622
- [17] Khoshnevisan, Davar, David A. Levin, and Pedro J. Méndez–Hernández (2005). Exceptional times and invariance for dynamical random walks, *Probab. Th. Rel. Fields* (to appear)
- [18] Kôno, Norio (1984). 4-dimensional Brownian motion is recurrent with positive capacity, *Proc. Japan Acad. Ser. A Math. Sci.* **60**(2), 57–59
- [19] Lacey, Michael T. (1990). Limit laws for local times of the Brownian sheet, *Probab. Th. Rel. Fields* **86**(1), 63–85
- [20] Lyons, Terry J. (1986). The critical dimension at which quasi-every Brownian path is self-avoiding, *Adv. in Appl. Probab. suppl.*, 87–99
- [21] Marstrand, J. M. (1954). The dimension of Cartesian product sets, *Proc. Cambridge Philos. Soc.* **50**, 198–202
- [22] Mattila, Pertti (1995). *Geometry of sets and measures in Euclidean spaces*, Cambridge University Press, Cambridge
- [23] Mountford, T. S. (1990). Double points and the Ornstein-Uhlenbeck process on Wiener space. *Illinois J. Math.* **34**(1) 38–48
- [24] Nualart, D. and E. Pardoux (1994). Markov field properties of solutions of white noise driven quasi-linear parabolic PDEs. *Stochastics Stochastics Rep.* **48**(1-2) 17–44
- [25] Orey, Steven and William E. Pruitt (1973). Sample functions of the N -parameter Wiener process, *Ann. Probab.* **1**(1), 138–163
- [26] Penrose, M. D. (1990). Quasi-everywhere properties of Brownian level sets and multiple points. *Stoch. Proc. Appl.* **36**(1), 33–43
- [27] Penrose, M. D. (1989). On the existence of self-intersections for quasi-every Brownian path in space, *Ann. Probab.* **17**(2), 482–502
- [28] Pitt, Loren D. (1978). Local times for Gaussian vector fields, *Indiana Univ. Math. J.* **27**(2), 309–330
- [29] Rosen, Jay (1984). Self-intersections of random fields, *Ann. Probab.* **12**(1), 108–119
- [30] Rosen, Jay (1984). A local time approach to the self-intersections of Brownian paths in space, *Comm. Math. Phys.* **88**(3), 327–338
- [31] Shigekawa, Ichiō (1984). On a quasi-everywhere existence of the local time of the 1-dimensional Brownian motion, *Osaka J. Math.* **21**(3), 621–627
- [32] Stein, Elias M. (1970). *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, New Jersey
- [33] Taylor, S. James (1961). On the connexion between Hausdorff measures and generalized capacity, *Proc. Cambridge Philos. Soc.* **57**, 524–531
- [34] Wong, Eugene and Moshe Zakai (1974). Martingales and stochastic integrals for processes with a multi-dimensional parameter, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **9**, 109–122
- [35] Xiao, Yimin (2005). Strong local non-determinism and sample path properties of Gaussian random fields, Preprint

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