Semi-discrete semi-linear parabolic SPDEs*

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Abstract

Consider the semi-discrete semi-linear Itô stochastic heat equation,

$$\partial_t u_t(x) = (\mathscr{L} u_t)(x) + \sigma(u_t(x))\, \partial_t B_t(x),$$

started at a non-random bounded initial profile $u_0 : \mathbf{Z}^d \to \mathbf{R}_+$. Here: $\{B(x)\}_{x\in\mathbf{Z}^d}$ is an field of i.i.d. Brownian motions; $\mathscr{L}$ denotes the generator of a continuous-time random walk on $\mathbf{Z}^d$; and $\sigma : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous and non-random with $\sigma(0) = 0$. The main findings of this paper are:

(i) The kth moment Lyapunov exponent of u grows exactly as k^2;

(ii) The following random Radon–Nikodým theorem holds:

$$\lim_{\tau\downarrow 0} \frac{u_{t+\tau}(x) - u_t(x)}{B_{t+\tau}(x) - B_t(x)} = \sigma(u_t(x)) \quad \text{in probability;}$$

(iii) Under some non-degeneracy conditions, there often exists a "scale function" $S : \mathbf{R} \to (0\,,\infty)$, such that the finite-dimensional distributions of $x \mapsto \{S(u_{t+\tau}(x)) - S(u_t(x))\}/\sqrt{\tau}$ converge to those of white noise as $\tau \downarrow 0$; and

(iv) When the underlying walk is transient and the "'noise level is sufficiently low," the solution can be a.s. uniformly dissipative provided that $u_0 \in \ell^1(\mathbf{Z}^d)$.

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1 Introduction

Consider the following semi-discrete stochastic heat equation,

$$\frac{\mathrm{d}u_t(x)}{\mathrm{d}t} = (\mathscr{L}u_t)(x) + \sigma(u_t(x))\frac{\mathrm{d}B_t(x)}{\mathrm{d}t}, \tag{SHE}$$

where $\{B(x)\}_{x\in\mathbf{Z}^d}$ is a field of independent standard linear Brownian motions, $\mathscr{L}$ denotes the generator of a continuous-time random walk $X := \{X_t\}_{t\geqslant 0} := \{\sum_{j=1}^{N_t} Z_j\}_{t\geqslant 0}$ on $\mathbf{Z}^d$ where N_t is a Poisson process with jump-rate one and the Z_j's are i.i.d. random variables taking values on $\mathbf{Z}^d$, and $\sigma : \mathbf{R} \to \mathbf{R}$ is a Lipschitz-continuous non-random function with

$$\sigma(0) = 0. \tag{1.1}$$

It is well-known that if the initial state $u_0 : \mathbf{Z}^d \to \mathbf{R}$ is non-random and bounded, then (SHE) has an a.s.-unique solution in the sense of K. Itô; see for example Shiga and Shimizu [36]. We will concentrate only on the case that

$$u_0(x) \geqslant 0 \text{ for all } x \in \mathbf{Z}^d, \text{ and } \sup_{x\in\mathbf{Z}^d} u_0(x) > 0, \tag{1.2}$$

though some of our theory works for more general initial functions, as well.

Semi-discrete stochastic partial differential equations such as (SHE) have been studied at great length [12, 13, 17–21, 25, 28, 31, 35, 36], most commonly in the context of well-established models of statistical mechanics or population genetics.

The purpose of this article is to highlight some subtle local and global features of the solution to (SHE). For our first result, consider the [maximal] *kth moment Lyapunov exponents*

$$\begin{aligned} \underline{\gamma}_k(u) &:= \liminf_{t\to\infty} \sup_{x\in\mathbf{Z}^d} \frac{1}{t} \log \mathrm{E}\left(|u_t(x)|^k\right), \\ \overline{\gamma}_k(u) &:= \limsup_{t\to\infty} \sup_{x\in\mathbf{Z}^d} \frac{1}{t} \log \mathrm{E}\left(|u_t(x)|^k\right). \end{aligned} \tag{1.3}$$

In the very important special case that $\sigma(x) \propto x$ and $\mathscr{L} :=$ the generator of a simple symmetric walk on $\mathbf{Z}^d$—this is the socalled *parabolic Anderson model*—it is the frequently the case that

$$\underline{\gamma}_k(u) = \overline{\gamma}_k(u) < \infty \qquad \text{for all } k \geqslant 2. \tag{1.4}$$

More significantly, it is frequently the case that $\underline{\gamma}_k(u) > 0$ for all $k \geqslant 2$ if and only $d \in \{1, 2\}$; see the memoir of Carmona and Molchanov [13] for these results in the case that u_0 is a constant, for instance.

In the present non-linear setting, one does not expect the equality of the Lyapunov exponents $\underline{\gamma}_k(u)$ and $\overline{\gamma}_k(u)$. Still, our first result shows that, under some "intermittency conditions," the Lyapunov exponents are always positive and finite, and that the kth moment Lyapunov exponents grow as k^2, as $k \to \infty$. This is contrast with continuous SPDEs where the Lyapunov exponents typically grow exactly as k^3 as $k \to \infty$ [4–7, 24].

With the preceding aims in mind, let us define

$$\mathrm{Lip}_\sigma := \sup_{-\infty<x\neq y<\infty} \left| \frac{\sigma(x)-\sigma(y)}{x-y} \right|, \qquad \ell_\sigma := \inf_{x\in\mathbf{R}} \left| \frac{\sigma(x)}{x} \right|. \tag{1.5}$$

Note, in particular, that $\ell_\sigma |x| \leqslant |\sigma(x)| \leqslant \mathrm{Lip}_\sigma |x|$ ($x \in \mathbf{R}$) by (1.1); the upper bound is always finite, and the lower bound is > 0 for $x \neq 0$ iff $\ell_\sigma > 0$.

The following result makes the previous assertions more precise. For the sake of completeness, we include also a careful existence-uniqueness and positivity statements, since those assertions are free byproducts of the proofs of the main part of the theorem, which involves the numerical [upper and lower] bounds on the growth of the Lyapunov exponents.

Theorem 1.1. *The non-linear stochastic heat equation* (SHE) *has a solution u that is continuous in the variable t, and is unique among all predictable random fields that satisfy* $\sup_{t\in[0,T]} \sup_{x\in\mathbf{Z}^d} \mathrm{E}(|u_t(x)|^2) < \infty$ *for all $T > 0$. Moreover,*

$$\overline{\gamma}_k(u) \leqslant 8\mathrm{Lip}_\sigma^2 k^2 \qquad \textit{for all integers } k \geqslant 2. \tag{1.6}$$

Furthermore, $u_t(x) \geqslant 0$ for all $t \geqslant 0$ and $x \in \mathbf{Z}^d$ a.s., provided that $u_0(x) \geqslant 0$ for all $x \in \mathbf{Z}^d$. Finally, if $\ell_\sigma > 0$ and $\sigma(x) > 0$ for all $x > 0$, then for all $\varepsilon \in (0\,,1)$,

$$\underline{\gamma}_k(u) \geqslant (1-\varepsilon)\ell_\sigma^2 k^2 \qquad \textit{for every integer } k \geqslant \varepsilon^{-1} + (\varepsilon\ell_\sigma^2)^{-1}. \tag{1.7}$$

Standard moment methods—which we will have to reproduce here as well—show that $t \mapsto u_t(x)$ is almost surely a Hölder-continuous random function for every Hölder exponent $< 1/2$. The following proves that the Hölder exponent $1/2$ is sharp.

Theorem 1.2 (A Radon–Nikodým property)**.** *For every $t \geqslant 0$ and $x \in \mathbf{Z}^d$,*

$$\lim_{\tau\downarrow 0} \frac{u_{t+\tau}(x) - u_t(x)}{B_{t+\tau}(x) - B_t(x)} = \sigma\left(u_t(x)\right) \quad \text{almost surely.} \tag{1.8}$$

In addition,

$$\limsup_{\tau\downarrow 0} \frac{u_{t+\tau}(x) - u_t(x)}{\sqrt{2\tau \log\log(1/\tau)}} = -\liminf_{\tau\downarrow 0} \frac{u_{t+\tau}(x) - u_t(x)}{\sqrt{2\tau \log\log(1/\tau)}} = \left|\sigma\left(u_t(x)\right)\right|, \tag{1.9}$$

almost surely.

Remark 1.3. Local iterated-logarithm laws, such as (1.9) are well known in the context of finite-dimensional diffusions; see for instance Anderson [3, Theorem 4.1]. The time-change methods employed in the finite-dimensional setting will, however, not work effectively in the present infinite-dimensional context. Here, we obtain (1.9) as a ready consequence of the proof of the "random Radon–Nikodým property" (1.8). □

Remark 1.4. Let us fix an $x \in \mathbf{Z}^d$ and a $t > 0$, and let us consider $R(\tau) := [u_{t+\tau}(x) - u_t(x)]/[B_{t+\tau}(x) - B_t(x)]$; this is a well-defined random variable for every $\tau > 0$, since $B_{t+\tau}(x) - B_t(x) \neq 0$ with probability one for every $\tau > 0$. However, $\{R(\tau)\}_{\tau > 0}$ is not a well-defined stochastic process since there exists random times $\tau > 0$ such that $B_{t+\tau}(x) - B_t(x) = 0$ a.s. Thus, one does not expect that the mode of convergence in (1.8) can be improved to almost-sure convergence. This statement can be strengthened further still, but we will not do so here. □

According to (1.8), the solution to the stochastic heat equation behaves as the non-interacting system "$\mathrm{d}u_t(x) \approx \sigma(u_t(x))\mathrm{d}B_t(x)$" of diffusions, locally to first order. This might seem to suggest the [false] assertion that $x \mapsto u_t(x)$ ought to be a sequence of independent random variables. That is not the case, as can be seen by looking more closely at the time increments of $t \mapsto u_t(x)$. In fact, our arguments can be extended to show that the spatial correlation structure of u appears at second-order approximation levels in the sense of the following three–term stochastic Taylor expansion in the scale $\tau^{1/2}$:

$$u_{t+\tau}(x) \simeq u_t(x) + \tau^{1/2}\sigma(u_t(x))Z_1 + \tau Z_2 + \tau^{3/2}U(\tau) \qquad \text{as } \tau \downarrow 0, \tag{1.10}$$

where: (i) "$\simeq$" denotes approximation in the sense of distributions; (ii) Z_1 is a standard normal variable independent of $u_t(x)$; (iii) Z_2 is a non-trivial random variable that depends on the entire random field $\{u_s(y)\}_{s\in[0,t],y\in\mathbf{Z}^d}$; and (iv) $U(\tau) = O_{\mathrm{P}}(1)$ as $\tau \downarrow 0$. The latter means that $\lim_{m\uparrow\infty} \limsup_{\tau\downarrow 0} \mathrm{P}\{|U(\tau)| \geqslant m\} = 0$. In particular, (1.10) tells us that the temporally-local interactions in the random field $x \mapsto u_t(x)$ are second order in nature.

Rather than prove these refined assertions, we next turn our attention to a different local property of the solution to (SHE) and show that, after a scale change, the local-in-time behavior of the solution to (SHE) is that of spatial white noise. Namely, we offer the following:

Theorem 1.5. *Suppose $\sigma(z) > 0$ for all $z \in \mathbf{R} \setminus \{0\}$, and define*

$$S(z) := \int_{z_0}^{z} \frac{\mathrm{d}w}{\sigma(w)} \qquad (z \geqslant 0), \tag{1.11}$$

where $z_0 \in \mathbf{R} \setminus \{0\}$ is a fixed number. Then, $S(u_t(x)) < \infty$ a.s. for all $t > 0$ and $x \in \mathbf{Z}^d$. Furthermore, if we choose and fix m distinct points $x_1, \ldots, x_m \in \mathbf{Z}^d$, then for all $t > 0$ and $q_1, \ldots, q_m \in \mathbf{R}$,

$$\lim_{\tau\downarrow 0} \mathrm{P}\left(\bigcap_{j=1}^{m} \left\{ S\left(u_{t+\tau}(x_j)\right) - S\left(u_t(x_j)\right) \leqslant q_j\sqrt{\tau} \right\} \right) = \prod_{j=1}^{m} \Phi(q_j), \tag{1.12}$$

where $\Phi(q) := (2\pi)^{-1/2} \int_{-\infty}^{q} \exp(-w^2/2)\,\mathrm{d}w$ denotes the standard Gaussian cumulative distribution function.

The preceding manifests itself in amusing ways for different choices of the non-linearity coefficient σ. Let us mention the following parabolic Anderson model, which has been a motivating example for us.

Example 1.6. Consider the semi-discrete parabolic Anderson model, which is (SHE) with $\sigma(x) = c|x|$ [for some fixed constant $c > 0$]. In that case, the solution to (SHE) is positive [if u_0 is] and the "scale function" S is $S(z) = c^{-1}\ln(z/z_0)$ for $z, z_0 > 0$. As such, $\sigma(u_t(x)) = cu_t(x)$ in (SHE), and we find the following log-normal limit law: For every $t > 0$ and $x_1, \ldots, x_m \in \mathbf{Z}^d$ fixed,

$$\left(\left[\frac{u_{t+\tau}(x_1)}{u_t(x_1)}\right]^{1/\sqrt{\tau}}, \ldots, \left[\frac{u_{t+\tau}(x_m)}{u_t(x_m)}\right]^{1/\sqrt{\tau}}\right) \Longrightarrow (\mathrm{e}^{cN_1}, \ldots, \mathrm{e}^{cN_m}), \tag{1.13}$$

as $\tau \downarrow 0$, where $N_1, \ldots, N_m$ are i.i.d. standard normal variables, and "$\Rightarrow$" denotes convergence in distribution.

Our final main result is a statement about the large-time behavior of the solution u to (SHE). We intend to prove a rigorous version of the following assertion: "*If the random walk X is transient and* Lip_σ *is sufficiently small —so that (SHE) is not very noisy—then a decay condition such as $u_0 \in \ell^1(\mathbf{Z}^d)$ on the initial profile is enough to ensure that* $\sup_{x\in\mathbf{Z}^d}|u_t(x)| \to 0$ *almost surely as* $t \to \infty$." This is new even for the parabolic Anderson model, where $\sigma(x) \propto x$ and $\mathscr{L} :=$ the generator of the simple walk on $\mathbf{Z}^d$. In fact, this result gives a partial [though strong] negative answer to an open problem of Carmona and Molchanov [13, p. 122] and rules out the existence of [the analogue of] a non-trivial "Anderson mobility edge" in the present non-stationary setting, when $u_0 \in \ell^1(\mathbf{Z}^d)$. We are aware only of one such non-existence theorem, this time for the original stationary Anderson model on "tree graphs"; see the recent paper by Aizenman and Warzel [2].

Recall that $X := \{X_t\}_{t\geqslant 0}$ is a continuous-time random walk on $\mathbf{Z}^d$ with generator $\mathscr{L}$. Let X' denote an independent copy of X, and define

$$\Upsilon(0) := \int_0^\infty \mathrm{P}\{X_t = X'_t\}\,\mathrm{d}t = \mathrm{E}\int_0^\infty \mathbf{1}_{\{0\}}(X_t - X'_t)\,\mathrm{d}t. \tag{1.14}$$

We can think of $\Upsilon(0)$ as the expected value of the total occupation time of $\{0\}$, as viewed by the symmetrized random walk $X - X'$. Although $\Upsilon(0)$ is always well defined, it is finite if and only if the symmetrized random walk $X - X'$ is transient [14]. We are ready to state our final result.

Theorem 1.7. *Suppose that*

$$\mathrm{Lip}_\sigma < [\Upsilon(0)]^{-1/2}, \tag{1.15}$$

and that there exists $\alpha \in (1, \infty)$ such that

$$\mathrm{P}\{X_t = X'_t\} = O(t^{-\alpha}) \qquad (t \to \infty), \tag{1.16}$$

where X' denotes an independent copy of X. If, in addition, $u_0 \in \ell^1(\mathbf{Z}^d)$ and the underlying probability space is complete, then

$$\lim_{t\to\infty}\sup_{x\in\mathbf{Z}^d}|u_t(x)| = \lim_{t\to\infty}\sum_{x\in\mathbf{Z}^d}|u_t(x)|^2 = 0 \qquad \textit{almost surely.} \tag{1.17}$$

Remark 1.8 (Hysteresis)**.** Consider the parabolic Anderson model [$\sigma(x) \propto x$], where the underlying symmetrized walk $X - X'$ is transient, the noise level is small, and u_0 is a constant. It is well known that, under these conditions, $u_t(x)$ converges weakly as $t \to \infty$ to a non-void random variable $u_\infty(x)$ for every $x \in \mathbf{Z}^d$. See, for example, Greven and den Hollander [28, Theorem 1.4], Cox and Greven [18], and Shiga [35]. These results provide a partial affirmative answer to a question of Carmona and Molchanov [13, p. 122] about the existence of long-term invariant laws in the low-noise regime of the transient parabolic Anderson model, in particular. By contrast, Theorem 1.7 shows that if u_0 is far from stationary [here, it decays at infinity], then the system is very strongly dissipative in the low-noise regime. Among other strange things, this result has the consequence that *the parabolic Anderson model remembers its initial state forever.* □

Remark 1.9. Continuous-time walks that have the property (1.16) include all transient finite-variance centered random walks on $\mathbf{Z}^d$ [$d > 2$, necessarily]. For those walks, $\alpha := d/2$, thanks to the local central limit theorem. There are more interesting examples as well. For instance, suppose $t^{-1/p}X_t$ converges in distribution to a stable random variable S as $t \to \infty$. [See Gnedenko and Kolmogorov [27, §35] for necessary and sufficient conditions.] Then, S is necessarily stable with index p, $p \in (0\,,2]$, and $t^{-1/p}(X_t - X_t')$ converges in law to a symmetric stable random variable S with stability index p. If, in addition, the group of all possible values of $X_t - X_t'$ generates all of $\mathbf{Z}^d$, then a theorem of Gnedenko [27, p. 236] ensures that $t^{1/p}\mathrm{P}\{X_t = X_t'\}$ converges to $f(0) < \infty$, where f denotes the probability density function of S, as long as $p \in (0\,,1)$. □

We close this introduction with some background on Burkholder's constants. According to the Burkholder–Davis–Gundy inequality [8–10],

$$z_p := \sup_x \sup_{t>0} \left[\frac{\mathrm{E}\left(|x_t|^p\right)}{\mathrm{E}\left(\langle x\rangle_t^{p/2}\right)} \right]^{1/p} < \infty, \tag{1.18}$$

where the supremum "$\sup_x$" is taken over all non-zero martinagles $x := \{x_t\}_{t\geqslant 0}$ that have continuous trajectories and are in $L^2(\mathrm{P})$ at all times, $\langle x\rangle_t$ denotes the quadratic variation of x at time t, and $0/0 := \infty/\infty := 0$. Davis [22] has computed the numerical value of z_p in terms of zeroes of special functions. In the special case that $p = k$ where $k \geqslant 2$ an integer, Davis' theorem implies that z_k is equal to the largest positive root of the modified Hermite polynomial He_k. Thus, for example, we obtain the following from direct evaluation of the zeros:

$$\begin{aligned} &z_2 = 1,\ z_3 = \sqrt{3},\ z_4 = \sqrt{3+\sqrt{6}} \approx 2.334, \\ &z_5 = \sqrt{5+\sqrt{10}} \approx 2.857, \quad z_6 \approx 3.324, \ldots . \end{aligned} \tag{1.19}$$

It is known that $z_p \sim 2\sqrt{p}$ as $p \to \infty$, and $\sup_{p\geqslant 2}(z_p/\sqrt{p}) = 2$; see Carlen and Kree [11, Appendix]. □

2 Preliminaries

2.1 The mild solution

As is customary, by a "solution" to (SHE) we mean a solution in integrated—or "mild"—form. That is, a predictable process $t \mapsto u_t$, with values in $\mathbf{R}^{\mathbf{Z}^d}$, that solves the following infinite system of Itô SDEs:

$$u_t(x) = (\tilde{p}_t * u_0)(x) + \sum_{y \in \mathbf{Z}^d} \int_0^t p_{t-s}(y-x)\sigma(u_s(y))\,\mathrm{d}B_s(y); \tag{2.1}$$

where $p_t(x) := \mathrm{P}\{X_t = x\}$,

$$(f * g)(x) := \sum_{y \in \mathbf{Z}^d} f(x-y)g(y) \qquad (x \in \mathbf{Z}^d) \tag{2.2}$$

denotes the convolution on $\mathbf{Z}^d$; and for every function $h : \mathbf{Z}^d \to \mathbf{R}$ we define a new function $\tilde{h}$,

$$\tilde{h}(x) := h(-x) \qquad (x \in \mathbf{Z}^d), \tag{2.3}$$

as the *reflection* of h.

It might be helpful to note also that $(P_t\phi)(x) := (\tilde{p}_t * \phi)(x)$ defines the semigroup of the random walk X via the identity $(P_t\phi)(x) = \mathrm{E}\phi(x + X_t)$. Thus we can write (2.1) in the following, perhaps more familar, form:

$$u_t(x) = (P_t u_0)(x) + \sum_{y \in \mathbf{Z}^d} \int_0^t p_{t-s}(y-x)\sigma(u_s(y))\,\mathrm{d}B_s(y), \tag{2.4}$$

2.2 A BDG inequality

Suppose $Z := \{Z_t(x)\}_{t \geqslant 0, x \in \mathbf{Z}^d}$ is a predictable random field, with respect to the infinite-dimensional Brownian motion $\{B_t(\bullet)\}_{t \geqslant 0}$, that satisfies the moment bound $\mathrm{E}\int_0^t \|Z_s\|^2_{\ell^2(\mathbf{Z}^d)}\,\mathrm{d}s < \infty$. Then, the Itô integral process defined by

$$\int_0^t Z_s \cdot \mathrm{d}B_s := \sum_{y \in \mathbf{Z}^d} \int_0^t Z_s(y)\,\mathrm{d}B_s(y) \qquad (t \geqslant 0) \tag{2.5}$$

exists and defines a continuous $L^2(\mathrm{P})$ martingale. This is part of the standard folklore of infinite-dimensional stochastic analysis; see for example Prévôt and Röckner [33]. The following variation of the Burkholder–Davis–Gundy inequality yields moment bounds for that martingale that also pay special attention to the constants in such inequalities.

Lemma 2.1 (BDG Lemma)**.** *For all finite real numbers $k \geqslant 2$ and $t \geqslant 0$,*

$$\mathrm{E}\left(\left|\int_0^t Z_s \cdot \mathrm{d}B_s\right|^k\right) \leqslant \left|4k \sum_{y \in \mathbf{Z}^d} \int_0^t \left\{\mathrm{E}\left(|Z_s(y)|^k\right)\right\}^{2/k} \mathrm{d}s\right|^{k/2}. \tag{2.6}$$

Proof. We follow a method of Foondun and Khoshnevisan [24].

A standard approximation argument tells us that it suffices to consider the case that $y \mapsto Z_s(y)$ has finite support. To be concrete, let $F \subset \mathbf{Z}^d$ be a finite set of cardinality $m \geqslant 1$, and suppose $Z_s(y) = 0$ for all $y \notin F$. Consider the [standard, finite-dimensional] Itô integral process $\int_0^t Z_s \cdot \mathrm{d}B_s := \sum_{y\in F} \int_0^t Z_s(y)\,\mathrm{d}B_s(y)$. According to Davis' [22] form of the Burkholder–Davis–Gundy inequality m-dimensional Brownian motion [8–10],

$$\mathrm{E}\left(\left|\int_0^t Z_s \cdot \mathrm{d}B_s\right|^k\right) \leqslant z_k^k \mathrm{E}\left(\left|\sum_{y\in F}\int_0^t [Z_s(y)]^2\,\mathrm{d}s\right|^{k/2}\right). \tag{2.7}$$

Finally, we use the Carlen–Kree bound $z_k \leqslant 2\sqrt{k}$ [11] together with the Minkowski inequality to finish the proof in the case that F is finite. A standard finite-dimensional approximation completes the proof. □

3 Proof of Theorem 1.1: Part 1

Existence and uniqueness, and also continuity, of the solution are dealt with extensively in the literature and are well known; see for example Shiga and Shimizu [36], and the general theory of Prévot and Röckner [33] for some of the latest developments. However, in order to derive our estimates of the Lyapunov exponents we will need *a priori* estimates which will also yield existence and uniqueness. Therefore, in this section, we hash out some—though not all—of the details.

Let us proceed by applying Picard iteration. Let $u_t^{(0)}(x) := u_0(x)$, and then define iteratively for all $n \geqslant 0$,

$$u_t^{(n+1)}(x) := (\tilde{p}_t * u_0)(x) + \sum_{y\in\mathbf{Z}^d}\int_0^t p_{t-s}(y-x)\sigma\left(u_s^{(n)}(y)\right)\,\mathrm{d}B_s(y). \tag{3.1}$$

It follows from the properties of the Itô integral that

$$M_t^{(n+1)} := \sup_{x\in\mathbf{Z}^d} \mathrm{E}\left(|u_t^{(n+1)}(x)|^k\right) \leqslant 2^{k-1} \sup_{x\in\mathbf{Z}^d}(I_x + J_x), \tag{3.2}$$

where

$$\begin{aligned} I_x &:= \left|(p_t * u_0)(x)\right|^k, \\ J_x &:= \mathrm{E}\left(\left|\sum_{y\in\mathbf{Z}^d}\int_0^t p_{t-s}(y-x)\sigma\left(u_s^{(n)}(y)\right)\,\mathrm{d}B_s(y)\right|^k\right). \end{aligned} \tag{3.3}$$

The first term is easy to bound:

$$\sup_{x\in\mathbf{Z}^d} I_x \leqslant \|u_0\|_{\ell^\infty(\mathbf{Z}^d)}^k, \tag{3.4}$$

since $\sum_x p_t(x) = 1$. Next we bound J_x.

Because σ is Lipschitz continuous and $\sigma(0) = 0$, we can see that $|\sigma(z)| \leqslant \mathrm{Lip}_\sigma |z|$ for all $z \in \mathbf{R}$. Thus, we may use the BDG lemma [Lemma 2.1] in order to see that

$$J_x^{2/k} \leqslant 4k\mathrm{Lip}_\sigma^2 \sum_{y\in\mathbf{Z}^d} \int_0^t [p_{t-s}(y-x)]^2 \left\{ \mathrm{E}\left(\left| u_s^{(n)}(y) \right|^k \right) \right\}^{2/k} \mathrm{d}s. \tag{3.5}$$

Therefore, we may recall the inductive definition (3.2) of M to see that

$$\begin{aligned} J_x^{2/k} &\leqslant 4k\mathrm{Lip}_\sigma^2 \sum_{y\in\mathbf{Z}^d} \int_0^t [p_{t-s}(y-x)]^2 \left(M_s^{(n)} \right)^{2/k} \mathrm{d}s \\ &\leqslant 4k\mathrm{Lip}_\sigma^2 \int_0^t \left(M_s^{(n)} \right)^{2/k} \mathrm{d}s, \end{aligned} \tag{3.6}$$

since

$$\sum_{z\in\mathbf{Z}^d} [p_r(z)]^2 = \mathrm{P}\{X_r = X'_r\} \leqslant 1, \tag{3.7}$$

where X' denotes an independent copy of X. [This last bound might appear to be quite crude, and it is when r is large. However, it turns out that the behavior of r near zero matters more to us. Therefore, the inequality is tight in the regime $r \approx 0$ of interest to us.]

We may combine (3.2), (3.4), and (3.6) in order to see that for all $\beta, t > 0$,

$$\begin{aligned} &\mathrm{e}^{-\beta t} M_t^{(n+1)} \\ &\leqslant 2^{k-1}\|u_0\|_{\ell^\infty(\mathbf{Z}^d)}^k + (16k\mathrm{Lip}_\sigma^2)^{k/2} \left| \int_0^t \mathrm{e}^{-2\beta(t-s)/k} \left(\mathrm{e}^{-\beta s} M_s^{(n)} \right)^{2/k} \mathrm{d}s \right|^{k/2}. \end{aligned} \tag{3.8}$$

Consequently, the sequence defined by

$$N_\beta^{(m)} := \sup_{t\geqslant 0} \left(\mathrm{e}^{-\beta t} M_t^{(m)} \right) \qquad (m \geqslant 0) \tag{3.9}$$

satisfies the recursive inequality

$$\begin{aligned} N_\beta^{(n+1)} &\leqslant 2^{k-1}\|u_0\|_{\ell^\infty(\mathbf{Z}^d)}^k + (16k\mathrm{Lip}_\sigma^2)^{k/2} \left| \int_0^t \mathrm{e}^{-2\beta s/k} \, \mathrm{d}s \right|^{k/2} N_\beta^{(n)} \\ &\leqslant 2^{k-1}\|u_0\|_{\ell^\infty(\mathbf{Z}^d)}^k + \left(\frac{8k^2\mathrm{Lip}_\sigma^2}{\beta} \right)^{k/2} N_\beta^{(n)}. \end{aligned} \tag{3.10}$$

In particular, if we denote [temporarily for this proof]

$$\alpha := 8(1+\delta)\mathrm{Lip}_\sigma^2, \tag{3.11}$$

where $\delta > 0$ is fixed but arbitrary, then

$$N_{\alpha k^2}^{(n+1)} \leqslant 2^{k-1}\|u_0\|_{\ell^\infty(\mathbf{Z}^d)}^k + (1+\delta)^{-k/2} N_{\alpha k^2}^{(n)}. \tag{3.12}$$

We may apply induction on n now in order to see that $\sup_{n\geqslant 0} N^{(n)}_{\alpha k^2} < \infty$; equivalently, for all $k \geqslant 2$ there exists $c_k \in (0\,,\infty)$ such that

$$\sup_{x\in\mathbf{Z}^d} \mathrm{E}\left(\left|u_t^{(n)}(x)\right|^k\right) \leqslant c_k \mathrm{e}^{8(1+\delta)\mathrm{Lip}_\sigma^2 k^2 t} \qquad \text{for all } t \geqslant 0. \tag{3.13}$$

Similarly,

$$\begin{aligned}
&\mathrm{E}\left(\left|u_t^{(n+1)}(x) - u_t^{(n)}(x)\right|^k\right) \qquad (3.14)\\
&= \mathrm{E}\left(\left|\sum_{y\in\mathbf{Z}^d}\int_0^t p_{t-s}(y-x)\left\{\sigma\left(u_s^{(n)}(y)\right) - \sigma\left(u_s^{(n-1)}(y)\right)\right\}\,\mathrm{d}B_s(y)\right|^k\right)\\
&\leqslant (4k\mathrm{Lip}_\sigma^2)^{k/2}\mathrm{E}\left(\left|\sum_{y\in\mathbf{Z}^d}\int_0^t [p_{t-s}(y-x)]^2\left\{u_s^{(n)}(y) - u_s^{(n-1)}(y)\right\}^2\,\mathrm{d}s\right|^{k/2}\right).
\end{aligned}$$

Define

$$L_t^{(n+1)} := \sup_{x\in\mathbf{Z}^d}\mathrm{E}\left(\left|u_t^{(n+1)}(x) - u_t^{(n)}(x)\right|^k\right) \tag{3.15}$$

to deduce from the preceding, (3.7), and Minkowski's inequality that

$$\begin{aligned}
L_t^{(n+1)} &\leqslant (4k\mathrm{Lip}_\sigma^2)^{k/2}\left(\sum_{y\in\mathbf{Z}^d}\int_0^t [p_{t-s}(y-x)]^2\left(L_s^{(n)}\right)^{2/k}\,\mathrm{d}s\right)^{k/2}\\
&\leqslant (4k\mathrm{Lip}_\sigma^2)^{k/2}\left(\int_0^t \left(L_s^{(n)}\right)^{2/k}\,\mathrm{d}s\right)^{k/2}.
\end{aligned} \tag{3.16}$$

Therefore,

$$K^{(m)}_{\alpha k^2} = \sup_{t\geqslant 0}\left(\mathrm{e}^{-\alpha k^2 t} L_t^{(m)}\right) \tag{3.17}$$

satisfies

$$\begin{aligned}
K^{(n+1)}_{\alpha k^2} &\leqslant (4k\mathrm{Lip}_\sigma^2)^{k/2}\left(\int_0^t \mathrm{e}^{-2\alpha k(t-s)}\,\mathrm{d}s\right)^{k/2} K^{(n)}_{\alpha k^2} \leqslant \left(\frac{4\mathrm{Lip}_\sigma^2}{2\alpha}\right)^{k/2} K^{(n)}_{\alpha k^2}\\
&\leqslant 2^{-k} K^{(n)}_{\alpha k^2}.
\end{aligned} \tag{3.18}$$

From this we can conclude that $\sum_{n=0}^\infty K^{(n)}_{\alpha k^2} < \infty$. Therefore, there exists a random field $u_t(x)$ such that $\lim_{n\to\infty} u_t^{(n)}(x) = u_t(x)$, where the limit takes place in $L^k(\mathrm{P})$. It follows readily that u solves (SHE), and u satisfies (1.6) thanks to (3.13) and Fatou's lemma. Uniqueness is proved by similar means, and we skip the details. □

4 A local approximation theorem

In this section we develop a description of the local dynamics of the random field $t \mapsto u_t(\bullet)$ in the form of several approximation results.

Our first approximation lemma is a standard sample-function continuity result; it states basically that outside a single null set,

$$u_{t+\tau}(x) = u_t(x) + O\left(\tau^{(1+o(1))/2}\right) \quad \text{as } \tau \to 0, \text{ for all } t \geqslant 0 \text{ and } x \in \mathbf{Z}^d. \tag{4.1}$$

The result is well known, but we need to be cautious with various constants that crop up in the proof. Therefore, we include the details to account for the dependencies of the implied constants.

Lemma 4.1. *There exists a version of u that is a.s. continuous in t with critical Hölder exponent $\geqslant 1/2$. In fact, for every $T \geqslant 1$, $\varepsilon \in (0\,,1)$ and $k \geqslant 2$,*

$$\sup_{x\in\mathbf{Z}^d}\sup_I \mathrm{E}\left(\sup_{\substack{s,t\in I\\ s\neq t}}\left[\frac{|u_t(x)-u_s(x)|}{|t-s|^{(1-\varepsilon)/2}}\right]^k\right) < \infty, \tag{4.2}$$

where "$\sup_I$" denotes the supremum over all closed subintervals I of $[0\,,T]$ that have length $\leqslant 1$.

Proof. Owing to Minkowski's inequality,

$$\left\{\mathrm{E}\left(|u_{t+\tau}(x)-u_t(x)|^k\right)\right\}^{1/k} \leqslant |Q_1| + Q_2 + Q_3, \tag{4.3}$$

where

$$\begin{aligned}
Q_1 &:= (\tilde{p}_{t+\tau} * u_0)(x) - (\tilde{p}_t * u_0)(x),\\
Q_2 &:= \left\{\mathrm{E}\left(\left|\sum_{y\in\mathbf{Z}^d}\int_0^t [p_{t+\tau-s}(y-x) - p_{t-s}(y-x)]\,\sigma(u_s(y))\,\mathrm{d}B_s(y)\right|^k\right)\right\}^{1/k},\\
Q_3 &:= \left\{\mathrm{E}\left(\left|\sum_{y\in\mathbf{Z}^d}\int_t^{t+\tau} p_{t+\tau-s}(y-x)\sigma(u_s(y))\,\mathrm{d}B_s(y)\right|^k\right)\right\}^{1/k}.
\end{aligned} \tag{4.4}$$

We estimate each item in turn.

Let $J_{t,t+\tau}$ denote the event that the random walk X jumps some time during the time interval $(t\,,t+\tau)$. Because

$$\begin{aligned}
\sum_{x\in\mathbf{Z}^d} |p_{t+\tau}(x) - p_t(x)| &= \sum_{x\in\mathbf{Z}^d}\left|\mathrm{E}\left(\mathbf{1}_{\{X_{t+\tau}=x\}} - \mathbf{1}_{\{X_t=x\}};\ J_{t,t+\tau}\right)\right|\\
&\leqslant 2\mathrm{P}(J_{t,t+\tau}) = 2\left(1-\mathrm{e}^{-\tau}\right) \leqslant 2\tau,
\end{aligned} \tag{4.5}$$

we obtain the following estimate for $|Q_1|$:

$$|Q_1| \leqslant 2\|u_0\|_{\ell^\infty(\mathbf{Z}^d)}\tau. \tag{4.6}$$

By the BDG Lemma 2.1,

$$\begin{aligned} Q_2^2 &\leqslant 4k \sum_{y\in\mathbf{Z}^d} \int_0^t \left[p_{t+\tau-s}(y-x) - p_{t-s}(y-x)\right]^2 \left\{\mathrm{E}\left(|\sigma(u_s(y))|^k\right)\right\}^{2/k} \mathrm{d}s \\ &\leqslant 4k \int_0^t \mathscr{Q}(s) \sup_{y\in\mathbf{Z}^d} \left\{\mathrm{E}\left(|\sigma(u_s(y))|^k\right)\right\}^{2/k} \mathrm{d}s, \end{aligned} \tag{4.7}$$

where

$$\mathscr{Q}(s) := \sum_{z\in\mathbf{Z}^d} |p_{t+\tau-s}(z) - p_{t-s}(z)|^2 \qquad (0 < s < t). \tag{4.8}$$

It is possible to find a real-variable estimate for $\mathscr{Q}(s)$ using (4.5); namely, $\mathscr{Q}(s) \leqslant \sum_{z\in\mathbf{Z}^d} |p_{t+\tau-s}(z) - p_{t-s}(z)| \leqslant 2\tau$. Unfortunately, this is not good enough for our present needs; we need to do a little better by showing that $\mathscr{Q}(s) \leqslant 2\tau^2$: Recall that we can represent $X_t := \sum_{j=0}^{N(t)} Y_j$, where $Y_0 := 0$, $\{Y_j\}_{j=1}^\infty$ is a sequence of i.i.d. random variables and $\{N(t)\}_{t\geqslant 0}$ is an independent rate-one Poisson process. Let $\varphi(\xi) := \mathrm{E}\exp(i\xi\cdot Y_1)$ denote the characteristic function of the increments of the continuous-time random walk X. It is an exercise in Poissonization that

$$\mathrm{E}\mathrm{e}^{i\xi\cdot X_t} = \mathrm{e}^{-t(1-\varphi(\xi))} \qquad \text{for all } \xi\in\mathbf{R}^d \text{ and } t\geqslant 0. \tag{4.9}$$

Therefore, we appeal to the Parseval identity and find that

$$\mathscr{Q}(s) = (2\pi)^{-d} \int_{[-\pi,\pi]^d} \left|\mathrm{e}^{-(t+\tau-s)(1-\varphi(\xi))} - \mathrm{e}^{-(t-s)(1-\varphi(\xi))}\right|^2 \mathrm{d}\xi \leqslant 2\tau^2, \tag{4.10}$$

uniformly for all $s\in(0\,,t)$. This shows that

$$Q_2^2 \leqslant 8k\tau^2 \int_0^t \sup_{y\in\mathbf{Z}^d} \left\{\mathrm{E}\left(|\sigma(u_s(y))|^k\right)\right\}^{2/k} \mathrm{d}s.$$

Because $|\sigma(z)| \leqslant \mathrm{Lip}_\sigma |z|$ for all $z\in\mathbf{R}$, the already-proved bound (1.6) tells us that there exist constants $c, c_k \in (0\,,\infty)$ $[k\geqslant 2]$ such that

$$\sup_{y\in\mathbf{Z}^d} \mathrm{E}\left(|\sigma(u_s(y))|^k\right) \leqslant c_k^k \mathrm{e}^{ck^2 s} \quad \text{for all integers } k\geqslant 2 \text{ and } s\geqslant 0. \tag{4.11}$$

Therefore,

$$Q_2^2 \leqslant 4c^{-1} c_k^2 \mathrm{e}^{2ckt}\tau^2. \tag{4.12}$$

Finally, we apply the BDG Lemma 2.1 to see that

$$\begin{aligned} Q_3^2 &\leqslant 4k \sum_{y\in\mathbf{Z}^d} \int_t^{t+\tau} \left[p_{t+\tau-s}(y-x)\right]^2 \left\{\mathrm{E}\left(|\sigma(u_s(y))|^k\right)\right\}^{2/k} \mathrm{d}s \\ &\leqslant 4kc_k^2 \sum_{y\in\mathbf{Z}^d} \int_t^{t+\tau} \left[p_{t+\tau-s}(y-x)\right]^2 \mathrm{e}^{2cks}\,\mathrm{d}s, \end{aligned} \tag{4.13}$$

owing to (4.11). Because $\sum_{y\in\mathbf{Z}^d}[p_h(y-x)]^2 \leqslant 1$ for all $h \geqslant 0$, we find that

$$Q_3^2 \leqslant 2c_k^2 \mathrm{e}^{2ck(t+\tau)}\tau. \tag{4.14}$$

We combine (4.6), (4.12), and (4.14) and find that for all integers $k \geqslant 2$, there exists a finite and positive constant $\tilde{a} := \tilde{a}(T,k)$ such that for every $\tau \in (0\,,1)$,

$$\sup_{x\in\mathbf{Z}^d}\sup_{t\in(0,T)} \mathrm{E}\left(|u_{t+\tau}(x) - u_t(x)|^k\right) \leqslant \tilde{a}\mathrm{e}^{\tilde{a}T}\tau^{k/2}. \tag{4.15}$$

The lemma follows from this bound, and an application of a quantitative form of the Kolmogorov continuity theorem [34, Theorem 2.1, p. 25]. We omit the remaining details, as they are nowadays standard. □

Our second approximation lemma yields a truncation error estimate for the nonlinearity σ.

Lemma 4.2. *Let $U_t^{(N)}(x)$ denote the a.s.-unique solution to* (SHE) *where σ is replaced by $\sigma^{(N)}$, where $\sigma^{(N)} = \sigma$ on $(-N\,,N)$, $\sigma^{(N)} = 0$ on $[-N-1\,,N+1]^c$, and defined by linear interpolation on $[-N-1\,,-N] \cup [N\,,N+1]$. Then, $\lim_{N\to\infty} U_t^{(N)}(x) = u_t(x)$ almost surely and in $L^k(\mathrm{P})$ for all $k \geqslant 2$, $t \geqslant 0$, and $x \in \mathbf{Z}^d$.*

Proof. Since $\sigma^{(N)}$ is Lipschitz continuous, Theorem 1.1 ensures the existence and uniqueness of $U^{(N)}$ for every $N \geqslant 1$. Next we note, using (2.1), that

$$u_t(x) - U_t^{(N)}(x) = T_1 + T_2, \tag{4.16}$$

where:

$$\begin{aligned} T_1 &:= \sum_{y\in\mathbf{Z}^d}\int_0^t p_{t-s}(y-x)\{\sigma(u_s(y)) - \sigma^{(N)}(u_s(y)\}\mathrm{d}B_s(y);\\ T_2 &:= \sum_{y\in\mathbf{Z}^d}\int_0^t p_{t-s}(y-x)\left\{\sigma^{(N)}(u_s(y)) - \sigma^{(N)}\left(U_s^{(N)}(y)\right)\right\}\mathrm{d}B_s(y). \end{aligned} \tag{4.17}$$

Because $|\sigma(z)| \leqslant \mathrm{Lip}_\sigma|z|$, Lemma 2.1 implies that $\left\{\mathrm{E}(|T_1|^k)\right\}^{2/k}$ is at most

$$4k\mathrm{Lip}_\sigma^2 \sum_{y\in\mathbf{Z}^d}\int_0^t [p_{t-s}(y-x)]^2\left\{\mathrm{E}\left(|u_s(y)|^k\,;\,|u_s(y)| \geqslant N\right)\right\}^{2/k}\mathrm{d}s. \tag{4.18}$$

We have $\mathrm{E}(|Y|^k;\,|Y| \geqslant N) \leqslant N^{-k}\mathrm{E}(Y^{2k})$, valid for all $2k$-times integrable random variables Y. Therefore,

$$\left\{\mathrm{E}(|T_1|^k)\right\}^{2/k} \leqslant \frac{4k\mathrm{Lip}_\sigma^2}{N^2}\sum_{y\in\mathbf{Z}^d}\int_0^t [p_{t-s}(y-x)]^2\left\{\mathrm{E}\left(|u_s(y)|^{2k}\right)\right\}^{2/k}\mathrm{d}s. \tag{4.19}$$

Because $\sum_{y\in\mathbf{Z}^d}[p_{t-s}(y-x)]^2 \leqslant 1$, the already-proved bound (1.6) tells us that

$$\left\{\mathrm{E}(|T_1|^k)\right\}^{2/k} \leqslant \frac{a_k}{N^2}\int_0^t \mathrm{e}^{128\mathrm{Lip}_\sigma^2 ks}\,\mathrm{d}s \leqslant \frac{Aa_k\mathrm{e}^{Akt}}{N^2}, \tag{4.20}$$

where a_k and A are uninteresting finite and positive constants; moreover, a_k depends only on k. This estimates the norm of T_1.

As for T_2, we use the simple inequality $|\sigma^{(N)}(r) - \sigma^{(N)}(\rho)| \leqslant C|r-\rho|$, together with the BDG Lemma 2.1 in order to find that

$$\left\{\mathrm{E}\left(|T_2|^k\right)\right\}^{2/k} \leqslant b_k\int_0^t \sup_{y\in\mathbf{Z}^d}\left\{\mathrm{E}\left(\left|u_s(y) - U_s^{(N)}(y)\right|^k\right)\right\}^{2/k}\mathrm{d}s, \tag{4.21}$$

where b_k is a constant dependent on σ and k. Consequently, we combine these bounds to deduce that

$$D_t^{(N)} := \sup_{x\in\mathbf{Z}^d}\left\{\mathrm{E}\left(\left|u_t(x) - U_t^{(N)}(x)\right|^k\right)\right\}^{2/k} \tag{4.22}$$

satisfies the recursion

$$D_t^{(N)} \leqslant \frac{\tilde{a}_k\mathrm{e}^{\tilde{A}k^2t}}{N^2} + \tilde{b}_k\int_0^t D_s^{(N)}\,\mathrm{d}s, \tag{4.23}$$

where $\tilde{a}_k$, $\tilde{b}_k$, and $\tilde{A}$ are positive and finite constants, and the first two depend only on k [whereas the latter is universal]. An application of the Gronwall inequality shows that $\sup_{t\in[0,T]} D_t^{(N)} = O(N^{-2})$ as $N\to\infty$, for every fixed value $T\in(0\,,\infty)$. This is more than enough to yield the lemma. □

Our next approximation result is the highlight of this section, and refines (4.1) by inspecting more closely the main contribution to the $O(\tau^{(1+o(1))/2})$ error term in (4.1). In order to describe the next approximation result, we first define for every fixed $t\geqslant 0$ an infinite-dimensional Brownian motion $B^{(t)}$ as follows:

$$B_\tau^{(t)}(x) := B_{\tau+t}(x) - B_t(x) \qquad (x\in\mathbf{Z}^d,\ \tau\geqslant 0). \tag{4.24}$$

If we continue to hold t fixed, then it is easy to see that $\{B_\bullet^{(t)}(x)\}_{x\in\mathbf{Z}^d}$ is a collection of independent d-dimensional Brownian motions. Furthermore, the entire process $B^{(t)}$ is independent of the infinite-dimensional random variable $u_t(\bullet)$, since it is easy to see from the proof of the first part of Theorem 1.1 that u_t is a measurable function of $\{B_s(y)\}_{s\in[0,t],y\in\mathbf{Z}^d}$, which is therefore independent of $B^{(t)}$ by the Markov property of B. Now for every fixed $t\geqslant 0$ and $x\in\mathbf{Z}^d$, consider the solution $u_\bullet^{(t)}(x)$ to the following [autonomous/non-interacting] Itô stochastic differential equation:

$$\left[\begin{aligned}&\frac{\mathrm{d}u_\tau^{(t)}(x)}{\mathrm{d}\tau} = \frac{\mathrm{d}(\tilde{p}_\tau * u_t)(x)}{\mathrm{d}\tau} + \sigma\left(u_\tau^{(t)}(x)\right)\frac{\mathrm{d}B_\tau^{(t)}(x)}{\mathrm{d}\tau},\\ &\text{subject to } u_0^{(t)}(x) = u_t(x).\end{aligned}\right. \tag{4.25}$$

Note, once again, that $B^{(t)}$ is independent of $u_t(\bullet)$. Moreover,

$$\sup_{\tau>0} \mathrm{E}\left(\left|(\tilde{p}_\tau * u_t)(x)\right|^2\right) \leqslant \sup_{y\in\mathbf{Z}^d} \mathrm{E}\left(|u_t(y)|^2\right) < \infty, \tag{4.26}$$

thanks to the already-proved bound (1.6) and Cauchy–Schwarz inequality. Therefore, (4.25) is a standard Itô-type SDE, and hence has a unique strong solution.

Theorem 4.3 (The local-diffusion property)**.** *For every $t \geqslant 0$, the following holds a.s. for all $x \in \mathbf{Z}^d$:*

$$u_{t+\tau}(x) = u_\tau^{(t)}(x) + O\left(\tau^{3/2+o(1)}\right) \quad \text{as } \tau \downarrow 0. \tag{4.27}$$

The proof of Theorem 4.3 hinges on three technical lemmas that we state next.

Lemma 4.4. *Choose and fix $t \geqslant 0$, $\tau \in [0\,,1]$, and $x \in \mathbf{Z}^d$, and define*

$$\begin{aligned}
\mathscr{A} &:= \sum_{y\in\mathbf{Z}^d} \int_t^{t+\tau} p_{t+\tau-s}(y-x)\sigma(u_s(y))\,\mathrm{d}B_s(y),\\
\mathscr{B} &:= \int_t^{t+\tau} \sigma(u_s(x))\,\mathrm{d}B_s(x).
\end{aligned} \tag{4.28}$$

Then, for all real numbers $k \geqslant 2$ there exist a finite constant $C_k > 0$—depending on k but not on $(t\,,\tau,x)$—and a finite constant $C > 0$—not depending on $(t\,,\tau,x\,,k)$—such that

$$\mathrm{E}\left(|\mathscr{A} - \mathscr{B}|^k\right) \leqslant C_k \mathrm{e}^{Ck^2(t+1)}\tau^{3k/2}. \tag{4.29}$$

Lemma 4.5. *For every $k \geqslant 2$ and $T \geqslant 1$ there exists a finite constant $C(k\,,T)$ such that for every $\tau \in (0\,,1]$,*

$$\sup_{t\in[0,T]} \sup_{x\in\mathbf{Z}^d} \mathrm{E}\left(\left|u_{t+\tau}(x) - u_\tau^{(t)}(x)\right|^k\right) \leqslant C(k\,,T)\tau^{3k/2}. \tag{4.30}$$

Lemma 4.6. *There exists a version of $u^{(\bullet)}$ that is a.s. continuous in $(t\,,\tau)$. Moreover, for every $T \geqslant 1$, $\varepsilon \in (0\,,1)$ and $k \geqslant 2$,*

$$\sup_{t\in[0,T]} \sup_{x\in\mathbf{Z}^d} \sup_I \mathrm{E}\left(\sup_{\substack{\nu,\mu\in I\\ \nu\neq\mu}} \left[\frac{|u_\nu^{(t)}(x) - u_\mu^{(t)}(x)|}{|\nu-\mu|^{(1-\varepsilon)/2}}\right]^k\right) < \infty, \tag{4.31}$$

where "$\sup_I$" denotes the supremum over all closed subintervals I of $[0\,,T]$ that have length $\leqslant 1$.

In order to maintain the flow of the discussion we prove Theorem 4.3 first. Then we conclude this section by establishing the three supporting lemmas mentioned above.

Proof of Theorem 4.3. Throughout the proof we choose and fix some $t \in [0\,,T]$ and $x \in \mathbf{Z}^d$.

Our plan is to prove that for all $\delta \in (0\,,1/2)$,

$$u_{t+\tau}(x) - u_\tau^{(t)}(x) = O\left(\tau^{3/2-\delta}\right) \qquad \text{as } \tau \downarrow 0, \qquad \text{a.s.} \tag{4.32}$$

Henceforth, we choose and fix some $\delta \in (0\,,1/2)$, and denote by A_k, A'_k, A''_k, etc. finite constants that depend only on a parameter $k \geqslant 2$ that will be selected later on during the course of the proof.

Thanks to Lemma 4.5, for all $k \geqslant 2$ and $\tau \in [0\,,1]$,

$$\mathrm{P}\left\{\left|u_{t+\tau}(x) - u_\tau^{(t)}(x)\right| \geqslant \frac{1}{3}\tau^{3/2-\delta}\right\} \leqslant C(k\,,T)\tau^{\delta k}. \tag{4.33}$$

We can choose k large enough and then apply the Borel–Cantelli lemma in order to deduce that with probability one,

$$\left|u_{t+\tau_n}(x) - u_{\tau_n}^{(t)}(x)\right| < \tau_n^{3/2-\delta} \quad \text{for all but a finite number of } n\text{'s}, \tag{4.34}$$

where $\tau_n := n^{\delta-(1/2)}$. Because $\tau_n - \tau_{n+1} \sim \text{const} \times n^{-1}\tau_n$ as $n \to \infty$, Hölder continuity ensures the following [Lemmas 4.1 and 4.6]: Uniformly for all $\tau \in [\tau_{n+1}, \tau_n]$,

$$\begin{aligned} |u_{t+\tau}(x) - u_{t+\tau_n}(x)| + \left|u_{\tau_n}^{(t)}(x) - u_\tau^{(t)}(x)\right| &= O\left([\tau_n/n]^{1/2-\delta}\right) \quad \text{a.s.} \\ &= O\left(\tau_n^{3/2-\delta}\right), \end{aligned} \tag{4.35}$$

thanks to the particular choice of the sequence $\{\tau_n\}_{n=1}^\infty$. The preceding two displays can now be combined to imply (4.27). □

Proof of Lemma 4.4. We may rewrite $\mathscr{B}$ as follows:

$$\mathscr{B} = \sum_{y\in\mathbf{Z}^d} \int_t^{t+\tau} \mathbf{1}_{\{0\}}(y-x)\sigma(u_s(y))\,\mathrm{d}B_s(y). \tag{4.36}$$

Therefore, the BDG Lemma 2.1 can be used to show that

$$\begin{aligned} &\left\{\mathrm{E}\left(|\mathscr{A} - \mathscr{B}|^k\right)\right\}^{2/k} \\ &\leqslant 4k \sum_{y\in\mathbf{Z}^d} \int_t^{t+\tau} \left[p_{t+\tau-s}(y-x) - \mathbf{1}_{\{0\}}(y-x)\right]^2 \left\{\mathrm{E}\left(|\sigma(u_s(y))|^k\right)\right\}^{2/k}\,\mathrm{d}s \\ &\leqslant 4kc_k^2\mathrm{e}^{2ck(t+1)}\left(\sum_{y\in\mathbf{Z}^d\setminus\{0\}} \int_0^\tau [p_s(y)]^2\,\mathrm{d}s + \int_0^\tau [1-p_s(0)]^2\,ds\right) \\ &\leqslant 4kc_k^2\mathrm{e}^{2ck(t+1)}2\int_0^\tau [1-p_s(0)]^2\,ds, \end{aligned} \tag{4.37}$$

where c, c_k appear in (4.11). Next we might observe that $p_s(0) = \mathrm{P}\{X_s = 0\} \geqslant \mathrm{P}\{N_s = 0\} = \mathrm{e}^{-s}$, where $\{N_s\}_{s\geqslant 0}$ denotes the underlying Poisson clock. Therefore, we obtain $\int_0^\tau [1-p_s(0)]^2\,\mathrm{d}s \leqslant (1/3)\tau^3$, and hence

$$\mathrm{E}\left(|\mathscr{A}-\mathscr{B}|^k\right) \leqslant (8/3)^{k/2}k^{k/2}c_k^k\mathrm{e}^{ck^2(t+1)}\tau^{3k/2}. \tag{4.38}$$

This implies the lemma. □

Proof of Lemma 4.5. In accord with (2.1), we may write $u_{t+\tau}(x)$ as

$$\left(\tilde{p}_{t+\tau} * u_0\right)(x) + \sum_{y\in\mathbf{Z}^d}\int_0^t p_{t+\tau-s}(y-x)\sigma(u_s(y))\,\mathrm{d}B_s(y) + \mathscr{A}, \tag{4.39}$$

where $\mathscr{A}$ was defined in Lemma 4.4.

By the Chapman–Kolmogorov property of the transition functions $\{p_t\}_{t\geqslant 0}$,

$$(\tilde{p}_\tau * u_t)(x) = (\tilde{p}_{t+\tau} * u_0)(x) + \sum_{y\in\mathbf{Z}^d}\int_0^t p_{t+\tau-s}(y-x)\sigma\left(u_s(y)\right)\,\mathrm{d}B_s(y). \tag{4.40}$$

The exchange of summation with stochastic integration can be justified, using the already-proved moment bound (1.6) of Theorem 1.1; we omit the details. Instead, let us apply this in (4.39) to see that

$$\begin{aligned} u_{t+\tau}(x) &= (\tilde{p}_\tau * u_t)(x) + \int_t^{t+\tau}\sigma(u_s(x))\,\mathrm{d}B_s(x) + (\mathscr{A}-\mathscr{B}) \\ &= (\tilde{p}_\tau * u_t)(x) + \int_0^\tau \sigma(u_{t+s}(x))\,\mathrm{d}_sB_s^{(t)}(x) + (\mathscr{A}-\mathscr{B}). \end{aligned} \tag{4.41}$$

Lemma 4.4 implies that for all $k \geqslant 2$, $t, \tau \geqslant 0$, and $x \in \mathbf{Z}^d$,

$$\begin{aligned} &\mathrm{E}\left(\left|u_{t+\tau}(x) - (\tilde{p}_\tau * u_t)(x) - \int_0^\tau \sigma(u_{t+s}(x))\,\mathrm{d}_sB_s^{(t)}(x)\right|^k\right) \\ &\qquad\qquad \leqslant a_k\mathrm{e}^{ak^2(t+1)}\tau^{3k/2}, \end{aligned} \tag{4.42}$$

where $a \in (0\,,\infty)$ is universal and $a_k \in (0\,,\infty)$ depends only on k. On the other hand,

$$u_\tau^{(t)}(x) - (\tilde{p}_\tau * u_t)(x) - \int_0^\tau \sigma\left(u_s^{(t)}(x)\right)\,\mathrm{d}B_s^{(t)}(x) = 0 \qquad \text{a.s.}, \tag{4.43}$$

by the very definition of $u^{(t)}$, and thank to the fact that $u_0^{(t)}(y) = u_t(y)$. The preceding two displays and Minkowski's inequality that

$$\psi(\tau) := \left\{\mathrm{E}\left(\left|u_{t+\tau}(x) - u_\tau^{(t)}(x)\right|^k\right)\right\}^{1/k} \leqslant a_k^{1/k}\mathrm{e}^{ak(t+1)}\tau^{3/2} + Q, \tag{4.44}$$

where

$$Q := \left\{ \mathrm{E}\left(\left| \int_0^\tau \left[\sigma(u_{t+s}(x)) - \sigma\left(u_s^{(t)}(x)\right)\right] \,\mathrm{d}_s B_s^{(t)}(x) \right|^k \right) \right\}^{1/k}. \tag{4.45}$$

According to the BDG Lemma 2.1 [actually we need a one-dimensional version of that lemma only], and since $|\sigma(r) - \sigma(\rho)| \leqslant \mathrm{Lip}_\sigma |r - \rho|$,

$$\begin{aligned} Q^2 &\leqslant 4k\mathrm{Lip}_\sigma^2 \int_0^\tau \left\{ \mathrm{E}\left(\left| u_{t+s}(x) - u_s^{(t)}(x) \right|^k \right) \right\}^{2/k} \mathrm{d}s \\ &= 4k\mathrm{Lip}_\sigma^2 \int_0^\tau [\psi(s)]^2 \,\mathrm{d}s. \end{aligned} \tag{4.46}$$

Thus, we find that

$$[\psi(\tau)]^2 \leqslant 2a_k^{2/k} \mathrm{e}^{2ak(t+1)} \tau^3 + 8k\mathrm{Lip}_\sigma^2 \int_0^\tau [\psi(s)]^2 \,\mathrm{d}s \qquad \text{for all } 0 \leqslant \tau \leqslant 1. \tag{4.47}$$

The lemma follows from this and an application of Gronwall's lemma. □

Proof of Lemma 4.6. One can model closely a proof after that of Lemma 4.1. However we omit the details, since this is a result about finite-dimensional diffusions and as such simpler than Lemma 4.1. □

We conclude this section with a final approximation lemma. The next assertion shows that the solution to (SHE) depends continuously on its initial function [in a suitable topology].

Lemma 4.7. *Let u and v denote the unique solutions to* (SHE)*, corresponding respectively to initial functions u_0 and v_0. Then,*

$$\sup_{x \in \mathbf{Z}^d} \mathrm{E}\left(|u_t(x) - v_t(x)|^2 \right) \leqslant \|u_0 - v_0\|_{\ell^\infty(\mathbf{Z}^d)}^2 \mathrm{e}^{\mathrm{Lip}_\sigma^2 t} \quad \text{for all } t \geqslant 0. \tag{4.48}$$

Proof. Choose and fix $t \geqslant 0$. The fact that $\sum_{y \in \mathbf{Z}^d} p_t(y) = 1$ alone ensures that

$$\sup_{x \in \mathbf{Z}^d} |(\tilde{p}_t * u_0)(x) - (\tilde{p}_t * v_0)(x)| \leqslant \|u_0 - v_0\|_{\ell^\infty(\mathbf{Z}^d)}. \tag{4.49}$$

Therefore, (2.1) and Itô's isometry together imply that

$$\begin{aligned} &\mathrm{E}\left(|u_t(x) - v_t(x)|^2 \right) \\ &\leqslant \|u_0 - v_0\|_{\ell^\infty(\mathbf{Z}^d)}^2 + \mathrm{Lip}_\sigma^2 \int_0^t \|p_s\|_{\ell^2(\mathbf{Z}^d)}^2 \cdot \sup_{y \in \mathbf{Z}^d} \mathrm{E}\left(|u_s(y) - v_s(y)|^2 \right) \mathrm{d}s. \end{aligned} \tag{4.50}$$

Since $\|p_s\|_{\ell^2(\mathbf{Z}^d)}^2 = \mathrm{P}\{X_s = X_s'\} \leqslant 1$, where X' is an independent copy of X, we may conclude that $f(t) := \sup_{x \in \mathbf{Z}^d} \mathrm{E}(|u_t(x) - v_t(x)|^2)$ satisfies

$$f(t) \leqslant \|u_0 - v_0\|_{\ell^\infty(\mathbf{Z}^d)}^2 + \mathrm{Lip}_\sigma^2 \int_0^t f(s) \,\mathrm{d}s. \tag{4.51}$$

Therefore, the lemma follows from Gronwall's inequality. □

5 Proof of Theorem 1.1: Part 2

We now return to the proof of Theorem 1.1, and complete it by verifying the two remaining assertions of that theorem: (i) The solution is nonnegative because $u_0(x) \geqslant 0$ and $\sigma(0) = 0$; and (ii) The lower bound (1.7) for the lower Lyapunov exponent holds. It is best to keep the two parts separate, as they use different ideas.

Theorem 5.1 (Comparison principle)**.** *Suppose u and v are the solutions to* (SHE) *with respective initial functions u_0 and v_0. If $u_0(x) \geqslant v_0(x)$ for all $x \in \mathbf{Z}^d$, then $u_t(x) \geqslant v_t(x)$ for all $t \geqslant 0$ and $x \in \mathbf{Z}^d$ a.s.*

The nonnegativity assertion of Theorem 1.1 is well known [35], but also follows from the preceding comparison principle. This is because the condition (1.1) implies that $v_t(x) \equiv 0$ is the unique solution to (SHE) with initial condition $v_0(x) \equiv 0$. Therefore, the comparison principle yields $u_t(x) \geqslant v_t(x) = 0$ a.s.

Proof of Theorem 5.1. Consider the following infinite dimensional SDE:

$$w_t(x) = w_0(x) + \int_0^t (\mathscr{L} w_s)(x)\,\mathrm{d}s + \int_0^t \sigma(w_s(x))\mathrm{d}B_s(x) \qquad (x \in \mathbf{Z}^d). \quad (5.1)$$

It is a well-known fact that the mild solution to (SHE) is also a solution in the weak sense. See, for example, Theorem 3.1 of Iwata [30] and its proof. Therefore, $u_t(x)$ and $v_t(x)$ respectively solve (5.1) with initial conditions $u_0(x)$ and $v_0(x)$.

Let $\{S_n\}_{n=1}^{\infty}$ denote a growing sequence of finite subsets of $\mathbf{Z}^d$ that exhaust all of $\mathbf{Z}^d$. Consider, for every $n \geqslant 1$, the stochastic integral equation,

$$\begin{cases} w_t^{(n)}(x) = w_0(x) + \displaystyle\int_0^t (\mathscr{L} w_s^{(n)})(x)\,\mathrm{d}s + \int_0^t \sigma(w_s^{(n)}(x))\mathrm{d}B_s(x) & \text{if } x \in S_n; \\ w_t^{(n)}(x) = w_0(x) & \text{if } x \notin S_n. \end{cases} \quad (5.2)$$

Similarly, we let $v^{(n)}$ solve the same equation, but start it as $v_0(x)$.

Each of these equations is in fact a finite-dimensional SDE, and has a unique strong solution, by Itô's theory. Moreover, Shiga and Shimizu's proof of their Theorem 2.1 [36] shows that, for every $x \in \mathbf{Z}^d$ and $t > 0$, there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ of increasing integers such that

$$w_t^{(n_k)}(x) \xrightarrow{\mathrm{P}} w_t(x) \quad \text{and} \quad v_t^{(n_k)}(x) \xrightarrow{\mathrm{P}} v_t(x), \quad (5.3)$$

as $k \to \infty$. Therefore, we may appeal to a comparison principle for finite-dimensional SDEs, such as that of Geiß and Manthey [26, Theorem 1.2], in order to conclude the result; the quasi-monotonicity condition of [26] is met simply because $\mathscr{L}$ is the generator of a Markov chain. The verification of that small detail is left to the interested reader. ☐

We are now in position to establish the lower bound (1.7) on the bottom Lyapunov exponent of the solution to (SHE).

Proof of Theorem 1.1: Verification of (1.7). Let v solve the stochastic heat equation

$$\mathrm{d}v_t(x) = (\mathscr{L}v_t)(x)\,\mathrm{d}t + \ell_\sigma v_t(x)\,\mathrm{d}B_t(x), \tag{5.4}$$

subject to $v_0(x) := u_0(x)$. Also define $V^{(N)}$ to be the solution to

$$\mathrm{d}V_t^{(N)}(x) = (\mathscr{L}v_t)(x)\,\mathrm{d}t + \zeta^{(N)}\left(V_t^{(N)}(x)\right)\,\mathrm{d}B_t(x), \tag{5.5}$$

where $\zeta^{(N)}(x) = \ell_\sigma x$ on $(-N\,,N)$, $\zeta^{(N)}(x) = 0$ when $|x| \geqslant N+1$, and $\zeta^{(N)}$ is defined by linear interpolation everywhere else.

Define $\sigma^{(N)}$ and $U^{(N)}$ as in Lemma 4.2. Because $\sigma^{(N)} \geqslant \zeta^{(N)}$ everywhere on $\mathbf{R}_+$, and since both $U^{(N)}$ and $V^{(N)}$ are $\geqslant 0$ a.s. and pointwise, the comparison theorem of Cox, Fleischmann, and Greven [17, Theorem 1] shows us that

$$\mathrm{E}\left(\left|V_t^{(N)}(x)\right|^k\right) \leqslant \mathrm{E}\left(\left|U_t^{(N)}(x)\right|^k\right), \tag{5.6}$$

for all $t \geqslant 0$, $x \in \mathbf{Z}^d$, $k \geqslant 2$, and $N \geqslant 1$. Let $N \to \infty$ and apply Lemma 4.2 to find that $V_t^{(N)}(x) \to v_t(x)$ and $U_t^{(N)}(x) \to u_t(x)$ in $L^k(\mathrm{P})$ for all $k \geqslant 2$. Therefore, the preceding display shows us that

$$\mathrm{E}\left(|v_t(x)|^k\right) \leqslant \mathrm{E}\left(|u_t(x)|^k\right). \tag{5.7}$$

Therefore, it suffices to bound $\underline{\gamma}_k(v)$ from below.

Let $\{X^{(i)}\}_{i=1}^k$ denote k independent copies of the random walk X. Then it is possible to prove that

$$\mathrm{E}\left(|v_t(x)|^k\right) = \mathrm{E}\left(\prod_{j=1}^k u_0\left(X_t^{(j)} + x\right)\cdot \mathrm{e}^{M_k(t)}\right), \tag{5.8}$$

where $M_k(t)$ denotes the "multiple collision local time,"

$$M_k(t) := 2\ell_\sigma^2 \sum\sum_{1\leqslant i<j\leqslant k} \int_0^t \mathbf{1}_{\{0\}}\left(X_s^{(i)} - X_s^{(j)}\right)\,\mathrm{d}s. \tag{5.9}$$

In the case that X is the continuous-time simple random walk on $\mathbf{Z}^d$, this is a well-known consequence of a Feynman–Kac formula; see, for instance, Carmona and Molchanov [13, p. 19]. When X is replaced by a Lévy process, Conus [15] has found an elegant derivation of this formula. The class of all Lévy processes includes that of continuous-time random walks, whence follows (5.8).

Finally, we note that if every walk $X^{(1)}, \ldots, X^{(k)}$ does not jump in the time interval $[0\,,t]$, then certainly

$$\prod_{j=1}^k u_0\left(X^{(j)} + x\right)\mathrm{e}^{M_k(t)} \geqslant [u_0(x)]^k \mathrm{e}^{k(k-1)\ell_\sigma^2 t}. \tag{5.10}$$

Since the probability is $\exp(-t)$ that $X^{(j)}$ does not jump in $[0\,,t]$, it follows from the independence of $X^{(1)},\ldots,X^{(k)}$ that

$$\mathrm{E}\left(|v_t(x)|^k\right) \geqslant [u_0(x)]^k \exp\left\{\left[k(k-1)\ell_\sigma^2 - k\right] t\right\}, \tag{5.11}$$

whence

$$\underline{\gamma}_k(u) \geqslant \underline{\gamma}_k(v) \geqslant k(k-1)\ell_\sigma^2 - k, \tag{5.12}$$

since u_0 is not identically zero. If k is at least $\varepsilon^{-1} + (\varepsilon\ell_\sigma^2)^{-1}$, then we certainly have $k(k-1)\ell_\sigma^2 - k \geqslant (1-\varepsilon)k^2\ell_\sigma^2$, and the theorem follows. □

6 Proof of Theorem 1.2

Theorem 1.2 is a consequence of the following result.

Proposition 6.1. *For every $t \geqslant 0$, the following holds a.s. for all $x \in \mathbf{Z}^d$:*

$$u_{t+\tau}(x) - u_t(x) = \sigma\left(u_t(x)\right)\left\{B_{t+\tau}(x) - B_t(x)\right\} + o\left(\tau^{1+o(1)}\right) \quad \text{as } \tau \downarrow 0. \tag{6.1}$$

Indeed, we obtain (1.8) from this proposition, simply because well-known properties of Brownian motion imply that for all $\varepsilon \in (0\,,1/2)$ and $t \geqslant 0$,

$$\lim_{\tau\downarrow 0} \frac{\tau^{1-\varepsilon}}{B_{t+\tau}(x) - B_t(x)} = 0 \quad \text{in probability.} \tag{6.2}$$

Moreover, (1.9) follows from the local law of the iterated logarithm for Brownian motion. It remains to prove Proposition 6.1.

Proof. According to (4.42), for every integer $k \geqslant 2$, and all $t, \tau \geqslant 0$ and $x \in \mathbf{Z}^d$,

$$\begin{aligned}\mathrm{E}&\left(\left|u_{t+\tau}(x) - u_t(x) - \int_0^\tau \sigma(u_{t+s}(x))\,\mathrm{d}_s B_s^{(t)}(x)\right|^k\right)\\ &\qquad\leqslant 2^{k-1}\left[a_k \mathrm{e}^{ak^2(t+1)}\tau^{3k/2} + \mathrm{E}\left(\left|u_t(x) - \left(\tilde{p}_\tau * u_t\right)(x)\right|^k\right)\right].\end{aligned} \tag{6.3}$$

We may write

$$\begin{aligned}\mathrm{E}&\left(\left|u_t(x) - \left(\tilde{p}_\tau * u_t\right)(x)\right|^k\right)\\ &= \mathrm{E}\left(\left|u_t(x) - \sum_{y\in\mathbf{Z}^d} p_\tau(y-x)u_t(y)\right|^k\right)\\ &= \mathrm{E}\left(\left|u_t(x)\mathrm{P}\{X_\tau \neq 0\} - \sum_{y\in\mathbf{Z}^d\setminus\{x\}} p_\tau(y-x)u_t(y)\right|^k\right).\end{aligned} \tag{6.4}$$

Because $\mathrm{P}\{X_\tau \neq 0\} = 1 - \exp(-\tau) \leqslant \tau$, Minkowski's inequality shows that

$$\begin{aligned}
&\left\{\mathrm{E}\left(\left|u_t(x) - (\tilde{p}_\tau * u_t)(x)\right|^k\right)\right\}^{1/k} \\
&\qquad\leqslant \tau\left\{\mathrm{E}\left(|u_t(x)|^k\right)\right\}^{1/k} + \sum_{y\in\mathbf{Z}^d\setminus\{x\}} p_\tau(y-x)\left\{\mathrm{E}\left(|u_t(y)|^k\right)\right\}^{1/k} \\
&\qquad\leqslant 2\tau \sup_{y\in\mathbf{Z}^d}\left\{\mathrm{E}\left(|u_t(y)|^k\right)\right\}^{1/k}.
\end{aligned} \tag{6.5}$$

We can conclude from this development, and from Theorem 1.1, that there exists $A_k < \infty$, depending only on k, and a universal $A < \infty$ such that

$$\begin{aligned}
&\mathrm{E}\left(\left|u_{t+\tau}(x) - u_t(x) - \int_0^\tau \sigma(u_{t+s}(x))\,\mathrm{d}_s B_s^{(t)}(x)\right|^k\right) \\
&\qquad\qquad\leqslant A_k \mathrm{e}^{Ak^2(t+1)}\left[\tau^{3k/2} + \tau^k\right] \leqslant A_k\mathrm{e}^{Ak^2(t+1)}\tau^k,
\end{aligned} \tag{6.6}$$

for all $\tau \in [0\,,1]$. Now, we may apply the BDG Lemma 2.1 in order to see that

$$\begin{aligned}
&\left\{\mathrm{E}\left(\left|\int_0^\tau \sigma(u_{t+s}(x))\,\mathrm{d}_s B_s^{(t)}(x) - \sigma(u_t(x))B_\tau^{(t)}(x) + \sigma(u_t(x))B_0^t(x)\right|^k\right)\right\}^{2/k} \\
&\qquad = \left\{\mathrm{E}\left(\left|\int_0^\tau \left\{\sigma(u_{t+s}(x)) - \sigma(u_t(x))\right\}\,\mathrm{d}_s B_s^{(t)}(x)\right|^k\right)\right\}^{2/k} \\
&\qquad \leqslant 4k\mathrm{Lip}_\sigma^2 \int_0^\tau \left\{\mathrm{E}\left(\left|u_{t+s}(x) - u_t(x)\right|^k\right)\right\}^{2/k}\,\mathrm{d}s.
\end{aligned} \tag{6.7}$$

Thanks to (4.15),

$$\begin{aligned}
&\sup_{x\in\mathbf{Z}^d}\left\{\mathrm{E}\left(\left|\int_0^\tau \sigma(u_{t+s}(x))\,\mathrm{d}_s B_s^{(t)}(x) - \sigma(u_t(x))B_\tau^{(t)}(x)\right|^k\right)\right\}^{2/k} \\
&\qquad\qquad \leqslant \tilde{a}_k \mathrm{e}^{\tilde{a}t}\int_0^\tau s\,\mathrm{d}s \leqslant \mathrm{const}\cdot\tau^2.
\end{aligned} \tag{6.8}$$

Therefore, we can deduce from (6.6) that

$$\mathrm{E}\left(|D(\tau)|^k\right) \leqslant c_{k,t}\tau^k \qquad (0 \leqslant \tau \leqslant 1), \tag{6.9}$$

where

$$D(\tau) := u_{t+\tau}(x) - u_t(x) - \sigma(u_t(x))\left\{B_{t+\tau}(x) - B_t(x)\right\}, \tag{6.10}$$

and $c_{k,t}$ is a finite constant that depends only on k and t; in particular, $c_{k,t}$ does not depend on τ. Now we choose and fix some $\eta > \xi > 0$ such that

$\eta+\xi < 1/2$, and then apply the Chebyshev inequality, and the preceding with any choice of integer $k > \xi^{-1}$, in order to see that $\sum_{n=1}^{\infty} \mathrm{P}\{|D(n^{-\eta})| > n^{-(\eta-\xi)}\} \leqslant c_{k,t} \sum_{n=1}^{\infty} n^{-\xi k} < \infty$. Thus,

$$D\left(n^{-\eta}\right) = O\left(n^{-(\eta-\xi)}\right) \qquad \text{a.s.,} \tag{6.11}$$

thanks to the Borel–Cantelli lemma. Because $n^{-\eta} - (n+1)^{-\eta} = O(n^{-1-\eta})$, the modulus of continuity of Brownian motion, together with Lemma 4.1, imply that

$$\sup_{(n+1)^{-\eta} \leqslant \tau \leqslant n^{-\eta}} \left|D\left(n^{-\eta}\right) - D(\tau)\right| = O\left(n^{-1/2}\right) = o\left(n^{-(\eta+\xi)}\right) \qquad \text{a.s.} \tag{6.12}$$

Therefore a standard monotonicity argument and (6.11) together reveal that $D(t) = O(t^{(\eta-\xi)/\eta})$ as $t \downarrow 0$, a.s. Since $\eta > \xi$ are arbitrary positive numbers, it follows that $\limsup_{t\downarrow 0}(\log D(t)/\log t) \leqslant 1$ a.s. This is another way to state the result. □

7 Proof of Theorem 1.5.

First we prove a preliminary lemma that guarantees strict positivity of the solution to the (SHE). We follow the method described in Conus, Joseph, and Khoshnevisan [16, Theorem 5.1], which in turn borrowed heavily from ideas of Mueller [31] and Mueller and Nualart [32].

Lemma 7.1. $\inf_{0\leqslant t\leqslant T} u_t(x) > 0$ *a.s. for every* $T \in (0\,,\infty)$ *and all* $x \in \mathbf{Z}^d$ *that satisfy* $u_0(x) > 0$.

Proof. We are going to prove that if $u_0(x_0) > 0$ for a fixed $x \in \mathbf{Z}^d$, then there exist finite and positive constants A and C such that

$$\mathrm{P}\left\{\inf_{0<s<t} u_s(x_0) \leqslant \varepsilon\right\} \leqslant A\varepsilon^{C\log|\log\varepsilon|}, \tag{7.1}$$

for that same point x_0, uniformly for all $\varepsilon \in (0\,,1)$. It turns out to be convenient to prove the following equivalent formulation of the preceding:

$$\mathrm{P}\left\{\inf_{0<s<t} u_s(x_0) \leqslant \mathrm{e}^{-n}\right\} \leqslant An^{-Cn}, \tag{7.2}$$

simultaneously for all $n \geqslant 1$, after a possible relabeling of the constants $A, C \in (0\,,\infty)$. If so, then we can simply let $n \to \infty$ and deduce the lemma. Without loss of too much generality we assume that $u_0(0) > 0$, and aim to prove (7.2) with $x_0 = 0$. In fact, we will simplify the exposition further and establish (7.2) in the case that $u_0(0) = 1$; the general case follows from this one and scaling. Finally, we appeal to the comparison theorem (Theorem 5.1) in order to reduce our problem further to the special case that

$$u_0(x) = \delta_0(x) \qquad \text{for all } x \in \mathbf{Z}^d. \tag{7.3}$$

Thus, we consider this case only from now on.

Let $\mathcal{F}_t := \sigma\{B_s(x) : x \in \mathbf{Z}^d, 0 < s \leqslant t\}$ describe the filtration generated by time t by all the Brownian motions, enlarged so that $t \mapsto u_t$ is a $C(\mathbf{R})$-valued [strong] Markov chain. Set $T_0 := 0$, and define iteratively for $k \geqslant 0$ the sequence of $\{\mathcal{F}_t\}_{t>0}$-stopping times

$$T_{k+1} := \inf\left\{s > T_k : u_s(0) \leqslant \mathrm{e}^{-k-1}\right\}, \tag{7.4}$$

using the usual convention that $\inf \varnothing := \infty$. We may observe that the preceding definitions imply that, almost surely on $\{T_k < \infty\}$,

$$u_{T_k}(x) \geqslant \mathrm{e}^{-k}\delta_0(x) \qquad \text{for all } x \in \mathbf{Z}^d. \tag{7.5}$$

We plan to apply the strong Markov property. In order to do that, we first define $u^{(k+1)}$ to be the unique continuous solution to the (SHE) (for same Brownian motions, pathwise), with initial data $u_0^{(k+1)}(x) := \mathrm{e}^{-k}\delta_0(x)$. Next we note that, for every $k \geqslant 0$, the random field

$$w_t^{k+1}(x) := \mathrm{e}^k u_t^{(k+1)}(x) \tag{7.6}$$

solves the system

$$\begin{cases} \dfrac{\mathrm{d}w_t^{(k+1)}(x)}{\mathrm{d}t} = (\mathscr{L} w_t^{(k+1)})(x) + \sigma_k\left(w_t^{(k+1)}(x)\right)\dfrac{\mathrm{d}B_t(x)}{\mathrm{d}t} \\ w_0^{(k+1)}(x) = \delta_0(x), \end{cases} \tag{7.7}$$

where $\sigma_k(y) := \mathrm{e}^k\sigma(\mathrm{e}^{-k}y)$. Because $\sigma(0) = 0$, we have $\mathrm{Lip}_{\sigma_k} = \mathrm{Lip}_\sigma$, uniformly for all $k \geqslant 1$. Thus, we can keep track of the constants in the proof of Lemma 4.1, in order to deduce the existence of a finite constant $K := K(\varepsilon)$ so that for all t, s with $|t - s| < 1$,

$$\mathrm{E}\left(\sup_{0<|t-s|<1} |w_t^{(k+1)}(0) - w_s^{(k+1)}(0)|^m\right) \leqslant Km^2\mathrm{e}^{Km^2}|t-s|^{m(1-\varepsilon)/2}, \tag{7.8}$$

for all real numbers $m \geqslant 2$.

For each $k \geqslant 0$ let us define

$$T_1^{(k+1)} = \inf\left\{t > 0 : w_t^{(k+1)}(0) \leqslant \mathrm{e}^{-1}\right\}. \tag{7.9}$$

Equation (7.5), the strong Markov property, and the comparison principle [Theorem 5.1] together imply that outside of a null set, the solution to the revised SPDE (7.7) satisfies

$$\mathrm{e}^{-k} w_t^{(k+1)}(x) \leqslant u_{T_k+t}(x). \tag{7.10}$$

Therefore, in particular,

$$T_1^{(k+1)} \leqslant T_{k+1} - T_k, \tag{7.11}$$

and the stopping times $T_1^{(k+1)}$ and $T_1^{(\ell+1)}$ are independent if $k \neq \ell$. For every $t < 1$,

$$\begin{aligned} \mathrm{P}\left\{T_1^{(k+1)} < t\right\} &\leqslant \mathrm{P}\left\{\sup_{0<s<t}\left|w_t^{(k+1)}(0) - w_s^{(k+1)}(0)\right| \geqslant 1 - \mathrm{e}^{-1}\right\} \\ &\leqslant K(\varepsilon) m^2 \mathrm{e}^{Km^2}\left(1 - \mathrm{e}^{-1}\right)^{-m} t^{(1-\varepsilon)m/2}, \end{aligned} \tag{7.12}$$

where the last inequality follows by Chebyshev's inequality and (7.8), and is valid for all $0 < \varepsilon < 1$. Let us emphasize that the constant of the bound in (7.12) does not depend on the parameter k which appears in the superscript of the random variable $T_1^{(k+1)}$. Now we compute

$$\begin{aligned} \mathrm{P}\left\{\inf_{0<s\leqslant t} u_s(0) \leqslant \mathrm{e}^{-n}\right\} &= \mathrm{P}\{T_n \leqslant t\} \\ &= \mathrm{P}\{(T_n - T_{n-1}) + \ldots + (T_1 - T_0) \leqslant t\} \\ &\leqslant \mathrm{P}\left\{T_1^{(n)} + T_1^{(n-1)} + \cdots + T_1^{(1)} \leqslant t\right\}, \end{aligned} \tag{7.13}$$

owing to (7.11).

The terms $T_1^{(n)}, \ldots, T_1^{(1)}$, that appear in the ultimate line of (7.13), are independent non-negative random variables. Thanks to the pigeon-hole principle, if the sum of those terms is at most t, then certainly it must be that at least $n/2$ of those terms are at most $t/2n$. If n is an even integer, larger than $t > 2$, then a simple union bound on (7.13) and (7.12) yields

$$\begin{aligned} &\mathrm{P}\left\{\inf_{0<s\leqslant t} u_s(0) \leqslant \mathrm{e}^{-n}\right\} \\ &\leqslant \binom{n}{n/2} K(\varepsilon)^{n/2} m^n \mathrm{e}^{Km^2 n}\left(1 - \mathrm{e}^{-1}\right)^{-mn} t^{(1-\varepsilon)mn/2} (2n)^{-(1-\varepsilon)mn/2} \\ &\leqslant \widetilde{K}(\varepsilon)^n 4^n m^n \mathrm{e}^{Km^2 n}\left(1 - \mathrm{e}^{-1}\right)^{-mn} t^{(1-\varepsilon)mn/4} (2n)^{-(1-\varepsilon)mn/4}. \end{aligned} \tag{7.14}$$

Now we set $m := \log n / \log\log n$ in (7.14) in order to deduce (7.2) for $x_0 = 0$ and every $n \geqslant 1$ sufficiently large. This readily yields (7.2) in its entirety, and concludes this demonstration. □

Next we show that if we start with an initial profile u_0 such that $u_0(x) > 0$ *for at least one point* $x \in \mathbf{Z}^d$, then $u_t(z) > 0$ *for all* $z \in \mathbf{Z}^d$ and $t > 0$ a.s. Because we are interested in establishing a lower bound, we may apply scaling and a comparison theorem (Theorem 5.1) in order to reduce our problem to the special case that

$$u_0 = \delta_0. \tag{7.15}$$

In this way, we are led to the following representation of the solution:

$$u_t(x) = p_t(x) + \int_0^t \sum_{y \in \mathbf{Z}^d} p_{t-s}(y - x)\sigma\big(u_s(y)\big)\,\mathrm{d}B_s(y). \tag{7.16}$$

Proposition 7.2. *If $u_0 = \delta_0$, then $u_t(x) > 0$ for all $x \in \mathbf{Z}^d$ and $t > 0$ a.s.*

Proposition 7.2 follows from a few preparatory lemmas.

Lemma 7.3. *If $u_0 = \delta_0$, then*

$$\mathrm{E}\left(|u_s(y)|^2\right) \leqslant \mathrm{e}^{\mathrm{Lip}_\sigma^2 s}[p_s(y)]^2 \qquad \text{for all } s > 0 \text{ and } y \in \mathbf{Z}^d. \tag{7.17}$$

Proof. We begin with the representation (7.16) of the solution u, in integral form, and appeal to Picard's iteration in order to prove the lemma.

Let $u_t^{(0)}(x) := 1$ for all $t \geqslant 0$, $x \in \mathbf{Z}^d$, and then let $\{u^{(n+1)}\}_{n\geqslant 0}$ be defined iteratively by

$$u_t^{(n+1)}(x) := p_t(x) + \int_0^t \sum_{y\in\mathbf{Z}^d} p_{t-s}(y-x)\sigma\left(u_s^{(n)}(y)\right) \mathrm{d}B_s(y). \tag{7.18}$$

Let us define

$$M_t^{(k)} := \sup_{x\in\mathbf{Z}^d} \mathrm{E}\left(\left|\frac{u_t^{(n+1)}(x)}{p_t(x)}\right|^2\right), \tag{7.19}$$

and apply Itô's isometry in order to deduce the recursive inequality for the $M^{(k)}$'s:

$$M_t^{(n+1)} \leqslant 1 + \mathrm{Lip}_\sigma^2 \cdot \sup_{x\in\mathbf{Z}^d} \int_0^t \sum_y \left[\frac{p_{t-s}(y-x)p_s(y)}{p_t(x)}\right]^2 M_s^{(n)}\, \mathrm{d}s. \tag{7.20}$$

Because $\sum_{y\in\mathbf{Z}^d}[f(y)]^2 \leqslant [\sum_{y\in\mathbf{Z}^d} f(y)]^2$ for all $f : \mathbf{Z}^d \to \mathbf{R}_+$, the semigroup property of $\{p_t\}_{t>0}$ yields the bound

$$\sum_{y\in\mathbf{Z}^d} [p_{t-s}(y-x)p_s(y)]^2 \leqslant [p_t(x)]^2, \tag{7.21}$$

whence $M_t^{(n+1)} \leqslant 1 + \mathrm{Lip}_\sigma^2 \cdot \int_0^t M_s^{(n)}\, \mathrm{d}s$ for all $t > 0$ and $n \geqslant 0$. It follows readily from this that $M_t^{(n)} \leqslant \exp(\mathrm{Lip}_\sigma^2 t)$, uniformly for all $n \geqslant 0$ and $t > 0$; equivalently,

$$\mathrm{E}\left(|u_t^{(n)}(x)|^2\right) \leqslant \mathrm{e}^{\mathrm{Lip}_\sigma^2 t}[p_t(x)]^2, \tag{7.22}$$

uniformly for all $n \geqslant 0$, $x \in \mathbf{Z}^d$, and $t > 0$. The lemma follows from this and Fatou's lemma, since $u_t^{(n)}(x) \to u_t(x)$ in $L^2(\mathrm{P})$ as $n \to \infty$. □

Our next lemma shows that the random term on the right-hand side of (7.16) is small, for small time, as compared with the nonrandom term in (7.16).

Lemma 7.4. *Assume the conditions of Proposition 7.2. Then there exists a finite constant $C > 0$ such that for all $t \in (0\,,1)$,*

$$\sup_{x\in\mathbf{Z}^d} \mathrm{P}\left\{\left|\int_0^t \sum_{y\in\mathbf{Z}^d} p_{t-s}(y-x)\sigma\left(u_s(y)\right) \mathrm{d}B_s(y)\right| > \frac{p_t(x)}{2}\right\} \leqslant Ct. \tag{7.23}$$

Proof. By Lemma 7.3 and Itô's isometry,

$$\mathrm{E}\left(\left|\int_0^t \sum_{y\in\mathbf{Z}^d} p_{t-s}(y-x)\sigma\left(u_s(y)\right)\mathrm{d}B_s(y)\right|^2\right) \tag{7.24}$$
$$\leqslant \mathrm{Lip}_\sigma^2 \cdot \int_0^t \sum_{y\in\mathbf{Z}^d} [p_{t-s}(y-x)p_s(y)]^2\, \mathrm{e}^{\mathrm{Lip}_\sigma^2 s}\,\mathrm{d}s \leqslant \mathrm{Lip}_\sigma^2 [p_t(x)]^2 \cdot \int_0^t \mathrm{e}^{\mathrm{Lip}_\sigma^2 s}\,\mathrm{d}s,$$

where we have used (7.21) in the last inequality. Because $\int_0^t \exp(\mathrm{Lip}_\sigma^2 s)\,\mathrm{d}s \leqslant ct$ for all $t\in(0\,,1)$ with $c := \exp(\mathrm{Lip}_\sigma^2)$, the lemma follows from Chebyshev's inequality. ☐

Now we can establish Proposition 7.2.

Proof of Proposition 7.2. Let us choose and fix an arbitrary $x\in\mathbf{Z}^d$. By the strong Markov property of the solution, and thanks to Lemma 7.1, we know that once the solution becomes positive at a point, it remains positive at that point at all future times, almost surely. Thus, it suffices to show that $u_t(x)>0$ for all times of the form $t=2^{-k}$, when k is a large enough integer. But this is immediate from (7.16) and (7.23), thanks to the Borel-Cantelli lemma. ☐

The preceding lemmas lay the groundwork for the proof of Theorem 1.5. We now proceed with the main proof.

Proof of Theorem 1.5. Let us first consider the case that $m=1$ and without loss of generality, $x_1=0$. In that case, we write

$$\lim_{\tau\downarrow 0}\mathrm{P}\left\{S(u_{t+\tau}(0)) - S(u_t(0)) \leqslant q\sqrt{\tau}\right\} \tag{7.25}$$
$$= \lim_{\tau\downarrow 0}\mathrm{P}\left\{\int_{u_t(0)}^{u_{t+\tau}(0)} \frac{\mathrm{d}y}{\sigma(y)} \leqslant q\sqrt{\tau}\right\}$$
$$= \lim_{\tau\downarrow 0}\mathrm{P}\left\{\int_{u_t(0)}^{u_{t+\tau}(0)} \left(\frac{1}{\sigma(y)} - \frac{1}{\sigma(u_t(0))}\right)\mathrm{d}y + \frac{u_{t+\tau}(0)-u_t(0)}{\sigma(u_t(0))} \leqslant q\sqrt{\tau}\right\}.$$

Lemma 7.1 and the positivity condition on σ ensure that $\sigma(u_t(0))>0$ a.s. Therefore, the theorem follows from Theorem 1.2 if we were to show that

$$\frac{1}{\sqrt{\tau}}\int_{u_t(0)}^{u_{t+\tau}(0)} \left(\frac{1}{\sigma(y)} - \frac{1}{\sigma(u_t(0))}\right)\mathrm{d}y \to 0 \quad \text{almost surely, as } \tau\downarrow 0. \tag{7.26}$$

Let $\mathcal{I}(t,t+\tau)$ denote the random closed interval with endpoints $u_t(0)$ and $u_{t+\tau}(0)$. Our strict positivity result [Lemma 7.1] implies that

$$\mathcal{I}(t\,,t+\tau)\subset(0\,,\infty) \quad \text{for all } t,\tau>0 \text{ a.s.}, \tag{7.27}$$

and thus paves way for the a.s. bounds

$$\left|\int_{u_t(0)}^{u_{t+\tau}(0)}\left(\frac{1}{\sigma(y)}-\frac{1}{\sigma(u_t(0))}\right)\mathrm{d}y\right| \leqslant \mathrm{Lip}_\sigma\cdot\frac{|u_t(0)-u_{t+\tau}(0)|^2}{\inf_{y\in\mathcal{I}(t,t+\tau)}|\sigma(y)|^2}$$
$$= O\left(\tau\log|\log\tau|\right)\qquad(\tau\downarrow 0);$$

see (1.9) for the last part. This implies (7.26) and thus completes our proof for $m=1$. The proof for general m is an easy adaption since $\{B(x_j)\}_{j=1}^m$ are i.i.d. Brownian motions. □

8 Preliminaries for the proof of Theorem 1.7

The following function will play a prominent role in the ensuing analysis:

$$\bar{P}(\tau) := \|p_\tau\|^2_{\ell^2(\mathbf{Z}^d)} = \sum_{x\in\mathbf{Z}^d}[p_\tau(x)]^2 \qquad \text{for all } \tau\geqslant 0. \tag{8.1}$$

Because of the Chapman–Kolmogorov property, we can also think of $\bar{P}$ as

$$\bar{P}(\tau) := \mathrm{P}\{X_\tau - X'_\tau = 0\}, \tag{8.2}$$

where X' is an independent copy of X. There is another useful way to think of $\bar{P}$ as well. Namely, we apply (4.9) and the Plancherel theorem to see that

$$\begin{aligned}\bar{P}(\tau) &= (2\pi)^{-d}\int_{(-\pi,\pi)^d}\left|\mathrm{E}\exp(i\xi\cdot X_\tau)\right|^2\,\mathrm{d}\xi\\ &= (2\pi)^{-d}\int_{(-\pi,\pi)^d}\mathrm{e}^{-2\tau(1-\mathrm{Re}\,\varphi(\xi))}\,\mathrm{d}\xi,\end{aligned}\tag{8.3}$$

where $\varphi(\xi) = \mathrm{E}[\exp(i\xi\cdot Z_1)]$, recall that Z_1 is the distribution of jump size. Therefore, in particular, the Laplace transform of $\bar{P}$ is

$$\begin{aligned}\Upsilon(\beta) &:= \int_0^\infty \mathrm{e}^{-\beta\tau}\bar{P}(\tau)\,\mathrm{d}\tau \qquad (\beta\geqslant 0)\\ &= (2\pi)^{-d}\int_{(-\pi,\pi)^d}\frac{\mathrm{d}\xi}{\beta+2(1-\mathrm{Re}\,\varphi(\xi))}.\end{aligned}\tag{8.4}$$

The interchange of the integrals is justified by Tonelli's theorem, since $1-\mathrm{Re}\,\varphi(\xi)\geqslant 0$.

Note that $\Upsilon(0)$ agrees with (1.14). Also, the classical theory of random walks tells us that $X-X'$ is transient if and only if $\Upsilon(0)=\int_0^\infty\bar{P}(\tau)\,\mathrm{d}\tau<\infty$, which is in turn equivalent to the condition,

$$\int_{(-\pi,\pi)^d}\frac{\mathrm{d}\xi}{1-\mathrm{Re}\,\varphi(\xi)}<\infty; \tag{8.5}$$

this is the Chung–Fuchs theorem [14], transliterated to the setting of continuous-time symmetric random walks thanks to a standard Poissonization argument which we feel free to omit.

Lemma 8.1. *If $u_0 \in \ell^2(\mathbf{Z}^d)$, then $u_t \in \ell^2(\mathbf{Z}^d)$ a.s. for all $t \geqslant 0$. Moreover, for every $\beta \geqslant 0$ such that $\mathrm{Lip}_\sigma^2 \Upsilon(\beta) < 1$,*

$$\mathrm{E}\left(\|u_t\|_{\ell^2(\mathbf{Z}^d)}^2\right) \leqslant \frac{\|u_0\|_{\ell^2(\mathbf{Z}^d)}^2 \mathrm{e}^{\beta t}}{1 - \mathrm{Lip}_\sigma^2 \Upsilon(\beta)} \qquad \text{for all } t \geqslant 0. \tag{8.6}$$

Proof. Let $u_t^{(0)}(x) := u_0(x)$ for all $t \geqslant 0$ and $x \in \mathbf{Z}^d$, and define $u^{(k)}$ to be the resulting kth-step approximation to u via Picard iteration. It follows that

$$\begin{aligned}
&\mathrm{E}\left(|u_t^{(n+1)}(x)|^2\right) \\
&= |(\tilde{p}_t * u_0)(x)|^2 + \sum_{y \in \mathbf{Z}^d} \int_0^t [p_{t-s}(y-x)]^2 \mathrm{E}\left(\left|\sigma\left(u_s^{(n)}(y)\right)\right|^2\right) \mathrm{d}s \\
&\leqslant |(\tilde{p}_t * u_0)(x)|^2 + \mathrm{Lip}_\sigma^2 \sum_{y \in \mathbf{Z}^d} \int_0^t [p_{t-s}(y-x)]^2 \mathrm{E}\left(\left|u_s^{(n)}(y)\right|^2\right) \mathrm{d}s.
\end{aligned} \tag{8.7}$$

We may add over all $x \in \mathbf{Z}^d$ to deduce from this and Young's inequality that

$$\mathrm{E}\left(\left\|u_t^{(n+1)}\right\|_{\ell^2(\mathbf{Z}^d)}^2\right) \leqslant \|u_0\|_{\ell^2(\mathbf{Z}^d)}^2 + \mathrm{Lip}_\sigma^2 \int_0^t \bar{P}(t-s) \mathrm{E}\left(\left\|u_s^{(n)}\right\|_{\ell^2(\mathbf{Z}^d)}^2\right) \mathrm{d}s. \tag{8.8}$$

Since $\Upsilon(\beta) = \beta^{-1} \int_0^\infty \exp(-s) \bar{P}(s/\beta)\,\mathrm{d}s \leqslant \beta^{-1} < \infty$, we can find $\beta > 0$ large enough to guarantee that $\mathrm{Lip}_\sigma^2 \Upsilon(\beta) < 1$.

We multiply both sides of (8.8) by $\exp(-\beta t)$—for this choice of β—and notice from (8.8) that

$$A_k := \sup_{t \geqslant 0} \left[\mathrm{e}^{-\beta t} \mathrm{E}\left(\left\|u_t^{(k)}\right\|_{\ell^2(\mathbf{Z}^d)}^2\right)\right] \qquad (k \geqslant 0) \tag{8.9}$$

satisfies

$$A_{n+1} \leqslant \|u_0\|_{\ell^2(\mathbf{Z}^d)}^2 + \mathrm{Lip}_\sigma^2 \Upsilon(\beta) A_n \qquad \text{for all } n \geqslant 0. \tag{8.10}$$

Since $A_0 = \|u_0\|_{\ell^2(\mathbf{Z}^d)}^2$, the preceding shows that $\sup_{n \geqslant 0} A_n$ is bounded above by $(1 - \mathrm{Lip}_\sigma^2 \Upsilon(\beta))^{-1} \|u_0\|_{\ell^2(\mathbf{Z}^d)}^2$. □

Proposition 8.2. *If $u_0 \in \ell^1(\mathbf{Z}^d)$, then for every $\beta \geqslant 0$ such that $\mathrm{Lip}_\sigma^2 \Upsilon(\beta) < 1$,*

$$\int_0^\infty \mathrm{e}^{-\beta t} \mathrm{E}\left(\|u_t\|_{\ell^2(\mathbf{Z}^d)}^2\right) \mathrm{d}t \leqslant \frac{\|u_0\|_{\ell^1(\mathbf{Z}^d)}^2 \Upsilon(\beta)}{1 - \mathrm{Lip}_\sigma^2 \Upsilon(\beta)}. \tag{8.11}$$

Moreover,

$$\int_0^\infty \mathrm{e}^{-\beta t} \mathrm{E}\left(\|u_t\|_{\ell^2(\mathbf{Z}^d)}^2\right) \mathrm{d}t = \infty, \tag{8.12}$$

for all $\beta \geqslant 0$ such that $\ell_\sigma^2 \Upsilon(\beta) \geqslant 1$.

Proof. We proceed as we did for lemma 8.1. But instead of deducing (8.8) from (8.7), we use a different bound for $\|\tilde{p}_t * u_0\|_{\ell^2(\mathbf{Z}^d)}$

$$\begin{aligned}
&\mathrm{E}\left(\left\|u_t^{(n+1)}\right\|^2_{\ell^2(\mathbf{Z}^d)}\right) \\
&\qquad\leqslant \|p_t\|^2_{\ell^2(\mathbf{Z}^d)}\|u_0\|^2_{\ell^1(\mathbf{Z}^d)} + \mathrm{Lip}^2_\sigma \int_0^t \bar{P}(t-s)\mathrm{E}\left(\left\|u_s^{(n)}\right\|^2_{\ell^2(\mathbf{Z}^d)}\right)\mathrm{d}s \qquad (8.13)\\
&\qquad= \bar{P}(t)\|u_0\|^2_{\ell^1(\mathbf{Z}^d)} + \mathrm{Lip}^2_\sigma \int_0^t \bar{P}(t-s)\mathrm{E}\left(\left\|u_s^{(n)}\right\|^2_{\ell^2(\mathbf{Z}^d)}\right)\mathrm{d}s,
\end{aligned}$$

thanks to a slightly different application of Young's inequality. If we integrate both sides $[\exp(-\beta t)\mathrm{d}t]$, then we find that

$$I_k := \int_0^\infty \mathrm{e}^{-\beta t}\mathrm{E}\left(\left\|u_t^{(k)}\right\|^2_{\ell^2(\mathbf{Z}^d)}\right)\mathrm{d}t \qquad (k \geqslant 0) \tag{8.14}$$

satisfies

$$\begin{aligned}
I_{n+1} &\leqslant \|u_0\|^2_{\ell^1(\mathbf{Z}^d)} \int_0^\infty \mathrm{e}^{-\beta t}\bar{P}(t)\,\mathrm{d}t + I_n \times \mathrm{Lip}^2_\sigma \int_0^\infty \mathrm{e}^{-\beta t}\bar{P}(t)\,\mathrm{d}t \\
&= \|u_0\|^2_{\ell^1(\mathbf{Z}^d)}\Upsilon(\beta) + I_n \mathrm{Lip}^2_\sigma \Upsilon(\beta);
\end{aligned} \tag{8.15}$$

see (8.4). The first portion of the lemma follows from this, induction, and Fatou's lemma since $\mathrm{Lip}^2_\sigma\Upsilon(\beta) < 1$.

Next, let us suppose that $\ell^2_\sigma\Upsilon(\beta) \geqslant 1$. The following complimentary form of (8.13) holds [for the same reasons that (8.13) held]:

$$\mathrm{E}\left(\|u_t\|^2_{\ell^2(\mathbf{Z}^d)}\right) \geqslant \|\tilde{p}_t * u_0\|^2_{\ell^2(\mathbf{Z}^d)} + \ell^2_\sigma \int_0^t \bar{P}(t-s)\mathrm{E}\left(\|u_s\|^2_{\ell^2(\mathbf{Z}^d)}\right)\mathrm{d}s. \tag{8.16}$$

It is not hard to verify directly that

$$\|\tilde{p}_t * u_0\|^2_{\ell^2(\mathbf{Z}^d)} \geqslant u_0^2(x_0)\|p_t\|^2_{\ell^2(\mathbf{Z}^d)}, \tag{8.17}$$

whence, by $u_0(x_0) > 0$ for some $x_0 > 0$, it follows that

$$F(t) := \mathrm{E}\left(\|u_t\|^2_{\ell^2(\mathbf{Z}^d)}\right) \qquad (t \geqslant 0) \tag{8.18}$$

solves the renewal inequality,

$$F(t) \geqslant u_0^2(x_0)\bar{P}(t) + \ell^2_\sigma \int_0^t \bar{P}(t-s)F(s)\,\mathrm{d}s. \tag{8.19}$$

Therefore, $\tilde{F}(\beta) := \int_0^\infty \exp(-\beta t)F(t)\,\mathrm{d}t$ satisfies

$$\tilde{F}(\beta) \geqslant u_0^2(x_0)\Upsilon(\beta) + \ell^2_\sigma\Upsilon(\beta)\tilde{F}(\beta). \tag{8.20}$$

Since $u_0(x_0) > 0$ and $\Upsilon(\beta) > 0$ for all $\beta \geqslant 0$, it follows that $\tilde{F}(\beta) = \infty$ whenever $\ell^2_\sigma\Upsilon(\beta) \geqslant 1$. □

Proposition 8.3. *If $u_0 \in \ell^1(\mathbf{Z}^d)$, then*

$$\sup_{t\geqslant 0}\sup_{x\in\mathbf{Z}^d} u_t(x) < \infty, \qquad \sum_{y\in\mathbf{Z}^d}\int_0^\infty |\sigma(u_s(y))|^2\,\mathrm{d}s < \infty \quad \text{a.s.} \tag{8.21}$$

Moreover: (i) If, in addition, $q := \mathrm{Lip}_\sigma^2\Upsilon(0) < 1$, then

$$\mathrm{E}\left(\sup_{t\geqslant 0}\sup_{x\in\mathbf{Z}^d}|u_t(x)|^2\right) \leqslant \mathrm{E}\left(\sup_{t\geqslant 0}\|u_t\|_{\ell^1(\mathbf{Z}^d)}^2\right) \leqslant \|u_0\|_{\ell^1(\mathbf{Z}^d)}^2 + \frac{4q}{1-q}. \tag{8.22}$$

(ii) If, in addition, $\ell_\sigma^2\Upsilon(0) \geqslant 1$, then

$$\mathrm{E}\left(\sup_{t\geqslant 0}\|u_t\|_{\ell^1(\mathbf{Z}^d)}^2\right) = \int_0^\infty \mathrm{E}\left(\|u_s\|_{\ell^2(\mathbf{Z}^d)}^2\right)\mathrm{d}s = \infty. \tag{8.23}$$

Remark 8.4. Clearly, (8.21) implies that if $u_0 \in \ell^1(\mathbf{Z}^d)$, then

$$\liminf_{t\to\infty}\sup_{x\in\mathbf{Z}^d}|\sigma(u_t(x))|^2 \leqslant \liminf_{t\to\infty}\sum_{x\in\mathbf{Z}^d}|\sigma(u_t(x))|^2 = 0 \qquad \text{a.s.} \tag{8.24}$$

Suppose, in addition, that $\ell_\sigma > 0$ [say]. Then, we can deduce from the preceding fact that $\liminf_{t\to\infty}\sup_{x\in\mathbf{Z}^d}|u_t(x)| = 0$ a.s. □

Recall that $X - X'$ is transient if and only if $\Upsilon(0) < \infty$. Therefore, in order for the condition $\mathrm{Lip}_\sigma^2\Upsilon(0) < 1$ to hold, it is necessary—though not sufficient—that $X - X'$ be transient.

Proof of Proposition 8.3. First of all, Theorem 1.1 assures us that $u_t(x) \geqslant 0$ a.s., and hence $\|u_t\|_{\ell^1(\mathbf{Z}^d)} = \sum_{x\in\mathbf{Z}^d} u_t(x)$. Therefore, if we add both sides of (2.1) then we find that

$$\|u_t\|_{\ell^1(\mathbf{Z}^d)} = \|u_0\|_{\ell^1(\mathbf{Z}^d)} + \sum_{y\in\mathbf{Z}^d}\int_0^t \sigma(u_s(y))\,\mathrm{d}B_s(y). \tag{8.25}$$

[It is easy to apply the moment bound of Theorem 1.1 to justify the interchange of the sum and the stochastic integral.] In particular, it follows that

$$M_t := \|u_t\|_{\ell^1(\mathbf{Z}^d)} \qquad (t \geqslant 0) \tag{8.26}$$

defines a non-negative continuous martingale with mean $\|u_0\|_{\ell^1(\mathbf{Z}^d)}$. Its quadratic variation satisfies the following relations:

$$\langle M\rangle_t = \sum_{y\in\mathbf{Z}^d}\int_0^t |\sigma(u_s(y))|^2\,\mathrm{d}s \leqslant \mathrm{Lip}_\sigma^2\int_0^t \|u_s\|_{\ell^2(\mathbf{Z}^d)}^2\,\mathrm{d}s. \tag{8.27}$$

The bound (1.6) of Theorem 1.1 is more than enough to show that $M := \{M_t\}_{t\geqslant 0}$ is a continuous $L^2(\mathrm{P})$ martingale. Since $M_t \geqslant 0$ a.s. [Theorem

1.1] it follows from the martingale convergence theorem that $\lim_{t\to\infty} M_t$ exists a.s. and is finite a.s., which proves the first part of (8.21). And therefore, $\langle M\rangle_\infty = \sum_{y\in\mathbf{Z}^d}\int_0^t |\sigma(u_s(y))|^2\,\mathrm{d}s$ has to be also a.s. finite., since we can realize M_t as $W(\langle M\rangle_t)$ for some Brownian motion W, thanks to the Dubins, Dambis–Schwartz representation theorem [34, p. 170].

(i) If we know also that $\mathrm{Lip}_\sigma^2\Upsilon(0) < 1$, then Proposition 8.2 guarantees that $\mathrm{E}\langle M\rangle_\infty$ is bounded from above by $(1-\mathrm{Lip}_\sigma^2\Upsilon(0))^{-1}\mathrm{Lip}_\sigma^2\Upsilon(0) < \infty$, whence it follows that $M := \{M_t\}_{t\geqslant 0}$ is a continuous $L^2(\mathrm{P})$-bounded martingale with

$$\mathrm{E}\left(\sup_{t\geqslant 0} M_t^2\right) \leqslant \|u_0\|^2_{\ell^1(\mathbf{Z}^d)} + \frac{4\mathrm{Lip}_\sigma^2\Upsilon(0)}{1-\mathrm{Lip}_\sigma^2\Upsilon(0)}, \tag{8.28}$$

thanks to Doob's maximal inequality. This proves part *(i)* because $\|u_t\|_{\ell^\infty(\mathbf{Z}^d)}$ is bounded above by $\|u_t\|_{\ell^1(\mathbf{Z}^d)}$.

(ii) Finally consider the case that $\ell_\sigma\Upsilon(0) \geqslant 1$. Since

$$\begin{aligned}\mathrm{E}\left(\|u_t\|^2_{\ell^1(\mathbf{Z}^d)}\right) = \mathrm{E}\left(M_t^2\right) &= \|u_0\|^2_{\ell^1(\mathbf{Z}^d)} + \sum_{y\in\mathbf{Z}^d}\int_0^t \mathrm{E}\left(|\sigma(u_s(y))|^2\right)\mathrm{d}s \\ &\geqslant \|u_0\|^2_{\ell^1(\mathbf{Z}^d)} + \ell_\sigma^2\int_0^t \mathrm{E}\left(\|u_s\|^2_{\ell^2(\mathbf{Z}^d)}\right)\mathrm{d}s,\end{aligned} \tag{8.29}$$

it suffices to show that this final integral is unbounded [as a function of t]. But that follows from the second part of Proposition 8.2. □

Corollary 8.5. *If $u_0 \in \ell^1(\mathbf{Z}^d)$, then the following is a* P*-null set:*

$$\left\{\omega:\ \lim_{t\to\infty}\sup_{x\in\mathbf{Z}^d}|u_t(x)(\omega)| = 0\right\} \triangle \left\{\omega:\ \lim_{t\to\infty}\|u_t\|_{\ell^2(\mathbf{Z}^d)}(\omega) = 0\right\}. \tag{8.30}$$

Proof. Let E_1 denote the event that $\lim_{t\to\infty}\sup_{x\in\mathbf{Z}^d}|u_t(x)| = 0$ and E_2 the event that $\lim_{t\to\infty}\|u_t\|_{\ell^2(\mathbf{Z}^d)} = 0$. Because of the real-variable bounds, $\|u_t\|^2_{\ell^\infty(\mathbf{Z}^d)} \leqslant \|u_t\|^2_{\ell^2(\mathbf{Z}^d)} \leqslant \|u_t\|_{\ell^\infty(\mathbf{Z}^d)}\cdot\|u_t\|_{\ell^1(\mathbf{Z}^d)}$, we have

$$E_1\triangle E_2 \subseteq \left\{\omega:\ \limsup_{t\to\infty}\|u_t\|_{\ell^1(\mathbf{Z}^d)}(\omega) = \infty\right\}. \tag{8.31}$$

But we have noted already that $M_t := \|u_t\|_{\ell^1(\mathbf{Z}^d)}$ defines a non-negative martingale, under the conditions of this corollary. Therefore, the final event in (8.31) is P-null, thanks to Doob's martingale convergence theorem. Thus, we find that $E_1\triangle E_2$ is a measurable subset of a P-null set, and is hence P-null. □

Proposition 8.6. *Suppose $u_0 \in \ell^1(\mathbf{Z}^d)$ and the random walk X is transient; i.e., $\Upsilon(0) < \infty$. Then,*

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathrm{E}\left(\|u_t\|^2_{\ell^1(\mathbf{Z}^d)}\right) \leqslant \inf\left\{\beta > 0:\ \mathrm{Lip}_\sigma^2\Upsilon(\beta) < 1\right\} < \infty. \tag{8.32}$$

If, in addition, $\ell_\sigma^2\Upsilon(0) > 1$*, then*

$$\liminf_{t\to\infty} \frac{1}{t}\log \mathrm{E}\left(\|u_t\|^2_{\ell^2(\mathbf{Z}^d)}\right) \geqslant \inf\left\{\beta > 0:\ \ell_\sigma^2\Upsilon(\beta) < 1\right\} > 0. \tag{8.33}$$

Proof. We have already proved a slightly weaker version of (8.32). Indeed, since $\ell^1(\mathbf{Z}^d) \subset \ell^2(\mathbf{Z}^d)$, (8.6) implies that

$$\limsup_{t\to\infty} \frac{1}{t}\log \mathrm{E}\left(\|u_t\|^2_{\ell^2(\mathbf{Z}^d)}\right) \leqslant \inf\left\{\beta > 0:\ \mathrm{Lip}_\sigma^2\Upsilon(\beta) < 1\right\}. \tag{8.34}$$

Then (8.25) and (8.34) together tell us that for every $C > \inf\{\beta > 0:\ \mathrm{Lip}_\sigma^2\Upsilon(\beta) < 1\}$, there exists $K = K(C) \in (0\,,\infty)$ such that

$$\begin{aligned}\mathrm{E}\left(\|u_t\|^2_{\ell^1(\mathbf{Z}^d)}\right) &\leqslant \|u_0\|^2_{\ell^1(\mathbf{Z}^d)} + \mathrm{Lip}_\sigma^2\int_0^t \mathrm{E}\left(\|u_s\|^2_{\ell^2(\mathbf{Z}^d)}\right)\mathrm{d}s \\ &\leqslant \|u_0\|^2_{\ell^1(\mathbf{Z}^d)} + K\int_0^t \mathrm{e}^{Cs}\,\mathrm{d}s = O\left(\mathrm{e}^{(C+o(1))t}\right) \qquad \text{as } t\to\infty.\end{aligned} \tag{8.35}$$

Thus follows the first bound of the proposition.

Because of (8.16) and (8.17), we find that

$$F(t) := \mathrm{E}\left(\|u_t\|^2_{\ell^2(\mathbf{Z}^d)}\right) \qquad (t \geqslant 0) \tag{8.36}$$

solves the renewal inequality,

$$F(t) \geqslant g(t) + \int_0^t h(t-s)F(s)\,\mathrm{d}s \qquad (t\geqslant 0), \tag{8.37}$$

where

$$g(t) := u_0^2(x_0)\bar{P}(t), \quad h(t) := \ell_\sigma^2\bar{P}(t) \qquad (t\geqslant 0). \tag{8.38}$$

A comparison result (Lemma A.2) tells us that $F(t) \geqslant f(t)$ for all $t \geqslant 0$, where f is the solution to the renewal equation

$$f(t) = g(t) + \int_0^t h(t-s)f(s)\,\mathrm{d}s \qquad (t\geqslant 0). \tag{8.39}$$

The condition that $\ell_\sigma^2\Upsilon(0) > 1$ is equivalent to $\int_0^\infty h(t)\,\mathrm{d}t > 1$. Because of transience $[\Upsilon(0) < \infty]$ and the fact that $\Upsilon(\beta)$ is strictly decreasing and continuous, we can find $\beta^* > 0$ such that $\int_0^\infty \exp(-\beta^* t)h(t)\,\mathrm{d}t = 1$. Note that $f_{\beta^*}(t) := \exp(-\beta^* t)f(t)$ solves the renewal equation

$$f_{\beta^*}(t) = g_{\beta^*}(t) + \int_0^t h_{\beta^*}(t-s)f_{\beta^*}(s)\,\mathrm{d}s \qquad (t\geqslant 0), \tag{8.40}$$

where $g_{\beta^*}(t) := \exp(-\beta^* t)g(t)$ and $h_{\beta^*}(t) := \exp(-\beta^* t)h(t)$. Since h_{β^*} is a probability density function and g_{β^*} is non increasing [see (8.3)], Blackwell's key renewal theorem [23] implies that

$$\begin{aligned}\liminf_{t\to\infty} \mathrm{e}^{-\beta^* t}F(t) \geqslant \lim_{t\to\infty} f_{\beta^*}(t) &= \left(\int_0^\infty s h_{\beta^*}(s)\,\mathrm{d}s\right)^{-1} \cdot \int_0^\infty g_{\beta^*}(s)\,\mathrm{d}s \\ &= u_0^2(x_0)\ell_\sigma^{-2}\left(\int_0^\infty s\mathrm{e}^{-\beta^* s}\bar{P}(s)\,\mathrm{d}s\right)^{-1} \cdot \Upsilon(\beta^*).\end{aligned} \tag{8.41}$$

Since $\bar{P}(s) \leqslant 1$, the right-most quantity is at least $u_0^2(x_0)\ell_\sigma^{-2}(\beta^*)^2\Upsilon(\beta^*) > 0$. This completes the proof of (8.33). Note that we have used the fact that $\Upsilon(\beta)$ is continuous in β and strictly decreasing, so that $\beta^* = \inf\{\beta > 0 : \ell_\sigma^2\Upsilon(\beta) < 1\}$. ☐

Proposition 8.7. *If* $\mathrm{Lip}_\sigma^2\Upsilon(0) < 1$, *then* $\lim_{t\to\infty} \mathrm{E}(\|u_t\|_{\ell^2(\mathbf{Z}^d)}^2) = 0$. *Furthermore, as* $t \to \infty$*:*

$$\begin{aligned}&\bar{P}(t) = O\left(\mathrm{E}\left(\|u_t\|_{\ell^2(\mathbf{Z}^d)}^2\right)\right); \quad \text{and} \\ &\mathrm{E}\left(\|u_t\|_{\ell^2(\mathbf{Z}^d)}^2\right) = O\left(t^{-\alpha}\right) \text{ for all } \alpha \geqslant 0 \text{ such that } \bar{P}(t) = O(t^{-\alpha}).\end{aligned} \tag{8.42}$$

Proof. The first assertion of (8.42) is simple to prove; in fact, $\mathrm{E}(\|u_t\|_{\ell^2(\mathbf{Z}^d)}^2) \geqslant [u_0(x_0)]^2\bar{P}(t)$ $(t \geqslant 0)$ for any $x_0 \in \mathbf{Z}^d$ and all $t > 0$; see (8.16) and (8.17). We concentrate our efforts on the remaining statements.

Thanks to (8.13),

$$\mathrm{E}\left(\|u_t\|_{\ell^2(\mathbf{Z}^d)}^2\right) \leqslant \bar{P}(t)\|u_0\|_{\ell^1(\mathbf{Z}^d)}^2 + \mathrm{Lip}_\sigma^2 \int_0^t \bar{P}(t-s)\mathrm{E}\left(\|u_s\|_{\ell^2(\mathbf{Z}^d)}^2\right)\mathrm{d}s. \tag{8.43}$$

That is, $F(t) := \mathrm{E}(\|u_t\|_{\ell^2(\mathbf{Z}^d)}^2)$ is a sub solution to a renewal equation; viz.,

$$F(t) \leqslant g(t) + \int_0^t h(t-s)F(s)\,\mathrm{d}s \qquad (t \geqslant 0), \tag{8.44}$$

for

$$g(t) := \bar{P}(t)\|u_0\|_{\ell^1(\mathbf{Z}^d)}^2, \qquad h(t) := \mathrm{Lip}_\sigma^2\bar{P}(t). \tag{8.45}$$

A comparison lemma (Lemma A.2) shows that $0 \leqslant F(t) \leqslant f(t)$ for all $t \geqslant 0$, where

$$f(t) = g(t) + \int_0^t h(t-s)f(s)\,\mathrm{d}s \qquad (t \geqslant 0). \tag{8.46}$$

Therefore, it remains to prove that $f(t) \to 0$ as $t \to \infty$. It is easy, as well as classical, that we can write f in terms of the renewal function of h; that is,

$$f(t) = g(t) + \sum_{n=0}^\infty \int_0^t h^{*(n)}(s)g(t-s)\,\mathrm{d}s \qquad (t \geqslant 0), \tag{8.47}$$

where $h^{*(1)}(t) := \int_0^t h(t-s)h(s)\,\mathrm{d}s$ denotes the convolution of h with itself, and $h^{*(k+1)}(t) := \int_0^t h^{*(k)}(t-s)h(s)\,\mathrm{d}s$ for all $k \geqslant 0$. We might note that

$g(t) \leqslant g(0) = \|u_0\|^2_{\ell^2(\mathbf{Z}^d)}$ because $\bar{P}$ is non increasing [see (8.3)] and one at zero. Therefore,

$$\begin{aligned}
0 &\leqslant \int_0^t h^{*(n)}(s) g(t-s)\,\mathrm{d}s \leqslant \|u_0\|^2_{\ell^2(\mathbf{Z}^d)} \int_0^\infty h^{*(n)}(s)\,\mathrm{d}s \\
&\leqslant \|u_0\|^2_{\ell^2(\mathbf{Z}^d)} \left(\int_0^\infty h(s)\,\mathrm{d}s\right)^{n+1} \qquad \text{[Young's inequality]} \\
&= \|u_0\|^2_{\ell^2(\mathbf{Z}^d)} \left(\mathrm{Lip}_\sigma^2 \Upsilon(0)\right)^{n+1}.
\end{aligned} \tag{8.48}$$

It is not hard to see that $\lim_{t\to\infty} g(t) = \lim_{t\to\infty} \bar{P}(t) = 0$; this follows from (8.3) and the monotone convergence theorem. Because $\mathrm{Lip}_\sigma^2 \Upsilon(0) < 1$, we can deduce from (8.48) and (8.47), in conjunction with the dominated convergence theorem, that $f(t)$—hence $F(t) = \mathrm{E}(\|u_t\|^2_{\ell^2(\mathbf{Z}^d)})$—converges to zero as $t \to \infty$.

It remains to prove the second assertion in (8.42). With this in mind, let us suppose $\bar{P}$ satisfies the following: There exists $c \in (0\,,\infty)$ and $\alpha \in [0\,,\infty)$ such that

$$\bar{P}(t) \leqslant c(1+t)^{-\alpha}. \tag{8.49}$$

For there is nothing to consider otherwise. We aim to prove that

$$\mathrm{E}\left(\|u_t\|^2_{\ell^2(\mathbf{Z}^d)}\right) \leqslant \mathrm{const}\cdot(1+t)^{-\alpha}, \tag{8.50}$$

for some finite constant that does not depend on t. This proves the proposition.

Define $F_k(t) := \mathrm{E}(\|u_t^{(k)}\|^2_{\ell^2(\mathbf{Z}^d)})$, where $u^{(k)}$ denotes the kth approximation to u via Picard's iteration (3.1), starting at $u_t^{(0)}(x) \equiv 0$. We can write (8.13), in short hand, as follows:

$$F_{n+1}(t) \leqslant \bar{P}(t)\|u_0\|^2_{\ell^1(\mathbf{Z}^d)} + \mathrm{Lip}_\sigma^2 \int_0^t \bar{P}(t-s) F_n(s)\,\mathrm{d}s. \tag{8.51}$$

Now let us choose and fix $\varepsilon \in (0\,,1)$ and write

$$\begin{aligned}
&\int_0^t \bar{P}(t-s)F_n(s)\,\mathrm{d}s = \int_{t\varepsilon}^t \bar{P}(s)F_n(t-s)\,\mathrm{d}s + \int_{t(1-\varepsilon)}^t \bar{P}(t-s)F_n(s)\,\mathrm{d}s \\
&\leqslant c\int_{t\varepsilon}^t \frac{F_n(t-s)}{(1+s)^\alpha}\,\mathrm{d}s + \sup_{w\geqslant 0}\left[(1+w)^\alpha F_n(w)\right] \int_{t(1-\varepsilon)}^t \frac{\bar{P}(t-s)}{(1+s)^\alpha}\,\mathrm{d}s \\
&\leqslant \frac{c}{\varepsilon^\alpha(1+t)^\alpha}\int_0^\infty F_n(s)\,\mathrm{d}s + \sup_{w\geqslant 0}\left[(1+w)^\alpha F_n(w)\right]\frac{\Upsilon(0)}{(1-\varepsilon)^\alpha(1+t)^\alpha}.
\end{aligned} \tag{8.52}$$

The proof of Proposition 8.2 shows that

$$\sup_{n\geqslant 0}\int_0^\infty F_n(s)\,\mathrm{d}s \leqslant \frac{\|u_0\|^2_{\ell^1(\mathbf{Z}^d)}\Upsilon(0)}{1-\mathrm{Lip}_\sigma^2\Upsilon(0)}. \tag{8.53}$$

Consequently,

$$R_k := \sup_{w\geqslant 0}\left[(1+w)^\alpha F_k(w)\right] \qquad (k \geqslant 0) \tag{8.54}$$

satisfies

$$R_{n+1} \leqslant A + R_n \frac{\mathrm{Lip}_\sigma^2 \Upsilon(0)}{(1-\varepsilon)^\alpha} \qquad \text{for all } n \geqslant 0, \tag{8.55}$$

where

$$A = A(\varepsilon) := c\|u_0\|^2_{\ell^1(\mathbf{Z}^d)} + \frac{c\|u_0\|^2_{\ell^1(\mathbf{Z}^d)} \mathrm{Lip}_\sigma^2 \Upsilon(0)}{\varepsilon^\alpha (1 - \mathrm{Lip}_\sigma^2 \Upsilon(0))}. \tag{8.56}$$

Since $\mathrm{Lip}_\sigma^2 \Upsilon(0) < 1$, we can choose ε sufficiently close to zero to ensure that $\mathrm{Lip}_\sigma^2 \Upsilon(0) < (1-\varepsilon)^{1+\alpha}$. For this particular ε, we find that $R_{n+1} \leqslant A + (1-\varepsilon) R_n$ for all n. Since $R_0 = 0$, this proves that $\sup_{n \geqslant 0} R_n \leqslant A/\varepsilon$. Eq. (8.50)—whence the proposition—follows from the latter inequality and Fatou's lemma. □

9 Proof of Theorem 1.7

Let us begin with an elementary real-variable inequality.

Lemma 9.1. *For all real numbers $k \geqslant 2$ and $x, y, \delta > 0$,*

$$(x+y)^k \leqslant (1+\delta)^{k-1} x^k + \left(\frac{1+\delta}{\delta} \right)^{k-1} y^k. \tag{9.1}$$

This is a consequence of Jensen's inequality when $\delta = 1$. We are interested in the case that $\delta \ll 1$.

Proof. The function $f(z) := (z+1)^k - (1+\delta)^{k-1} z^k$ $(z > 0)$ is maximized at $z_* := \delta^{-1}$, and $\max_z f(z) = f(z_*) = \{(1+\delta)/\delta\}^{k-1}$; i.e., $f(x) \leqslant \{(1+\delta)/\delta\}^{k-1}$ for all $x > 0$. This is the desired result when $y = 1$. We can factor the variable y from both sides of (9.1) in order to reduce the problem to the already-proved case that $y = 1$. □

Lemma 9.2. $\int_0^\infty \|p_{s+\tau} - p_s\|^2_{\ell^2(\mathbf{Z}^d)}\, \mathrm{d}s \leqslant 4\Upsilon(0)\tau^2$ *for all $\tau \geqslant 0$.*

Proof. We apply the Plancherel theorem and (4.9) in order to deduce that

$$\begin{aligned} \|p_{s+\tau} - p_s\|^2_{\ell^2(\mathbf{Z}^d)} &= (2\pi)^{-d} \int_{(-\pi,\pi)^d} \left| \mathrm{e}^{-(s+\tau)(1-\varphi(\xi))} - \mathrm{e}^{-s(1-\varphi(\xi))} \right|^2 \mathrm{d}\xi \\ &= (2\pi)^{-d} \int_{(-\pi,\pi)^d} \mathrm{e}^{-2s(1-\mathrm{Re}\varphi(\xi))} \left| 1 - \mathrm{e}^{-\tau(1-\varphi(\xi))} \right|^2 \mathrm{d}\xi \\ &\leqslant \frac{4\tau^2}{(2\pi)^d} \int_{(-\pi,\pi)^d} \mathrm{e}^{-2s(1-\mathrm{Re}\varphi(\xi))}\, \mathrm{d}\xi. \end{aligned} \tag{9.2}$$

Integrate [ds] to finish; compare with (8.4). □

Recall that z_k denotes the optimal constant in the BDG inequality [(2.7)].

Lemma 9.3. *If $k \in (2\,,\infty)$ satisfies $z_k \mathrm{Lip}_\sigma \sqrt{\Upsilon(0)} < (1+\delta)^{-(k-1)/k}$ for some $\delta > 0$, then*

$$\sup_{t\geqslant 0} \mathrm{E}\left(\sup_{x\in\mathbf{Z}^d} |u_t(x)|^k\right) \leqslant \sup_{t\geqslant 0} \mathrm{E}\left(\|u_t\|_{\ell^k(\mathbf{Z}^d)}^k\right) < \infty. \tag{9.3}$$

Proof. Let $u_t^{(0)}(x) := u_0(x)$ and define $u^{(n)}$ to be the nth step Picard approximation to u, as in (3.1). Define

$$\bar{M}_t^{(n)} := \mathrm{E}\left(\left\|u_t^{(n)}\right\|_{\ell^k(\mathbf{Z}^d)}^k\right) \qquad \text{for all } t \geqslant 0 \text{ and } k \geqslant 1. \tag{9.4}$$

Then we can apply Lemma 9.1 and write, in analogy with (8.26),

$$\bar{M}_t^{(n+1)} \leqslant \left(\frac{1+\delta}{\delta}\right)^{k-1} \sum_{x\in\mathbf{Z}^d} I_x + (1+\delta)^{k-1} \sum_{x\in\mathbf{Z}^d} J_x, \tag{9.5}$$

where I_x and J_x were defined earlier in (3.3). One estimates $\sum_{x\in\mathbf{Z}^d} I_x$ via Jensen's inequality, using $p_t(\bullet - x)$ as the base measure, in order to find that

$$\sum_{x\in\mathbf{Z}^d} I_x \leqslant \|u_0\|_{\ell^k(\mathbf{Z}^d)}^k. \tag{9.6}$$

In order to estimate $\sum_{x\in\mathbf{Z}^d} J_x$, we define—for all $(t\,,x) \in \mathbf{R}_+ \times \mathbf{Z}^d$— a Borel measure $\rho_{t,x}$ on $\mathbf{R}_+ \times \mathbf{Z}^d$ as follows:

$$\rho_{t,x}(\mathrm{d}s\,\mathrm{d}y) := \left[p_{t-s}(y-x)\right]^2 \mathbf{1}_{[0,t]}(s)\,\mathrm{d}s\,\chi(\mathrm{d}y); \tag{9.7}$$

where χ denotes the counting measure on $\mathbf{Z}^d$. Because of the transience of $X - X'$, the measure $\rho_{t,x}$ is finite; in fact,

$$\rho_{t,x}(\mathbf{R}_+ \times \mathbf{Z}^d) = \int_0^t \|p_s\|_{\ell^2(\mathbf{Z}^d)}^2\,\mathrm{d}s = \int_0^t \bar{P}(s)\,\mathrm{d}s \leqslant \Upsilon(0). \tag{9.8}$$

Therefore, we apply (3.5) and Jensen's inequality, in conjunction, in order to see that

$$\begin{aligned} J_x &\leqslant z_k^k \left(\mathrm{Lip}_\sigma^2 \int_{[0,t]\times\mathbf{Z}^d} \left\{\mathrm{E}\left(\left|u_s^{(n)}(y)\right|^k\right)\right\}^{2/k} \rho_{t,x}(\mathrm{d}s\,\mathrm{d}y)\right)^{k/2} \\ &\leqslant (z_k \mathrm{Lip}_\sigma)^k [\Upsilon(0)]^{(k-2)/2} \int_{[0,t]\times\mathbf{Z}^d} \mathrm{E}\left(\left|u_s^{(n)}(y)\right|^k\right) \rho_{t,x}(\mathrm{d}s\,\mathrm{d}y). \end{aligned} \tag{9.9}$$

Thus,

$$\begin{aligned} \sum_{x\in\mathbf{Z}^d} J_x &\leqslant (z_k \mathrm{Lip}_\sigma)^k [\Upsilon(0)]^{(k-2)/2} \int_0^t \bar{P}(t-s) \mathrm{E}\left(\left\|u_s^{(n)}\right\|_{\ell^k(\mathbf{Z}^d)}^k\right) \mathrm{d}s \\ &\leqslant \left(z_k \mathrm{Lip}_\sigma \sqrt{\Upsilon(0)}\right)^k \cdot \sup_{r\geqslant 0} \mathrm{E}\left(\left\|u_r^{(n)}\right\|_{\ell^k(\mathbf{Z}^d)}^k\right), \end{aligned} \tag{9.10}$$

thanks to (8.4).

In summary, (9.5) has the following consequence: For all $n \geqslant 0$,

$$\sup_{t\geqslant 0} \bar{M}_t^{(n+1)} \leqslant \left(\frac{1+\delta}{\delta}\right)^{k-1} \|u_0\|_{\ell^k(\mathbf{Z}^d)}^k + (1+\delta)^{k-1}\left(z_k \mathrm{Lip}_\sigma \sqrt{\Upsilon(0)}\right)^k \sup_{t\geqslant 0} \bar{M}_t^{(n)}. \tag{9.11}$$

Since $(1+\delta)^{k-1}(z_k\mathrm{Lip}_\sigma\sqrt{\Upsilon(0)})^k < 1$ and $\sup_{t\geqslant 0} \bar{M}_t^{(0)} = \|u_0\|_{\ell^k(\mathbf{Z}^d)}^k$, this shows that $C := \sup_{n\geqslant 0}\sup_{t\geqslant 0} \bar{M}_t^{(n)} < \infty$. Fatou's lemma now implies half of the result, since it shows that $\mathrm{E}(\|u_t\|_{\ell^k(\mathbf{Z}^d)}^k) \leqslant \liminf_{n\to\infty} \mathrm{E}(\|u_t^{(n)}\|_{\ell^k(\mathbf{Z}^d)}^k) \leqslant C$. The remainder of the proposition follows simply because $\|\bullet\|_{\ell^\infty(\mathbf{Z}^d)} \leqslant \|\bullet\|_{\ell^k(\mathbf{Z}^d)}$. □

Proposition 9.4. *Assume that* $\limsup_{t\to\infty} t^\alpha \mathrm{P}\{X_t = X_t'\} < 1$ *for some* $\alpha > 1$, *where* X *and* X' *are two independent random walks with generator* $\mathscr{L}$. *If* $k \in (2\,,\infty)$ *satisfies* $z_k\mathrm{Lip}_\sigma\sqrt{\Upsilon(0)} < (1+\delta)^{-(k-1)/k}$ *for some* $\delta > 0$, *then there exists a finite constant* A*—depending only on* δ, Lip_σ, $\Upsilon(0)$, *and* $\|u_0\|_{\ell^1(\mathbf{Z}^d)}$*—such that*

$$\mathrm{E}\left(\|u_{t+\tau} - u_t\|_{\ell^k(\mathbf{Z}^d)}^k\right) \leqslant \frac{A\tau^{k/2}}{(1+t)^\alpha} \qquad \text{for every } t, \tau \geqslant 0. \tag{9.12}$$

Consequently, there exists a Hölder-continuous modification of the process $t \mapsto u_t(\bullet)$ *with values in* $\ell^\infty(\mathbf{Z}^d)$. *Moreover, for that modification, there a finite constant* A'*—depending only on* δ, Lip_σ, $\Upsilon(0)$, *and* $\|u_0\|_{\ell^1(\mathbf{Z}^d)}$*—such that*

$$\mathrm{E}\left(\sup_{s\neq r\in[t,t+1]}\sup_{x\in\mathbf{Z}^d}\left|\frac{u_r(x)-u_s(x)}{|r-s|^\eta}\right|^k\right) \leqslant \frac{A'}{(1+t)^\alpha} \tag{9.13}$$

as long as $0 \leqslant \eta < (k-2)/(2k)$.

Proof. Thanks to Lemma 9.3, $\|u_t\|_{\ell^k(\mathbf{Z}^d)}$ has a finite kth moment. This observation justifies the use of these moments in the ensuing discussion. Now we begin our proof in earnest.

The proof requires us to make a few small adjustments to the derivation of Lemma 4.1; specifically we now incorporate the fact that $\mathrm{Lip}_\sigma^2\Upsilon(0) < 1$ into that proof. Therefore, we mention only the required changes.

We use the notation of the proof of Lemma 4.1 and write

$$\left\{\mathrm{E}\left(|u_{t+\tau}(x) - u_t(x)|^k\right)\right\}^{1/k} \leqslant |Q_1| + Q_2 + Q_3, \tag{9.14}$$

whence

$$\mathrm{E}\left(|u_{t+\tau}(x) - u_t(x)|^k\right) \leqslant 3^{k-1}\left(|Q_1|^k + Q_2^k + Q_3^k\right). \tag{9.15}$$

Note that

$$\sum_{x\in\mathbf{Z}^d}|Q_1|^k \leqslant \sum_{x\in\mathbf{Z}^d}\left(\sum_{y\in\mathbf{Z}^d} u_0(y)\,|p_{t+\tau}(y-x)-p_t(y-x)|\right)^k$$
$$\leqslant \|u_0\|_{\ell^1(\mathbf{Z}^d)}^{k-1}\cdot\sum_{x\in\mathbf{Z}^d}\sum_{y\in\mathbf{Z}^d} u_0(y)\,|p_{t+\tau}(y-x)-p_t(y-x)|^k \tag{9.16}$$
$$= \|u_0\|_{\ell^1(\mathbf{Z}^d)}^{k}\cdot\|p_{t+\tau}-p_t\|_{\ell^k(\mathbf{Z}^d)}^k \leqslant \|u_0\|_{\ell^1(\mathbf{Z}^d)}^{k}\cdot\|p_{t+\tau}-p_t\|_{\ell^2(\mathbf{Z}^d)}^k\,,$$

thanks to Jensen's inequality. We observe that

$$\|p_{t+\tau}-p_t\|_{\ell^2(\mathbf{Z}^d)}^2 = (2\pi)^{-d}\int_{[-\pi,\pi]^d}\left|\mathrm{e}^{-t(1-\varphi(\xi))}\right|^2\left|\mathrm{e}^{-\tau(1-\varphi(\xi))}-1\right|^2\mathrm{d}\xi$$
$$\leqslant \text{const}\cdot\tau^2\int_{[-\pi,\pi]^d}\left|\mathrm{e}^{-t(1-\varphi(\xi))}\right|^2\mathrm{d}\xi = \text{const}\cdot\tau^2\mathrm{P}\{X_t=X_t'\}$$
$$\leqslant \text{const}\cdot\frac{\tau^2}{(1+t)^\alpha}. \tag{9.17}$$

Consequently,

$$\sum_{x\in\mathbf{Z}^d}|Q_1|^k \leqslant \frac{\text{const}\cdot\tau^k}{(1+t)^{\alpha k/2}} \tag{9.18}$$

We estimate Q_2 slightly differently from the proof of Lemma 4.1 as well.

For every $(t\,,x)\in\mathbf{R}_+\times\mathbf{Z}^d$, let us define a similar Borel measure $R_{t,x}$ to $\rho_{t,x}$ [see (9.7)] as follows:

$$R_{t,x}(\mathrm{d}s\,\mathrm{d}y) := [p_{t+\tau-s}(y-x)-p_{t-s}(y-x)]^2\,\mathbf{1}_{[0,t]}(s)\,\mathrm{d}s\,\chi(\mathrm{d}y). \tag{9.19}$$

Now we re-examine the first line of (4.7), and note that

$$Q_2^2 \leqslant (z_k\mathrm{Lip}_\sigma)^2\sum_{y\in\mathbf{Z}^d}\int_0^t [p_{t+\tau-s}(y-x)-p_{t-s}(y-x)]^2\left\{\mathrm{E}\left(|u_s(y)|^k\right)\right\}^{2/k}\mathrm{d}s$$
$$= (z_k\mathrm{Lip}_\sigma)^2\int_{\mathbf{R}_+\times\mathbf{Z}^d}\left\{\mathrm{E}\left(|u_s(y)|^k\right)\right\}^{2/k}R_{t,x}(\mathrm{d}s\,\mathrm{d}y). \tag{9.20}$$

This follows from (1.1) and (4.7), but we use the optimal constant z_k in place of the slightly-weaker one $2\sqrt{k}$ that came from Lemma 2.1.

Lemma 9.2 implies that $R_{t,x}(\mathbf{R}_+\times\mathbf{Z}^d) = \int_0^t \|p_{s+\tau}-p_s\|_{\ell^2(\mathbf{Z}^d)}^2\,\mathrm{d}s \leqslant 4\Upsilon(0)\tau^2$. This bound and Jensen's inequality together show that $\sum_{x\in\mathbf{Z}^d}Q_2^k$ is bounded from above by

$$(z_k\mathrm{Lip}_\sigma)^k(4\Upsilon(0)\tau^2)^{(k-2)/2}\sum_{x\in\mathbf{Z}^d}\int_{\mathbf{R}_+\times\mathbf{Z}^d}\mathrm{E}\left(|u_s(y)|^k\right)R_{t,x}(\mathrm{d}s\,\mathrm{d}y) \tag{9.21}$$
$$= (z_k\mathrm{Lip}_\sigma)^k(4\Upsilon(0)\tau^2)^{(k-2)/2}\int_0^t\|p_{t+\tau-s}-p_{t-s}\|_{\ell^2(\mathbf{Z}^d)}^2\,\mathrm{E}\left(\|u_s\|_{\ell^k(\mathbf{Z}^d)}^k\right)\mathrm{d}s.$$

By an argument similar to the one used in Proposition 8.7, one is able to show that $\mathrm{E}(\|u_s\|_{\ell^k(\mathbf{Z}^d)}^k) \leqslant \text{const}\cdot(1+s)^{-\alpha}$. Here is an outline of the proof: We can follow the proof of Lemma 9.3, but derive a better bound on $\sum_{x\in\mathbf{Z}^d} I_x \leqslant \bar{P}(t)\|u_0\|_{\ell^1(\mathbf{Z}^d)}^k$, in order to obtain

$$\begin{aligned}&\mathrm{E}\left(\|u_t^{(n+1)}\|_{\ell^k(\mathbf{Z}^d)}^k\right) \leqslant \left(\frac{1+\delta}{\delta}\right)^{k-1} \bar{P}(t)\|u_0\|_{\ell^1(\mathbf{Z}^d)}^k \\ &+ (1+\delta)^{k-1}(z_k\mathrm{Lip}_\sigma)^k[\Upsilon(0)]^{(k-2)/2}\int_0^t \bar{P}(t-s)\mathrm{E}\left(\left\|u_s^{(n)}\right\|_{\ell^k(\mathbf{Z}^d)}^k\right)\,\mathrm{d}s.\end{aligned} \tag{9.22}$$

From here, we proceed along similar lines, as was done from (8.51) onwards. We follow the proof of Proposition 8.2, using (9.22), in order to derive the following analog of (8.53):

$$\sup_{n\geqslant 0}\int_0^\infty F_n(s)\,\mathrm{d}s \leqslant \frac{((1+\delta)/\delta)^{k-1}\|u_0\|_{\ell^1(\mathbf{Z}^d)}^k\Upsilon(0)}{1-(1+\delta)^{k-1}(z_k\mathrm{Lip}_\sigma\Upsilon(0))^k}, \tag{9.23}$$

where $F_n(t) := \mathrm{E}(\|u_t^{(n+1)}\|_{\ell^k(\mathbf{Z}^d)}^k)$. In this way, we can obtain the bound, $\mathrm{E}(\|u_s\|_{\ell^k(\mathbf{Z}^d)}^k) \leqslant \text{const}\cdot(1+s)^{-\alpha}$, as was needed. We use this bound, as well as (9.17) in (9.21), and split the integral into two parts (0 to $t/2$ and $t/2$ to t), in order to obtain the following:

$$\sum_{x\in\mathbf{Z}^d} Q_2^k \leqslant \frac{\text{const}\cdot\tau^k}{(1+t)^\alpha}. \tag{9.24}$$

Finally we estimate $\sum_{x\in\mathbf{Z}^d} Q_3^k$ by first modifying (4.13) as follows:

$$\begin{aligned}Q_3^2 &\leqslant (z_k\mathrm{Lip}_\sigma)^2\sum_{y\in\mathbf{Z}^d}\int_t^{t+\tau}[p_{t+\tau-s}(y-x)]^2\left\{\mathrm{E}\left(|u_s(y)|^k\right)\right\}^{2/k}\,\mathrm{d}s \\ &= (z_k\mathrm{Lip}_\sigma)^2\int_{\mathbf{R}_+\times\mathbf{Z}^d}\left\{\mathrm{E}\left(|u_s(y)|^k\right)\right\}^{2/k}\mathcal{R}_{t,\tau,x}(\mathrm{d}s\,\mathrm{d}y),\end{aligned} \tag{9.25}$$

where the Borel measures $\mathcal{R}_{t,\tau,x}$ are defined in a similar manner as in (9.7); that is,

$$\mathcal{R}_{t,\tau,x}(\mathrm{d}s\,\mathrm{d}y) := \sum_{y\in\mathbf{Z}^d}[p_{t+\tau-s}(y-x)]^2\,\mathbf{1}_{[t,t+\tau]}(s)\,\mathrm{d}s\,\chi(\mathrm{d}y). \tag{9.26}$$

Because $\mathcal{R}_{t,\tau,x}(\mathbf{R}_+\times\mathbf{Z}^d) = \int_0^\tau \bar{P}(s)\mathrm{d}s \leqslant \tau$, Jensen's inequality assures us that

$$\begin{aligned}\sum_{x\in\mathbf{Z}^d} Q_3^k &\leqslant (z_k\mathrm{Lip}_\sigma)^k\tau^{(k-2)/2}\sum_{x\in\mathbf{Z}^d}\int_{\mathbf{R}_+\times\mathbf{Z}^d}\mathrm{E}\left(|u_s(y)|^k\right)\mathcal{R}_{t,\tau,x}(\mathrm{d}s\,\mathrm{d}y) \\ &= (z_k\mathrm{Lip}_\sigma)^k\tau^{(k-2)/2}\int_t^{t+\tau}\bar{P}(t+\tau-s)\mathrm{E}\left(\|u_s\|_{\ell^k(\mathbf{Z}^d)}^k\right)\,\mathrm{d}s \\ &\leqslant \frac{\text{const}\cdot\tau^{k/2}}{(1+t)^\alpha},\end{aligned} \tag{9.27}$$

thanks to the bounds $\mathrm{E}(\|u_s\|^k_{\ell^k(\mathbf{Z}^d)}) \leqslant \text{const}\cdot(1+s)^{-\alpha}$ and $\bar{P}(t+\tau-s) \leqslant 1$. Since $\|u_0\|_{\ell^k(\mathbf{Z}^d)} \leqslant \|u_0\|_{\ell^1(\mathbf{Z}^d)}$, displays (9.18), (9.24), and (9.27) together imply (9.12). This yields the first estimate of the proposition. The remaining assertions follow (9.12), using a suitable form of the Kolmogorov continuity theorem [34, Theorem 2.1, p. 25] and the fact that $\sup_{x\in\mathbf{Z}^d}|u_t(x)-u_s(x)| \leqslant \|u_t-u_s\|_{\ell^k(\mathbf{Z}^d)}$. □

Proof of Theorem 1.7. We apply Proposition 8.7 and Chebyshev's inequality in conjunction in order to see that,

$$\begin{aligned}\sum_{n=1}^\infty \mathrm{P}\left\{\sup_{x\in\mathbf{Z}^d}|u_n(x)|>\varepsilon\right\} &\leqslant \frac{1}{\varepsilon^2}\sum_{n=1}^\infty \mathrm{E}\left(\|u_n\|^2_{\ell^2(\mathbf{Z}^d)}\right)\\ &\leqslant \frac{\text{const}}{\varepsilon^2}\cdot\sum_{n=1}^\infty n^{-\alpha}<\infty.\end{aligned} \tag{9.28}$$

Therefore, the Borel–Cantelli lemma implies that

$$\lim_{n\to\infty}\sup_{x\in\mathbf{Z}^d}|u_n(x)|=0 \qquad \text{a.s.} \tag{9.29}$$

We next note that the Burkholder's constants z_k vary continuously for $k \geqslant 2$ and $z_2 = 1$ is the minimum, see Davis [22]. Davis [22] obtains z_k as the largest positive zero of the parabolic cylinder function of parameter k and this varies continuously in k, see Abramowitz and Stegun [1].

If $\mathrm{Lip}_\sigma\sqrt{\Upsilon(0)}<1$, we can find $k>2$ and $\delta>0$ such that

$$z_k\mathrm{Lip}_\sigma\sqrt{\Upsilon(0)}<(1+\delta)^{-(k-1)/k}. \tag{9.30}$$

We can now use Proposition 9.4 (with $\eta=0$) along with Chebyshev's inequality to control the spacings

$$\begin{aligned}&\mathrm{P}\left\{\sup_{s\in[n,n+1]}\sup_{x\in\mathbf{Z}^d}|u_s(x)-u_n(x)|>\varepsilon\right\} \qquad (9.31)\\ &\leqslant \frac{1}{\varepsilon^k}\sup_I \mathrm{E}\left(\sup_{s\in[n,n+1]}\sup_{x\in\mathbf{Z}^d}|u_t(x)-u_s(x)|^k\right)=O\left(n^{-\alpha}\right) \qquad \text{as } n\to\infty.\end{aligned}$$

We may use the Borel–Cantelli lemma and (9.29) in order to deduce that $\lim_{t\to\infty}\sup_{x\in\mathbf{Z}^d}|u_t(x)|=0$ a.s. Thanks to this fact, Corollary 8.5 implies the seemingly-stronger assertion that $\lim_{t\to\infty}\|u_t\|^2_{\ell^2(\mathbf{Z}^d)}=0$ a.s., and completes the proof. □

A Some renewal theory

In this appendix we state and prove a few facts from [linear] renewal theory. These facts ought to be well known, but we have not succeeded to find concrete references, and so will describe them in some detail.

Let us suppose that the functions $h, g : (0\,,\infty) \to \mathbf{R}_+$ are locally integrable [say] and pre-defined, and let us look for a measurable solution $f : (0\,,\infty) \to \mathbf{R}_+$ to the renewal equation,

$$f(t) = g(t) + \int_0^t h(t-s) f(s)\,\mathrm{d}s \qquad (t \geqslant 0). \tag{A.1}$$

If $h \in L^1(0\,,\infty)$, then this is a classical subject [23]. For a more general treatment, we may proceed with Picard's iteration: Let $f^{(0)}(t) : (0\,,\infty) \to \mathbf{R}_+$ be a fixed measurable function, and iteratively define

$$f^{(n+1)}(t) := g(t) + \int_0^t h(t-s) f^{(n)}(s)\,\mathrm{d}s \qquad (t > 0,\ n \geqslant 0). \tag{A.2}$$

Lemma A.1. *Suppose that there exists a constant $\beta \in \mathbf{R}$ that satisfies the following three conditions: (i) $\gamma := \sup_{t\geqslant 0}[\exp(-\beta t) g(t)] < \infty$; (ii) $\rho := \int_0^\infty \exp(-\beta t) h(t)\,\mathrm{d}t < 1$; and (iii) $\sup_{t\geqslant 0}[\exp(-\beta t) f^{(0)}(t)] < \infty$. Then* (A.1) *has a unique non-negative solution f that satisfies the following:*

$$f(t) \leqslant \frac{\gamma \mathrm{e}^{\beta t}}{1-\rho} \qquad (t \geqslant 0). \tag{A.3}$$

Moreover, $\lim_{n\to\infty} \sup_{t\geqslant 0}(\mathrm{e}^{-\beta t}|f^{(n)}(t) - f(t)|) = 0$.

Proof. Choose such a $\beta \in \mathbf{R}$ and define

$$\gamma := \sup_{t\geqslant 0}\left[\mathrm{e}^{-\beta t} g(t)\right], \qquad \rho := \int_0^\infty \mathrm{e}^{-\beta t} h(t)\,\mathrm{d}t < 1, \tag{A.4}$$

and

$$C_k := \sup_{t\geqslant 0}\left(\mathrm{e}^{-\beta t} f^{(k)}(t)\right), \qquad D_k := \sup_{t\geqslant 0}\left(\mathrm{e}^{-\beta t}\left|f^{(k)}(t) - f^{(k-1)}(t)\right|\right), \tag{A.5}$$

for integers $k \geqslant 1$. Thanks to the definition of the $f^{(k)}$'s,

$$C_{n+1} \leqslant \gamma + \rho C_n, \quad D_{n+1} \leqslant \rho D_n \qquad (n \geqslant 0). \tag{A.6}$$

Consequently, $\sup_{n\geqslant 0} C_n \leqslant \gamma(1-\rho)^{-1}$ and $D_n = O(\rho^n)$. Since $\sum_{n=0}^\infty D_n < \infty$, it follows that there exists a function f such that $\sup_{t\geqslant 0}(\mathrm{e}^{-\beta t}|f^{(n)}(t) - f(t)|) \to 0$ as $n \to \infty$, and $\sup_{t\geqslant 0}(\mathrm{e}^{-\beta t} f(t)) \leqslant \sup_{n\geqslant 0} C_n$. These observations together prove the lemma. □

The following is the main result of this appendix.

Lemma A.2 (Comparison lemma)**.** *Suppose there exists $\beta \in \mathbf{R}$ such that: (i) $\gamma := \sup_{t\geqslant 0}[\exp(-\beta t) g(t)] < \infty$; and (ii) $\rho := \int_0^\infty \exp(-\beta t) h(t)\,\mathrm{d}t < 1$; and let f denote the unique non-negative solution to* (A.1) *that satisfies* (A.3)*. If $F : \mathbf{R}_+ \to \mathbf{R}_+$ satisfies: (a) $\sup_{t\geqslant 0}[\exp(-\beta t) F(t)] < \infty$; and (b)*

$$F(t) \geqslant g(t) + \int_0^t h(t-s) F(s)\,\mathrm{d}s \qquad (t \geqslant 0), \tag{A.7}$$

then $f(t) \leqslant F(t)$ *for all* $t \geqslant 0$*. Finally, if we replace condition* (A.7) *by*

$$F(t) \leqslant g(t) + \int_0^t h(t-s)F(s)\,\mathrm{d}s \qquad (t \geqslant 0), \tag{A.8}$$

then $f(t) \geqslant F(t)$ *for all* $t \geqslant 0$*.*

Proof. We will prove (A.7); (A.8) is proved similarly.

We apply Picard's iteration with initial function $f^{(0)} := F$, and note that

$$f^{(1)}(t) = g(t) + \int_0^t h(t-s)F(s)\,\mathrm{d}s \leqslant F(t) \qquad (t \geqslant 0). \tag{A.9}$$

This and induction together show that $f^{(n+1)}(t) \leqslant f^{(n)}(t)$ for all $t \geqslant 0$ and $n \geqslant 0$. Let $n \to \infty$ to deduce the lemma from Lemma A.1. □

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