

PACKING DIMENSION OF THE RANGE OF A LÉVY PROCESS

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ABSTRACT. Let $\{X(t)\}_{t \geq 0}$ denote a Lévy process in \mathbf{R}^d with exponent Ψ . Taylor (1986) proved that the packing dimension of the range $X([0, 1])$ is given by the index

$$(0.1) \quad \gamma' = \sup \left\{ \alpha \geq 0 : \liminf_{r \rightarrow 0^+} \int_0^1 \frac{\mathbf{P}\{|X(t)| \leq r\}}{r^\alpha} dt = 0 \right\}.$$

We provide an alternative formulation of γ' in terms of the Lévy exponent Ψ . Our formulation, as well as methods, are Fourier-analytic, and rely on the properties of the Cauchy transform. We show, through examples, some applications of our formula.

1. INTRODUCTION

Let $X := \{X(t)\}_{t \geq 0}$ denote a d -dimensional Lévy process (Bertoin, 1998; Sato, 1999) which starts at the origin. Define Ψ to be the Lévy exponent of X , normalized so that $\mathbf{E}[\exp(izX(t))] = \exp(-t\Psi(z))$ for all $t \geq 0$ and $z \in \mathbf{R}^d$, and let \dim_p denote the packing dimension (Tricot, 1982; Sullivan, 1984). Taylor (1986) has proved that with probability one, $\dim_p X([0, 1]) = \gamma'$, where γ' is the index of Hendricks (1983); see (0.1).

Usually, one defines a Lévy process by constructing its Lévy exponent Ψ . From this perspective, formula (0.1) is difficult to apply in concrete settings. Primarily this is because the small- r behavior of $\int_0^1 \mathbf{P}\{|X(t)| \leq r\} dt$ is only well-understood when X is a nice Lévy process. For instance, when X is a subordinator γ' can be shown to be equal to the Blumenthal and Gettoor (1961) upper index β (Fristedt and Taylor, 1992; Bertoin, 1999); see also Theorem 3.3 below. When X is a general Lévy process Pruitt and Taylor (1996) find several quantitative relationships between γ' and other known fractal indices of Lévy processes.

The principle goal of this article is to describe $\gamma' = \dim_p X([0, 1])$ more explicitly than (0.1), and solely in terms of the Lévy exponent Ψ . For all $r > 0$ define

$$(1.1) \quad W(r) := \int_{\mathbf{R}^d} \frac{\kappa(x/r)}{\prod_{j=1}^d (1 + x_j^2)} dx,$$

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where κ is the following well-known function (Orey, 1967; Kesten, 1969):

$$(1.2) \quad \kappa(z) := \operatorname{Re} \left(\frac{1}{1 + \Psi(z)} \right) \quad (z \in \mathbf{R}^d).$$

This function is symmetric (i.e., $\kappa(-z) = \kappa(z)$ for all $z \in \mathbf{R}^d$) and satisfies the pointwise bounds $0 \leq \kappa \leq 1$, which we use tacitly throughout. The following contains our formula for γ' .

Theorem 1.1. *For all d -dimensional Lévy processes X ,*

$$(1.3) \quad \dim_{\mathbf{p}} X([0, 1]) = \sup \left\{ \alpha \geq 0 : \liminf_{r \rightarrow 0^+} \frac{W(r)}{r^\alpha} = 0 \right\} = \limsup_{r \rightarrow 0^+} \frac{\log W(r)}{\log r},$$

almost surely, where $\sup \emptyset := 0$.

Xiao (2004, Question 4.16) has asked if we can write $\dim_{\mathbf{p}} X([0, 1])$ explicitly in terms of Ψ . Theorem 1.1 answers this question in the affirmative.

The following is one of the many consequences of Theorem 1.1.

Theorem 1.2. *Let X be a d -dimensional Lévy process that has a non-trivial, non-degenerate Gaussian part. That is, $X = G + Y$, where G is a non-degenerate Gaussian Lévy process, and Y is an independent pure-jump Lévy process. Then, $\dim_{\mathbf{p}} X([0, 1]) = \dim_{\mathbf{p}} G([0, 1])$ and $\dim_{\mathbf{H}} X([0, 1]) = \dim_{\mathbf{H}} G([0, 1])$ a.s., where $\dim_{\mathbf{H}}$ denotes the Hausdorff dimension.*

We do not know of a direct proof of this result, although it is a very natural statement. However, some care is needed as the result can fail when G is degenerate (Example 4.1). Our methods will make clear that in general we can say only that $\dim X([0, 1]) \geq \dim G([0, 1])$ a.s., where “dim” stands for either “ $\dim_{\mathbf{p}}$ ” or “ $\dim_{\mathbf{H}}$.”

We also mention the following ready consequence of Theorem 1.1:

Corollary 1.3. *Let X be a Lévy process in \mathbf{R}^d and X' its symmetrization; i.e., $X'(t) = X(t) - X''(t)$ where X'' is an independent copy of X . Then,*

$$(1.4) \quad \dim_{\mathbf{p}} X([0, 1]) \geq \dim_{\mathbf{p}} X'([0, 1]) \quad a.s.$$

It has been shown that Corollary 1.3 continues to hold if we replace $\dim_{\mathbf{p}}$ by $\dim_{\mathbf{H}}$ everywhere; see Khoshnevisan, Xiao, and Zhong (2003) and/or (1.5) below. Thus we have further confirmation of the somewhat heuristic observation of Kesten (1969, p. 7) that the range of X is larger than the range of its symmetrization.

Theorem 1.1 is proved in Section 2. Our proof also yields the following almost-sure formula for the Hausdorff dimension of $X([0, 1])$:

$$(1.5) \quad \dim_{\mathbf{H}} X([0, 1]) = \sup \left\{ \alpha \geq 0 : \limsup_{r \rightarrow 0^+} \frac{W(r)}{r^\alpha} = 0 \right\} = \liminf_{r \rightarrow 0^+} \frac{\log W(r)}{\log r};$$

see Remark 2.4. Recently, Khoshnevisan, Xiao, and Zhong (2003) established an equivalent formulation of this formula. Whereas their derivation is long and complicated, ours is direct and fairly elementary. Section 3 contains non-trivial examples wherein we compute $\dim_p X([0, 1])$ for anisotropic Lévy processes X . Finally, Theorem 1.2 is proved in Section 4.

2. THE INCOMPLETE RENEWAL MEASURE

Define U to be the *incomplete renewal measure* of X . That is, for all Borel sets $A \subset \mathbf{R}^d$,

$$(2.1) \quad U(A) := \int_0^1 \mathbf{P}\{X(t) \in A\} dt.$$

We may deduce from (0.1) that

$$(2.2) \quad \dim_p X([0, 1]) = \limsup_{r \rightarrow 0^+} \frac{\log U(B(0, r))}{\log r},$$

where $B(a, r) := \{z \in \mathbf{R}^d : |z - a| \leq r\}$ for all $a \in \mathbf{R}^d$ and $r \geq 0$, and $|y| := \max_{1 \leq j \leq d} |y_j|$ for all $y \in \mathbf{R}^d$.

Let ζ denote an independent, mean-one exponential random variable. The killed occupation measure of $B(0, r)$ can then be defined by

$$(2.3) \quad T(r) := \int_0^\zeta \mathbf{1}_{B(0, r)}(X(t)) dt \quad (r > 0).$$

Proposition 2.1. *For all $r > 0$,*

$$(2.4) \quad \frac{e}{4^d(e-1)} \mathbf{E}[T(r)] \leq U(B(0, r)) \leq e \mathbf{E}[T(r)].$$

In order to prove this we first recall the notion of weak unimodality (Khoshnevisan and Xiao, 2003).

Definition 2.2. A Borel measure μ on \mathbf{R}^d is *c-weakly unimodal* if $c > 0$ is a constant that satisfies $\sup_{a \in \mathbf{R}^d} \mu(B(a, r)) \leq c\mu(B(0, r))$ for all $r > 0$.

The following is a variant of Lemma 4.1 of Khoshnevisan and Xiao (2003).

Lemma 2.3. *U is 4^d -weakly unimodal.*

Proof. Let us fix $a \in \mathbf{R}^d$ and $r > 0$, and define

$$(2.5) \quad \sigma := \inf\{s > 0 : |X(s) - a| < r\},$$

where $\inf \emptyset := \infty$. Clearly, σ is a stopping time, and by the strong Markov property,

$$(2.6) \quad \begin{aligned} U(B(a, r)) &= \mathbb{E} \left[\int_0^{(1-\sigma)^+} \mathbf{1}_{B(0, r)}(X(u + \sigma) - a) du \right] \\ &\leq \mathbb{E} \left[\int_0^1 \mathbf{1}_{B(0, 2r)}(X(u + \sigma) - X(\sigma)) du \right] = U(B(0, 2r)). \end{aligned}$$

Euclidean topology dictates that there are points $z_1, \dots, z_{4^d} \in B(0, 2r)$ that have the property that $\cup_{i=1}^{4^d} B(z_i, r/2) = B(0, 2r)$. According to (2.6), we have the following “volume doubling” property:

$$(2.7) \quad U(B(0, 2r)) \leq \sum_{j=1}^{4^d} U(B(z_j, r/2)) \leq 4^d U(B(0, r)).$$

The desired result follows from this and (2.6). \square

Proof of Proposition 2.1. The upper bound is immediate. To prove the other half we note that

$$(2.8) \quad \mathbb{E}[T(r)] = \int_0^\infty \mathbb{P}\{|X(t)| \leq r\} e^{-t} dt \leq \sum_{j=0}^\infty e^{-j} \mathbb{E}[U(B(X(j), r))],$$

thanks to the Markov property. Lemma 2.3 implies that the preceding is at most $4^d \sum_{j=0}^\infty e^{-j} U(B(0, r))$. The proposition follows from this. \square

Proof of Theorem 1.1. We derive only the second identity of (1.3); the first is manifestly an equivalent statement.

Let $(\mathcal{F}f)(z) := \int_{\mathbf{R}^d} e^{izx} f(x) dx$ denote the Fourier transform of $f \in L^1(\mathbf{R}^d)$. For all fixed $r > 0$ and $x \in \mathbf{R}^d$ define

$$(2.9) \quad \phi_r(x) := \prod_{j=1}^d \frac{1 - \cos(2rx_j)}{2\pi r x_j^2}.$$

Then ϕ_r is a non-negative function and $(\mathcal{F}\phi_r)(z) = \prod_{j=1}^d (1 - |z_j|/(2r))^+$ for all $z \in \mathbf{R}^d$, where $a^+ := \max(a, 0)$. Evidently, $\phi_r \in L^1(\mathbf{R}^d)$, and $0 \leq \mathcal{F}\phi_r \leq 1$ pointwise.

Let $\mathbf{1}_A$ denote the indicator function of A . Note that $z \in B(0, r)$ implies that $1 - (2r)^{-1}|z_j| \geq \frac{1}{2}$. This implies that $\mathbf{1}_{B(0, r)}(z) \leq 2^d (\mathcal{F}\phi_r)(z)$ for all $z \in \mathbf{R}^d$. Therefore, by the Fubini–Tonelli theorem, $\mathbb{E}[T(r)] \leq 2^d \int_{\mathbf{R}^d} \kappa(x) \phi_r(x) dx \leq 2^d W(r)$. The last inequality follows readily from the elementary bound, $(1 - \cos z)/z^2 \leq 2\pi/(1 + z^2)$, valid for all $z \in \mathbf{R}$. Thanks to (2.2) and Proposition 2.1 we can deduce that almost surely,

$$(2.10) \quad \dim_{\mathbf{P}} X([0, 1]) \geq \limsup_{r \rightarrow 0^+} \frac{\log W(r)}{\log r}.$$

In order to establish the converse inequality we introduce the process $S(t) := (S_1(t), \dots, S_d(t))$, where S_1, \dots, S_d are independent symmetric Cauchy processes in \mathbf{R} , all with the same characteristic function $\mathbb{E}[e^{izS_1(t)}] = e^{-t|z|}$. We assume further that S is independent of X . Then for all $\lambda > 0$, $\mathbb{E}[\exp\{iX(t) \cdot S(\lambda)\}] = \mathbb{E}[\exp\{-\lambda \sum_{j=1}^d |X_j(t)|\}]$. On the other hand, the scaling property of S implies

$$(2.11) \quad \mathbb{E}[e^{iX(t) \cdot S(\lambda)}] = \mathbb{E}[e^{-t\Psi(S(\lambda))}] = \frac{1}{\pi^d} \int_{\mathbf{R}^d} \frac{e^{-t\Psi(\lambda x)}}{\prod_{j=1}^d (1 + x_j^2)} dx.$$

For all $r, k > 0$ and $x \in \mathbf{R}^d$, $\exp\{-(k/r) \sum_{j=1}^d |x_j|\} \leq \mathbf{1}_{B(0,r)}(x) + e^{-k} \mathbf{1}_{B(0,r)^c}(x)$. Therefore, $\mathbf{1}_{B(0,r)}(x) \geq \mathbb{E}[\exp\{ix \cdot S(k/r)\}] - e^{-k} \mathbf{1}_{B(0,r)^c}(x)$, whence $\mathbb{E}[T(r)]$ is at least $\int_0^\infty \mathbb{E}[\exp\{iX(t) \cdot S(k/r)\}] e^{-t} dt - e^{-k}(1 - \mathbb{E}[T(r)])$. This and (2.11) together imply that

$$(2.12) \quad (1 - e^{-k})\mathbb{E}[T(r)] \geq -e^{-k} + \frac{1}{\pi^d} W(r/k).$$

Now we choose $k = r^{-\varepsilon}$, for an arbitrary small $\varepsilon > 0$, to find that the inequality in (2.10) is an equality. This completes our proof. \square

Remark 2.4. From the proof of Theorem 1.1 we see that $\mathbb{E}[T(r)]$ and $W(r)$ are roughly comparable; i.e., for all $\varepsilon, r > 0$ sufficiently small,

$$(2.13) \quad \frac{1}{\pi^d} W(r^{1+\varepsilon}) - \exp(-r^{-\varepsilon}) \leq \mathbb{E}[T(r)] \leq 4^d W(r).$$

Thanks to Proposition 2.1 this yields (1.5).

3. SOME EXAMPLES

We illustrate the utility of Theorem 1.1 by specializing it to a large class of examples.

3.1. Anisotropic Examples. It is possible to construct examples of anisotropic Lévy “stable-like” processes whose $\dim_{\mathbf{p}} X([0, 1])$ are computable. The following furnishes most of the basic technical background that we shall need.

Theorem 3.1. *Let X be a Lévy process in \mathbf{R}^d with Lévy exponent Ψ . Suppose there exist constants β_1, \dots, β_d such that*

$$(3.1) \quad 2 \geq \beta_1 \geq \dots \geq \beta_d > 0 \quad \text{and} \quad \lim_{\|z\| \rightarrow \infty} \frac{1}{\ln \|z\|} \ln \operatorname{Re} \left(\frac{\sum_{j=1}^d |z_j|^{\beta_j}}{1 + \Psi(z)} \right) = 0.$$

If $N := \max\{1 \leq j \leq d : \beta_j = \beta_1\}$, then almost surely,

$$(3.2) \quad \dim_{\mathbf{p}} X([0, 1]) = \begin{cases} \beta_1, & \text{if } \beta_1 \leq N, \\ 1 + \beta_2 \left(1 - \frac{1}{\beta_1}\right), & \text{otherwise.} \end{cases}$$

Proof. Throughout this proof we write c for a generic constant whose value can change within, as well as between, lines. In order to simplify the exposition somewhat, we note that it is sufficient to prove 3.2 under the following [slightly] stronger form of (3.1):

$$(3.3) \quad \frac{c}{1 + \sum_{j=1}^d |z_j|^{\beta_j}} \leq \operatorname{Re} \left(\frac{1}{1 + \Psi(z)} \right) \leq \frac{c}{1 + \sum_{j=1}^d |z_j|^{\beta_j}} \quad \text{for all } z \in \mathbf{R}^d.$$

First we consider the case when $\beta_1 \leq N$. Condition (3.3) implies that if $r \in (0, 1)$, then

$$(3.4) \quad W(r) \geq cr^{\beta_1} \int_{\mathbf{R}^d} \frac{dx}{(1 + \sum_{j=1}^d |x_j|^{\beta_j}) \prod_{j=1}^d (1 + x_j^2)}.$$

It follows from this and Theorem 1.1 that $\dim_p X([0, 1]) \leq \beta_1$ a.s.

Recall that $N := \max\{1 \leq j \leq d : \beta_j = \beta_1\}$. From this and a few lines of computations it follows that $W(r) = O(r^{\beta_1} \log(1/r))$, whence $\dim_p X([0, 1]) \geq \beta_1$ a.s. [The $\log(1/r)$ accounts for the case that $\beta_1 = N$.]

Next we consider the case when $\beta_1 > N$. This implies $N = 1$ and $\beta_1 > \beta_2$. In order to prove that $\dim_p X([0, 1]) \leq 1 + \beta_2(1 - \beta_1^{-1})$ a.s. we first derive a lower bound for $W(r)$. We do this by restricting the integral to the domain $D := \{x \in \mathbf{R}^d : |x_j| \leq 1 \text{ for } 3 \leq j \leq d\}$. More precisely, by (3.3) we have

$$(3.5) \quad \begin{aligned} W(r) &\geq c \int_D \frac{dx}{(1 + \sum_{j=1}^d |x_j/r|^{\beta_j}) \prod_{j=1}^d (1 + x_j^2)} \\ &\geq c \int_D \frac{dx}{(1 + \sum_{j=1}^d |x_j/r|^{\beta_j}) \prod_{j=1}^2 (1 + x_j^2)}. \end{aligned}$$

Let $(x_3, \dots, x_d) \in [-1, 1]^{d-2}$ be fixed and let $A := 1 + \sum_{j=3}^d |x_j/r|^{\beta_j}$. Consider

$$(3.6) \quad \begin{aligned} \mathcal{J} &:= \int_{\mathbf{R}^2} \frac{dx_1 dx_2}{(A + |x_1/r|^{\beta_1} + |x_2/r|^{\beta_2}) \prod_{j=1}^2 (1 + x_j^2)} \\ &\geq r^{\beta_2} \int_{r^{1-(\beta_2/\beta_1)}}^{\infty} dx_1 \int_0^{\infty} \frac{dx_2}{(r^{\beta_2} A + r^{\beta_2-\beta_1} x_1^{\beta_1} + x_2^{\beta_2}) \prod_{j=1}^2 (1 + x_j^2)}. \end{aligned}$$

Observe that $r^{\beta_2-\beta_1} x_1^{\beta_1} \geq 1$ for all $x_1 \geq r^{1-(\beta_2/\beta_1)}$. On the other hand, $r^{\beta_2} A \leq d - 1$ for all $r \in (0, 1)$. It follows from these facts, and a change of variables, that $\mathcal{J} \geq c r^{1+\beta_2(1-\beta_1^{-1})}$. Combine this with (3.5) and (3.6) to deduce that $W(r) \geq c r^{1+\beta_2(1-\beta_1^{-1})}$ for all $r \in (0, 1)$, whence $\lim_{r \rightarrow 0^+} r^{-\alpha} W(r) = \infty$ for all $\alpha > 1 + \beta_2(1 - \beta_1^{-1})$. This implies that $\dim_p X([0, 1]) \leq 1 + \beta_2(1 - \beta_1^{-1})$ a.s.

Now we derive the lower bound for $\dim_p X([0, 1])$. It follows from (3.3) that

$$(3.7) \quad \begin{aligned} W(r) &\leq c \int_{\mathbf{R}^2} \frac{dx_1 dx_2}{(1 + |x_1/r|^{\beta_1} + |x_2/r|^{\beta_2}) \prod_{j=1}^2 (1 + x_j^2)} \\ &= 4c r^{\beta_2} \int_0^\infty \frac{dx_1}{1 + x_1^2} \int_0^\infty \frac{dx_2}{(B + x_2^{\beta_2})(1 + x_2^2)}, \end{aligned}$$

where $B := r^{\beta_2} + r^{\beta_2 - \beta_1} x_1^{\beta_1}$. It remains to verify that the last expression in (3.7) is at most $c r^{1 + \beta_2(1 - \beta_1^{-1})} \log(1/r)$, where $\log(1/r)$ appears because of the possibility that $\beta_2 = 1$.

By breaking the dx_2 -integral according to whether $|x_2| \leq 1$ or $|x_2| > 1$, and after a change of variables, we can verify the following elementary inequalities:

- (i) If $B \leq 1$, then $\int_0^\infty (B + x_2^{\beta_2})^{-1} (1 + x_2^2)^{-1} dx_2 \leq c g(x_1)$, where: $g(x_1) = 1$ if $\beta_2 < 1$, $\log(B^{-1})$ if $\beta_2 = 1$, and $B^{(1/\beta_2) - 1}$ if $\beta_2 < 1$.
- (ii) If $B > 1$, then $\int_0^\infty (B + x_2^{\beta_2})^{-1} (1 + x_2^2)^{-1} dx_2 \leq c/B$.

We return to (3.7) and split the dx_1 -integral respectively over the intervals $\{x_1 : B \leq 1\}$ and $\{x_1 : B > 1\}$. It follows from (3.7), (i) and (ii), and a direct computation, that $W(r) \leq c r^{1 + \beta_2(1 - \beta_1^{-1})} \log(1/r)$. Hence, Theorem 1.1 implies that $\dim_p X([0, 1]) \geq 1 + \beta_2(1 - \beta_1^{-1})$ a.s. This finishes the proof Theorem 3.1. \square

In the following, we apply Theorem 3.1 to operator-stable Lévy processes in \mathbf{R}^d with $d \geq 2$. Recall from Sharpe (1969) that a Lévy process X in \mathbf{R}^d is called *operator stable* if: (i) The law $\nu := P \circ (X(1))^{-1}$ of $X(1)$ is not supported on any $(d - 1)$ -dimensional hyper-plane [i.e., $X(1)$ is *full*]; and (ii) ν is *strictly operator stable*. That is, there exists a linear operator B on \mathbf{R}^d such that the t -fold convolution of ν satisfies $\nu^t = t^B \nu$ for all $t > 0$. Here, ν^t denotes the t -fold convolution power of ν , and $t^B \nu(dx) := \nu(t^{-B} dx)$ is the image measure of ν under the action of the linear operator $t^B := \sum_{n=0}^\infty (\log t)^n B^n / n!$. The linear operator B is called a *stability exponent* of X . The set of all possible exponents of an operator stable law is characterized by Meerschaert and Scheffler (2001, Theorem 7.2.11).

A stochastic process $Y := \{Y(t)\}_{t \in \mathbf{R}_+}$, with values in \mathbf{R}^d , is said to be *operator self-similar* if there exists a linear operator B on \mathbf{R}^d such that for every $c > 0$, $\{Y(ct)\}_{t \geq 0} \stackrel{d}{=} \{c^B Y(t)\}_{t \geq 0}$ where “ $\stackrel{d}{=}$ ” denotes the equality of finite-dimensional distributions. The linear operator B is called a *self-similarity exponent* of Y .

Hudson and Mason (1982) proved that if X is a Lévy process in \mathbf{R}^d such that the distribution of $X(1)$ is full, then X is operator self-similar if and only if $X(1)$ is strictly operator stable. In this case, every stability exponent B of X is also a self-similarity exponent of X . Hence, from now on we will refer to B simply as an *exponent* of X .

Operator-stable Lévy processes are scaling limits of d -dimensional random walks that are normalized by linear operators (Meerschaert and Scheffler, 2001, Chapter 11). All d -dimensional strictly stable Lévy processes of index α are operator-stable with exponent $B := \alpha^{-1}I$, where I denotes the $(d \times d)$ identity matrix.

More generally, let X_1, \dots, X_d be independent stable Lévy processes in \mathbf{R} with respective indices $\alpha_1, \dots, \alpha_d \in (0, 2]$. Define $X(t) := (X_1(t), \dots, X_d(t))$. One can then verify that X is an operator-stable Lévy process whose exponent B is the $(d \times d)$ diagonal matrix $\text{diag}(\alpha_1^{-1}, \alpha_2^{-1}, \dots, \alpha_d^{-1})$. These processes were first introduced by Pruitt and Taylor (1969) under the title of *Lévy processes with stable components*. These processes have been used to construct various counterexamples (Hendricks, 1983).

Let X be an operator-stable Lévy process in \mathbf{R}^d with exponent B . Factor the minimal polynomial of B into $q_1(x), \dots, q_p(x)$ where all roots of $q_i(x)$ have real part a_i , and $a_i < a_j$ for $i < j$. Define $\alpha_i := a_i^{-1}$, so that $\alpha_1 > \dots > \alpha_p$, and note that $0 < \alpha_i \leq 2$ (Meerschaert and Scheffler, 2001, Theorem 7.2.1). Define $V_i := \text{Ker}(q_i(B))$ and $d_i := \dim(V_i)$. Then $d_1 + \dots + d_p = d$, and $V_1 \oplus \dots \oplus V_p$ is a direct-sum decomposition of \mathbf{R}^d into B -invariant subspaces. We may write B as $B = B_1 \oplus \dots \oplus B_p$, where $B_i : V_i \rightarrow V_i$ and every eigenvalue λ of B_i has the property that $\text{Re } \lambda = a_i$. We can apply Theorem 3.1 for operator stable Lévy processes to obtain a wholly different proof of the following theorem of Meerschaert and Xiao (2005, Theorem 3.2).

Theorem 3.2 (Meerschaert and Xiao (2005)). *Let X be an operator-stable Lévy process in \mathbf{R}^d . Then almost surely,*

$$(3.8) \quad \dim_{\mathbf{P}} X([0, 1]) = \begin{cases} \alpha_1, & \text{if } \alpha_1 \leq d_1, \\ 1 + \alpha_2 \left(1 - \frac{1}{\alpha_1}\right), & \text{otherwise.} \end{cases}$$

Proof. Define $\beta_j := \alpha_\ell$ where ℓ is determined by $\sum_{i=0}^{\ell-1} d_i < j \leq \sum_{i=\ell-1}^{\ell} d_i$, and $d_0 := 0$. Because Meerschaert and Xiao (2005) have established (3.1), Theorem 3.1 implies (3.8) with $N := d_1$. \square

3.2. Subordinators. Let us consider the special case that X is a [non-negative] subordinator. We conclude this article by showing that our Theorem 1.1 includes the well-known formula for $\dim_{\mathbf{P}} X([0, 1])$; see Fristedt and Taylor (1992) and Bertoin (1999, §5.1.2). Let Φ denote the Laplace exponent of X , normalized so that $\mathbf{E}[\exp(-\lambda X(t))] = \exp(-t\Phi(\lambda))$ for all $\lambda, t \geq 0$. The following is an immediate consequence of Theorem 1.1.

Theorem 3.3 (Fristedt and Taylor (1992); Bertoin (1999)). *With probability one,*

$$(3.9) \quad \dim_{\mathbf{p}} X([0, 1]) = \limsup_{\lambda \rightarrow \infty} \frac{\log \Phi(\lambda)}{\log \lambda}.$$

Proof. Let S denote an independent Cauchy process such that $E[\exp(i\xi S(t))] = \exp(-t|\xi|)$ for all $t \geq 0$ and $\xi \in \mathbf{R}$. Then,

$$(3.10) \quad e^{-t\Phi(\lambda)} = E[e^{-\lambda X(t)}] = E[e^{iX(t)S(\lambda)}] = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-t\Psi(\lambda z)}}{1 + z^2} dz.$$

Multiply both sides by e^{-t} and integrate $[dt]$ to find that $(1 + \Phi(\lambda))^{-1} = \pi^{-1}W(1/\lambda)$. A direct appeal to Theorem 1.1 finishes the proof. \square

4. PROOF OF THEOREM 1.2

Throughout, $\|x\| := (x_1^2 + \cdots + x_d^2)^{1/2}$ for all $x \in \mathbf{R}^d$. This is the usual ℓ^2 -norm on \mathbf{R}^d , and should not be confused with the ℓ^∞ -norm $|x| = \max_{1 \leq j \leq d} |x_j|$ that we have used so far.

Because of the Lévy–Khintchine formula (Bertoin, 1998), X indeed has the stated representation. That is, we can write X as $X = G + Y$, where G is a Gaussian Lévy process, and Y is an independent, pure-jump Lévy process.

Thanks to the centered-ball inequality of T. W. Anderson (1955), $a \mapsto P\{|G(t) - a| \leq r\}$ is maximized at the origin. Apply this, conditionally on Y , to find that $P\{|X(t)| \leq r\} \leq P\{|G(t)| \leq r\}$ for all $t, r > 0$. Thus, it follows from (0.1) and (2.2) that $\dim_{\mathbf{p}} G([0, 1]) \leq \dim_{\mathbf{p}} X([0, 1])$ a.s. The analogous bound for $\dim_{\mathbf{H}}$ follows from this and the formula of Pruitt (1969).

In order to prove the converse bound we appeal to Theorem 1.1. Recall that $\Psi_X(z) = O(\|z\|^2)$ as $\|z\| \rightarrow \infty$ (Bochner, 1955, eq. (3.4.14), p. 67). [The subscript X refers to the process X , the subscript G to the process G , etc.] Therefore, there exists a constant C such that $\kappa_X(z) \geq C\{1 + \|z\|^2\}^{-1}$ for all $z \in \mathbf{R}^d$. The non-degeneracy of G implies that $\|z\|^{-2}\Psi_G(z)$ is bounded below uniformly for all $z \in \mathbf{R}^d$. Because $\operatorname{Re} \Psi_Y(z) \geq 0$, it follows that there exists a constant c such that $\kappa_X(z) \geq c\kappa_Y(z)$ for all $z \in \mathbf{R}^d$. Therefore, $W_X(r) \geq cW_G(r)$, and Theorem 1.1 shows that $\dim_{\mathbf{p}} X([0, 1]) \leq \dim_{\mathbf{p}} G([0, 1])$ a.s. The analogous bounds for $\dim_{\mathbf{H}}$ follows from (1.5), Remark 2.4 and Proposition 2.1. This completes the proof. \square

We conclude this section by mentioning a simple example wherein Theorem 1.2 fails because G is degenerate.

Example 4.1. Let Y be an isotropic stable Lévy process in \mathbf{R}^2 with index $\alpha \in (1, 2]$. Let G_1 be an independent one-dimensional Brownian motion, and define $G(t) := (G_1(t), 0)$. Then, $X := G + Y$ has the form of the process in Theorem 1.2, but now G is degenerate. Direct

calculations show that $\Psi_Y(z) = c\|z\|^\alpha$ for some $c > 0$, and $\Psi_G(z) = c'z_1^2$ for some $c' > 0$. It follows readily from this discussion that $\Psi_X(z) = c\|z\|^\alpha + c'z_1^2$, whence it follows that κ_X is bounded above and below by universal-constant multiplies of $(|z_1|^2 + |z_2|^\alpha)^{-1}$. Theorem 3.1 implies that with probability one, $\dim_{\mathbb{P}} X([0, 1]) = 1 + \alpha(1 - \frac{1}{2}) = 1 + (\alpha/2)$. On the other hand, according to Theorem 1.1, with probability one $\dim_{\mathbb{P}} Y([0, 1]) = \alpha$, whereas $\dim_{\mathbb{P}} G([0, 1]) = 1$. Therefore, if $\alpha \in (1, 2)$ then $\dim_{\mathbb{P}} X([0, 1])$ is almost surely *strictly* greater than both $\dim_{\mathbb{P}} Y([0, 1])$ and $\dim_{\mathbb{P}} G([0, 1])$.

Despite the preceding, it is not always necessary that G is non-degenerate, viz.,

Example 4.2. Let $Y := \{Y(t)\}_{t \in \mathbf{R}_+}$ be a Lévy process in \mathbf{R}^d with characteristic exponent $\Psi(\xi) = \|\xi\|^2 L(\xi)$, where $L : \mathbf{R}^d \rightarrow \mathbf{C}$ is slowly varying at infinity. Such exponents can be constructed via the Lévy–Khintchine formula. For any Gaussian process $G := \{G(t)\}_{t \in \mathbf{R}_+}$ in \mathbf{R}^d define $X := Y + G$. It follows from Theorem 3.1 that $\dim_{\mathbb{P}} X([0, 1]) = \dim_{\mathbb{P}} Y([0, 1]) = \min(2, d)$ almost surely.

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