# Non-linear noise excitation of intermittent stochastic PDEs and the topology of LCA groups*

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**Abstract**

Consider the stochastic heat equation $\partial_t u = \mathscr{L}u + \lambda\sigma(u)\xi$, where $\mathscr{L}$ denotes the generator of a Lévy process on a locally compact Hausdorff abelian group $G$, $\sigma : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous, $\lambda \gg 1$ is a large parameter, and $\xi$ denotes space-time white noise on $\mathbf{R}_+ \times G$.

The main result of this paper contains a near-dichotomy for the [expected squared] energy $\mathrm{E}(\|u_t\|^2_{L^2(G)})$ of the solution. Roughly speaking, that dichotomy says that, in all known cases where $u$ is intermittent, the energy of the solution behaves generically as $\exp\{\text{const} \cdot \lambda^2\}$ when $G$ is discrete and $\geqslant \exp\{\text{const} \cdot \lambda^4\}$ when $G$ is connected.

*Keywords:* The stochastic heat equation, intermittency, non-linear noise excitation, Lévy processes, locally compact abelian groups.

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## 1 An informal introduction

Consider a stochastic heat equation of the form

$$\frac{\partial}{\partial t} u = \mathscr{L}u + \lambda\sigma(u)\,\xi. \tag{SHE}$$

Here, $\sigma : \mathbf{R} \to \mathbf{R}$ is a nice function, $t > 0$ denotes the time variable, $x \in G$ is the space variable for a nice state space $G$—such as $\mathbf{R}$, $\mathbf{Z}^d$, or $[0\,,1]$—and the initial value $u_0 : G \to \mathbf{R}$ is non random and well behaved. The operator $\mathscr{L}$ acts on the variable $x$ only, and denotes the generator of a nice Markov process

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on $G$, and $\xi$ denotes space-time white noise on $(0\,,\infty)\times G$. The number $\lambda$ is a positive parameter; this is the so called *level of the noise*.

In this paper, we study the "noisy case." That is when $\lambda$ is a large quantity. The case that $\lambda$ is small is also interesting; see for example the deep theory of Freidlin and Wentzel [24].

We will consider only examples of (SHE) that are *intermittent*. Intuitively speaking, "intermittency" is the property that the solution $u_t(x)$ develops extreme oscillations at some values of $x$, typically when $t$ is large. Intermittency was announced first (1949) by Batchelor and Townsend in a WHO conference in Vienna; [1] and slightly later by Emmons [21] in the context of boundary-layer turbulence. Ever since that time, intermittency has been observed in an enormous number of scientific disciplines. Shortly we will point to concrete instances in theoretical physics. In the mean time, let us also mention that, in neuroscience, intermittency is observed as "spikes" in neural activity. [Tuckwell [43] contains a gentle introduction to SPDEs in neuroscience.] And in finance, intermittency is usually associated with financial "shocks."

The standard mathematical definition of intermittency (see Molchanov [35] and Zeldovich et al [45]) is that

$$\frac{\gamma(k)}{k} < \frac{\gamma(k')}{k'} \qquad \text{whenever } 2 \leqslant k < k' < \infty, \tag{1.1}$$

where $\gamma$ denotes any reasonable choice of a so-called Lyapunov exponents of the moments of the energy of the solution: We may use either

$$\gamma(k) := \limsup_{t\to\infty} t^{-1}\log \mathrm{E}\left(\|u_t\|_{L^2(G)}^k\right), \text{ or } \gamma(k) := \liminf_{t\to\infty} t^{-1}\log \mathrm{E}\left(\|u_t\|_{L^2(G)}^k\right).$$

Other essentially-equivalent choices are also possible. One can justify this definition either by making informal analogies with finite-dimensional non-random dynamical systems [34], or by making a somewhat informal appeal to the Borel–Cantelli lemma [3]. Gibbons and Titi [26] contains an exciting modern account of mathematical intermittency and its role in our present-day understanding of physical intermittency.

In the case that $G=\mathbf{R}$, $G=[0\,,1]$, or $G=\mathbf{Z}^d$, there is a huge literature that is devoted to the intermittency properties of (SHE) when $\sigma(x)=\text{const}\cdot x$; this particular model—the so-called *parabolic Anderson model*—is interesting in its own right, as it is connected deeply with a large number of diverse questions in probability theory and mathematical physics. See, for example, the ample bibliographies of Ref.s [3, 8, 25, 29, 30, 35, 45] and those of Ref.s [10, 11, 17, 19, 22] for the many more-recent developments.

The parabolic Anderson model arises in a surprisingly large number of diverse scientific problems; see Carmona and Molchanov [8, Introduction]. We mention quickly a few such instances: If $\sigma(0)=0$, $u_0(x)>0$ for all $x\in G$, and $G$ is either $\mathbf{R}$ or $[0\,,1]$ then Mueller's comparison principle [37] shows that $u_t(x)>0$ almost surely for all $t>0$ and $x\in G$; see also [14, p. 130]. In that case, $h_t(x):=\log u_t(x)$ is well defined and is the so-called Cole–Hopf solution to the KPZ equation of statistical mechanics [29, 30]. The parabolic Anderson model has direct connections also with the stochastic Burger's equation [8] and Majda's model of shear-layer flow in turbulent diffusion [33].

Foondun and Khoshnevisan [22] have shown that the solution to (SHE) is fairly generically intermittent even when $\sigma$ is non linear, as long as $\sigma$ behaves as a line in one form or another.

It was noticed early on, in NMR spectroscopy, that intermittency can be associated strongly to non-linear noise excitation. See for example Blümich [5]; Lindner et al [32] contains a survey of many related ideas in the physics literature. In the present context, this informal observation is equivalent to the existence of a non-linear relationship between the energy $\|u_t\|_{L^2(G)}$ of the solution at time $t$ and the level $\lambda$ of the noise. A precise form of such a relationship will follow as a ready consequence of our present work in all cases where the solution is known [and/or expected] to be intermittent. In fact, the main findings of this paper will imply that typically, when the solution is intermittent, there is a near-dichotomy:

- On one hand, if $G$ is discrete then the energy of the solution behaves roughly as $\exp\{\text{const}\cdot\lambda^2\}$;
- on the other hand, if $G$ is connected then the energy behaves at least as badly as $\exp\{\text{const}\cdot\lambda^4\}$.

And quite remarkably, these properties do not depend in an essential way on the operator $\mathscr{L}$; they depend only on the connectivity properties of the underlying state space $G$.

Every standard numerical method for solving (SHE) that is known to us begins by first discretizing $G$ and $\mathscr{L}$. Our results suggest that nearly all such methods will fail generically when we use them to predict the size of the biggest intermittency islands [or shocks, or spikes] of the solution to (SHE). In a separate project we hope to address this problem by presenting a problem-dependent "practical remedy."

Other SPDE models are analyzed in a companion paper [31] which should ideally be read before the present paper. That paper is less abstract than this one and, as such, has fewer mathematical prerequisites. We present in that paper the surprising result that the stochastic heat equation on an interval is typically significantly more noise excitable than the stochastic wave equation on the real line.

## 2 Main results

The main goal of this article is to describe the behavior of (SHE) for a locally compact Hausdorff abelian group $G$, where the initial value $u_0$ is non random and is in the group algebra $L^2(G)$.[1] Compelling, as well as easy to understand, examples can be found in Section 4 below.

[1]This is the usual space of all measurable functions $f : G \to \mathbf{R}$ that are square integrable with respect to the Haar measure on $G$.

We assume throughout that the operator $\mathscr{L}$ acts on the space variable only and denotes the generator of a Lévy process $X := \{X_t\}_{t\geqslant 0}$ on $G$, $\sigma : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous and non-random, and $\xi$ denotes space-time white noise on $(0\,,\infty) \times G$. That is, $\xi$ is a generalized centered Gaussian process that is indexed by $(0\,,\infty) \times G$ and whose covariance measure is described via

$$\mathrm{Cov}\left(\int \varphi\, \mathrm{d}\xi\ ,\int \psi\, \mathrm{d}\xi\right) = \int_0^\infty \mathrm{d}t \int_G m_G(\mathrm{d}x)\ \varphi_t(x)\psi_t(x), \tag{2.1}$$

for all $\varphi, \psi \in L^2(\mathrm{d}t \times \mathrm{d}m_G)$, where $m_G$ denotes the Haar measure on $G$, and $\int \varphi\, \mathrm{d}\xi$ and $\int \psi\, \mathrm{d}\xi$ are defined as Wiener integrals. Last but not the least, $\lambda > 0$ designates a fixed parameter that is generally referred to as the *level of the noise*.

One can adapt the method of Dalang [13] in order to show that, in the linear case—that is when $\sigma \equiv$ constant—(SHE) has a function solution iff

$$\int_{G^*} \left(\frac{1}{\beta + \mathrm{Re}\Psi(\chi)}\right) m_{G^*}(\mathrm{d}\chi) < \infty \quad \text{for one, hence all, } \beta > 0, \tag{D}$$

where $\Psi$ denotes the characteristic exponent of our Lévy process $\{X_t\}_{t\geqslant 0}$ and $m_{G^*}$ denotes the Haar measure on the dual $G^*$ to our group $G$. See also Brzeźniak and Jan van Neerven [6] and Peszat and Zabczyk [38]. Because we want (SHE) to have a function solution, at the very least in the linear case, we have no choice but to assume Dalang's condition (D) from now on. Henceforth, we assume (D) without further mention.

In some cases, Condition (D) always holds. For example, suppose $G$ is discrete. Because $G^*$ is compact, thanks to Pontryagin–van Kampen duality [36, 41], continuity of the function $\Psi$ implies its uniform boundedness, whence we find that the *Dalang Condition (D) always holds when $G$ is discrete.* This simple observation is characteristic of many interesting results about the heat equation (SHE) in the sense that a purely-topological property of the group $G$ governs important aspects of (SHE): In this case, we deduce the existence of a solution generically when $G$ is discrete. For a probabilistic proof of this particular fact see Lemma 10.1 below.

We wish to establish that "noise excitation" properties of (SHE) are "intrinsic to the group $G$." This goal forces us to try and produce solutions that take values in the group algebra $L^2(G)$. The following summarizes the resulting existence and regularity theorem that is good enough to help us begin our discussion of noise excitation.

**Theorem 2.1.** *Suppose, in addition, that either $G$ is compact or $\sigma(0) = 0$. Then for every non-random initial value $u_0 \in L^2(G)$ and $\lambda > 0$, the stochastic heat equation* (SHE) *has a mild solution $\{u_t\}_{t\geqslant 0}$, with values in $L^2(G)$, that satisfies the following: There exists a finite constant $c \geqslant 0$ that yields the energy inequality*

$$\mathrm{E}\left(\|u_t\|^2_{L^2(G)}\right) \leqslant c\mathrm{e}^{ct} \qquad \textit{for every } t \geqslant 0. \tag{2.2}$$

*Moreover, if $v$ is any mild solution that satisfies* (2.2) *as well as $v_0 = u_0$, then* $\mathrm{P}\{\|u_t - v_t\|_{L^2(G)} = 0\} = 1$ *for all $t \geqslant 0$.*

The preceding result is well-known for many euclidean examples; see in particular Dalang and Mueller [15].

Thus, we assume from now on, and without further mention, that

$$\text{either } G \text{ is compact, or } \sigma(0) = 0, \tag{2.3}$$

in order to know, *a priori*, that there exists an $L^2(G)$-valued solution to (SHE).

The principal aim of this paper is to study the energy of the solution when $\lambda$ is large. In order to simplify the exposition, let us denote the *energy of the solution at time $t$* by

$$\mathscr{E}_t(\lambda) := \sqrt{\mathrm{E}\left(\|u_t\|^2_{L^2(G)}\right)}. \tag{2.4}$$

To be more precise, $\mathscr{E}_t(\lambda)$ denotes the $L^2(\mathrm{P})$-norm of the energy of the solution. But we refer to it as the energy in order to save on the typography.

We begin our analysis of noise excitation by first noting the following fact: If $\sigma$ is essentially bounded and $G$ is compact, then the solution to (SHE) is *at most linearly noise excitable.* The following is the precise formulation of this statement.

**Proposition 2.2** (Linear noise excitation)**.** *If $\sigma \in L^\infty(\mathbf{R})$ and $G$ is compact, then*

$$\limsup_{\lambda\uparrow\infty} \frac{\mathscr{E}_t(\lambda)}{\lambda} < \infty \qquad \textit{for all } t > 0. \tag{2.5}$$

*This bound can be reversed in the following sense: If also* $\inf_{x\in G} |u_0(x)| > 0$ *and* $\inf_{z\in\mathbf{R}} |\sigma(z)| > 0$*, then*

$$\liminf_{\lambda\uparrow\infty} \frac{\mathscr{E}_t(\lambda)}{\lambda} > 0 \qquad \textit{for all } t > 0. \tag{2.6}$$

The bulk of this paper is concerned with the behavior of (SHE) when the energy $\mathscr{E}_t(\lambda)$ behaves as $\exp(\text{const}\cdot\lambda^q)$, for a fixed positive constant $q$, as $\lambda \uparrow \infty$. With this in mind, let us define for all $t > 0$,

$$\underline{\mathfrak{e}}(t) := \liminf_{\lambda\uparrow\infty} \frac{\log\log \mathscr{E}_t(\lambda)}{\log\lambda}, \quad \overline{\mathfrak{e}}(t) := \limsup_{\lambda\uparrow\infty} \frac{\log\log \mathscr{E}_t(\lambda)}{\log\lambda}. \tag{2.7}$$

**Definition 2.3.** We refer to $\overline{\mathfrak{e}}(t)$ and $\underline{\mathfrak{e}}(t)$ respectively as the *upper* and the *lower excitation indices* of $u$ at time $t$. In many cases of interest, $\underline{\mathfrak{e}}(t)$ and $\overline{\mathfrak{e}}(t)$ are equal and do not depend on the time variable $t > 0$ [N.B. *not* to be confused with $t \geqslant 0$]. In such cases, we tacitly write $\mathfrak{e}$ for that common value.

Thus, Proposition (2.2) can be summarized as the statement that $\mathfrak{e} = 0$ when $\sigma$ is essentially bounded.

We think of $\underline{\mathfrak{e}}(t)$ and $\bar{\mathfrak{e}}(t)$ respectively as the *lower* and *upper nonlinear excitation indices* of the solution at time $t$. And, if and when $\mathfrak{e}$ exists, then we think of $\mathfrak{e}$ as the *index of nonlinear noise excitation* of the solution to (SHE).

As a critical part of our analysis, we will prove that both of these indices are natural quantities, as they are "group invariants" in a sense that will be made clear later on. Moreover, one can deduce from our work that when $G$ is unimodular, the law of the solution to (SHE) is itself a "group invariant." A careful explanation of the quoted terms will appear later on in the paper. For now, we content ourselves with closing the introduction by stating the main three results of this paper.

**Theorem 2.4** (Discrete case)**.** *If $G$ is discrete, then $\bar{\mathfrak{e}}(t) \leqslant 2$ for all $t > 0$. In fact, $\mathfrak{e} = 2$, provided additionally that*

$$\ell_\sigma := \inf_{z \in \mathbf{R} \setminus \{0\}} |\sigma(z)/z| > 0. \tag{2.8}$$

**Theorem 2.5** (Connected case)**.** *Suppose that $G$ is connected and* (2.8) *holds. Then $\underline{\mathfrak{e}}(t) \geqslant 4$ for all $t > 0$, provided that in addition either $G$ is non compact or $G$ is compact, metrizable, and has more than one element.*

**Theorem 2.6** (Connected case)**.** *For every $\theta \geqslant 4$ there are models of the triple $(G\,, \mathscr{L}, u_0)$ for which $\mathfrak{e} = \theta$.*

Thus, when (2.8) holds in addition to the preceding conditions, then we can summarize Theorems 2.4, 2.5, and 2.6 as follows: Either the energy of the solution behaves as $\exp(\text{const} \cdot \lambda^2)$ or it is greater than $\exp(\text{const} \cdot \lambda^4)$ for large noise levels, and this lower bound cannot be improved upon in general. Moreover, the connectivity properties of $G$—and not the operator $\mathscr{L}$—alone determine the first-order strength of the growth of the energy, viewed as a function of the noise level $\lambda$.

Finally, we will soon see that when the energy behaves as $\exp(\text{const} \cdot \lambda^2)$, this means that (SHE) is only as noise excitable as a classical Itô stochastic differential equation. Martin Hairer has asked [private communication] whether intermittency properties of (SHE) are always related to those of the McKean exponential martingale [for Brownian motion]. A glance at Example 4.1 below shows in essence that, as far as nonlinear noise excitation is concerned, intermittent examples of (SHE) behave as the exponential martingale if and only if $G$ is discrete.

Throughout, $\mathrm{L}_\sigma$ designates the optimal Lipschitz constant of the function $\sigma$; that is,

$$\mathrm{L}_\sigma := \sup_{-\infty < x < y < \infty} \left| \frac{\sigma(x) - \sigma(y)}{x - y} \right| < \infty. \tag{2.9}$$

# 3 Analysis on LCA groups

We follow the usual terminology of the literature and refer to a *locally compact Hausdorff abelian group* as an *LCA group.* Morris [36] and Rudin [41] are two standard references for the theory of LCA groups.

If $G$ is an LCA group, then we let $m_G$ denote the Haar measure on $G$. The dual, or character, group to $G$ is denoted by $G^*$, and the *Fourier transform* on $L^1(G)$ is defined via the following normalization:

$$\hat{f}(\chi) := \int_G (x\,,\chi) f(x)\, m_G(\mathrm{d}x) \qquad \text{for all } \chi \in G^* \text{ and } f \in L^1(G), \tag{3.1}$$

where $(x\,,\chi) := \chi(x) := x(\chi)$ are interchangeable notations that all describe the natural pairing between $x \in G$ and $\chi \in G^*$. [Different authors use slightly different normalizations of Fourier transforms from us; see, for example Rudin [41].]

Of course, $m_G$ is defined uniquely only up to a multiplicative factor. Still, we always insist on a *standard normalization* of Haar measures; that is any normalization that ensures that the Fourier transform has a continuous isometric extension to $L^2(G) = L^2(G^*)$. Namely,

$$\|f\|_{L^2(G)} = \|\hat{f}\|_{L^2(G^*)} \qquad \text{for all } f \in L^2(G). \tag{3.2}$$

Our normalization of Haar measure translates to well-known normalizations of Haar measures via Pontryagin–van Kampen duality [36, 41]:

**Case 1.** If $G$ is compact, then $G^*$ is discrete; $m_G(G) = 1$; and $m_{G^*}$ denotes the counting measure on subsets of $\Gamma^*$.

**Case 2.** If $G$ is discrete, then $G^*$ is compact, $m_{G^*}(G^*) = 1$, and $m_G$ coincides with the counting measure on $G$.

**Case 3.** If $G = \mathbf{R}^n$ for some integer $n \geqslant 1$, then $G^* = \mathbf{R}^n$; we may choose $m_G$ and $m_{G^*}$, in terms of $n$-dimensional Lebesgue measure, as $m_G(\mathrm{d}x) = a\,\mathrm{d}x$ and $m_{G^*}(\mathrm{d}x) = b\,\mathrm{d}x$ for any two positive reals $a$ and $b$ that satisfy the relation $ab = (2\pi)^{-n}$.

# 4 Some examples

The stochastic PDEs introduced here are quite natural; in many cases, they are in fact well-established equations. In this section, we identify some examples to highlight the preceding claims. Of course, one can begin with the most obvious examples of stochastic PDEs; for instance, where $G = \mathbf{R}$, $\mathscr{L} = \Delta$, etc. But we prefer to have a different viewpoint: As far as interesting examples are concerned, it is helpful to sometimes think about concrete examples of LCA groups $G$; then try to understand the Lévy processes on $G$ [a kind of Lévy–Khintchine formula] in order to know which operators $\mathscr{L}$ are relevant. And only then one can think about the actual resulting stochastic partial differential equation. This slightly-different viewpoint produces interesting examples.

**Example 4.1** (The trivial group)**.** For our first example let us consider the trivial group $G$ with only one element $g$. The only Lévy process on this group is $X_t := g$. All functions on the group $G$ are, by default, constants. Therefore, $\mathscr{L}f = 0$ for all $f : G \to \mathbf{R}$, and hence $U_t := u_t(g)$ solves the Itô SDE

$$\mathrm{d}U_t = \lambda\sigma(U_t)\,\mathrm{d}B_t \quad \text{with } U_0 = u_0(g), \tag{4.1}$$

where $B_t := \int_{[0,t]\times G} \mathrm{d}\xi$ defines a Brownian motion. In other words, when $G$ is the trivial group, (SHE) characterizes all drift-free 1-dimensional Itô diffusions!

**Example 4.2** (Cyclic groups)**.** For a slightly more interesting example consider the cyclic group $G := \mathbf{Z}_2$ on two elements. We may think of $G$ as $\mathbf{Z}/2\mathbf{Z}$; i.e., the set $\{0\,, 1\}$ endowed with binary addition [addition mod 1] and discrete topology. It is an elementary fact that the group $G$ admits only one 1-parameter family of Lévy processes. Indeed, we can apply the strong Markov property to the first jump time of $X$ to see that if $X$ is a Lévy process on $\mathbf{Z}_2$, then there necessarily exists a number $\kappa \geqslant 0$ such that, at independent exponential times, the process $X$ changes its state at rate $\kappa$: From 0 to 1 if $X$ is at 0 at the jump time, and from 1 to 0 when $X$ is at 1 at the jump time [$\kappa = 0$ yields the constant process]. In this way we find that (SHE) is an encoding of the coupled two-dimensional SDE

$$\begin{aligned} \mathrm{d}u_t(0) &= \kappa\,[u_t(1) - u_t(0)]\,\mathrm{d}t + \lambda\sigma(u_t(0))\,\mathrm{d}B_t(0),\\ \mathrm{d}u_t(1) &= \kappa\,[u_t(0) - u_t(1)]\,\mathrm{d}t + \lambda\sigma(u_t(1))\,\mathrm{d}B_t(1), \end{aligned} \tag{4.2}$$

where $B(0)$ and $B(1)$ are two independent 1-dimensional Brownian motions. In other words, when $G = \mathbf{Z}_2$, (SHE) describes a 2-dimensional Itô diffusion with local diffusion coefficients where the particles [coordinate processes] feel an attractive linear drift toward their neighbors [unless $\kappa = 0$, which corresponds to two decoupled diffusions].

**Example 4.3** (Cyclic groups)**.** Let us consider the case that $G := \mathbf{Z}_n$ is the cyclic group on $n$ elements when $n \geqslant 3$. We may think of $G$ as $Z/n\mathbf{Z}$; that is, the set $\{0\,,\ldots,n-1\}$ endowed with addition $\pmod n$ and discrete topology. If $X$ is a Lévy process on $G$, then it is easy to see that there exist $n-1$ parameters $\kappa_1,\ldots,\kappa_{n-1} \geqslant 0$ such that $X$ jumps [at iid exponential times] from $i \in \mathbf{Z}/n\mathbf{Z}$ to $i+j \pmod{n-1}$ at rate $\kappa_j$ for every $i \in \{0\,,\ldots,n-1\}$ and $j \in \{1\,,\ldots,n-1\}$. In this case, our stochastic heat equation (SHE) is another way to describe the evolution of the $n$-dimensional Itô diffusion $(u(1)\,,\ldots u(n))$, where for all $i = 0,\ldots,n-1$,

$$\mathrm{d}u_t(i) = \sum_{j=1}^{n-1} \kappa_j\,[u_t(i+j \pmod n)) - u_t(i)]\,\mathrm{d}t + \sigma(u_t(i))\,\mathrm{d}B_t(i), \tag{4.3}$$

for an independent system $B(0),\ldots,B(n-1)$ of one-dimensional Brownian motions. Thus, in this example, (SHE) encodes all possible $n$-dimensional diffusions with local diffusion coefficients and Ornstein–Uhlenbeck type attractive drifts. Perhaps the most familiar example of this type is the simple symmetric case in which $\kappa_1 = \kappa_{n-1} := \kappa > 0$ and $\kappa_j = 0$ for $j \notin \{1, n-1\}$. In that case, (4.3) simplifies to

$$\mathrm{d}u_t(i) = \kappa(\Delta u_t)(i) + \sigma(u_t(i))\, \mathrm{d}B_t(i), \tag{4.4}$$

where $(\Delta f)(i) := f(i \boxplus 1) + f(i \boxminus 1) - 2f(i)$ is the "Laplacian" of $f : \mathbf{Z}_n \to \mathbf{R}$, and $a \boxplus b := a + b \pmod{n-1}$ and $a \boxminus b := a - b \pmod{n-1}$.

**Example 4.4** (Lattice groups)**.** In this example, $G$ denotes a lattice subgroup of $\mathbf{R}^d$. This basically means that $G = \delta\mathbf{Z}^d$ for some $\delta > 0$ and $d = 1, 2, \ldots$ . The class of all Lévy processes on $G$ coincides with the class of all continuous-time random walks on $G$. Thus, standard random walk theory tells us that there exists a constant $\kappa \geqslant 0$—the rate—and a probability function $\{J(y)\}_{y \in \delta\mathbf{Z}^d}$—the so-called jump measure—such that $(\mathscr{L}f)(x) = \kappa \sum_{y \in \delta\mathbf{Z}^d} \{f(y) - f(x)\}\, J(y)$, and hence (SHE) is an encoding of the following infinite system of interacting Itô-type stochastic differential equations:

$$\mathrm{d}u_t(x) = \kappa \sum_{y \in \delta\mathbf{Z}^d} [u_t(y) - u_t(x)]\, J(y) + \sigma(u_t(x))\, \mathrm{d}B_t(x), \tag{4.5}$$

for iid one-dimensional Brownian motions $\{B(z)\}_{z \in \delta\mathbf{Z}^d}$ and all $x \in \delta\mathbf{Z}^d$. A particularly well-known case is when $J(y)$ puts equal mass on the neighbors of the origin in $\delta\mathbf{Z}^d$. In that case,

$$\mathrm{d}u_t(x) = \frac{\kappa}{2d}(\Delta u_t)(x) + \sigma(u_t(x))\, \mathrm{d}B_t(x), \tag{4.6}$$

where $(\Delta f)(x) := \sum_{|y-x|=1} \{f(y) - f(x)\}$ denotes the graph Laplacian of $f : \delta\mathbf{Z}^d \to \mathbf{R}$ with $|y - x| := \sum_{i=1}^d |y_i - x_i|$.

**Example 4.5** (The real line)**.** As an example, let us choose $G := \mathbf{R}$ and $X :=$ 1-dimensional Brownian motion on $\mathbf{R}$. Then, $\mathscr{L}f = f''$ and (SHE) becomes the usual stochastic heat equation

$$\frac{\partial u_t(x)}{\partial t} = \kappa \frac{\partial^2 u_t(x)}{\partial x^2} + \sigma(u_t(x))\xi, \tag{4.7}$$

driven by space-time white noise on $(0, \infty) \times \mathbf{R}$.

**Example 4.6** (Tori)**.** Next, we may consider $G := [0, 1)$; as usual we identify the ends of $[0, 1)$ in order to obtain the torus $G := \mathbf{T}$, endowed with addition mod one. Let $X :=$ Brownian motion on $\mathbf{T}$. Its generator is easily seen to be the Laplacian on $[0, 1)$ with periodic boundary conditions. Hence, (SHE) encodes

$$\left[\begin{aligned} &\frac{\partial u_t(x)}{\partial t} = \kappa \frac{\partial^2 u_t(x)}{\partial x^2} + \sigma(u_t(x))\xi \quad \text{for all } 0 \leqslant x < 1, \\ &\text{subject to } u_t(0) = u_t(1-), \end{aligned}\right. \tag{4.8}$$

in this case.

**Example 4.7** (Totally disconnected examples)**.** Examples 4.1 through 4.6 are concerned with more or less standard SDE/SPDE models. Here we mention one among many examples where (SHE) is more exotic. Consider $G := \mathbf{Z}_2 \times \mathbf{Z}_2 \times \cdots$ to be a countable direct product of the cyclic group on two elements. Then, $G$ is a compact abelian group; this is a group that acts transitively on binary trees and is related to problems in fractal percolation. A Lévy process on $G$ is simply a process that has the form $X_t^1 \times X_t^2 \times \cdots$ at time $t \geqslant 0$, where $X^1 \times \cdots \times X^k$ is a Lévy process on $\prod_{i=1}^k \mathbf{Z}_2$ for every $k \geqslant 1$ [see Example 4.1]. It is easy to see then that if $f : G \to \mathbf{R}$ is a function that is constant in every coordinates but the coordinates in some finite set $F$, then the generator of $X$ is the composition $\prod_{j\in F} \mathscr{L}^j$, where $\mathscr{L}^j$ denotes the generator of $X^j$ [see Example 4.1]. The stochastic heat equation (SHE) is not the subject of our analysis here *per se*, but suffice it to say that it appears to have connections to interacting random walks on a random environment on a binary tree.

**Example 4.8** (Positive multiplicative reals)**.** Our next, and last, example, requires a slightly longer discussion than its predecessors. But we feel that this is an illuminating example and thus worth the effort.

Let
$$h(x) := \mathrm{e}^x \qquad (x \in \mathbf{R}). \tag{4.9}$$
The range $G := h(\mathbf{R})$ of the function $h$ is the multiplicative positive reals. Frequently, one writes $G$ as $\mathbf{R}^\times_{>0}$; this is an LCA group, and $h$ is an isomorphism between $\mathbf{R}$ and $\mathbf{R}^\times_{>0}$. [There are of course other topological isomorphisms from $\mathbf{R}$ to $\mathbf{R}^\times_{>0}$; in fact, $\mathbf{R} \ni x \mapsto \exp(qx) \in \mathbf{R}^\times_{>0}$ works for every real number $q \neq 0$.]

Recall that the dual group of $\mathbf{R}$ is $\mathbf{R}$ [36, 41]. The dual group of $\mathbf{R}^\times_{>0}$ is just the image under $h(x) = \exp(x)$ of $\mathbf{R}$. That is, the dual of $\mathbf{R}^\times_{>0}$ is $\mathbf{R}$ as well. The duality relation between $\mathbf{R}^\times_{>0}$ and itself is the image under $h$ of the one for the reals. Namely,
$$(x\,,\chi) := \chi(x) := x(\chi) := x^{i\chi} \quad \text{for } x \in \mathbf{R}^\times_{>0} \text{ and } \chi \in \mathbf{R} = (\mathbf{R}^\times_{>0})^*. \tag{4.10}$$
Thus, we can choose the Haar measure on $\mathbf{R}^\times_{>0}$ and its dual as the restrictions of the Haar measure on $\mathbf{R}$ and its dual, respectively. In particular, the choice is unique once we agree on the normalization of the Lebesgue measure on $\mathbf{R}$. In particular, if $\xi$ defines a white noise on $(0\,,\infty) \times \mathbf{R}$, then
$$\xi^\times_{>0}\left([0\,,T] \times B\right) := \xi([0\,,T] \times h^{-1}(B)), \tag{4.11}$$
valid for $T > 0$ and Borel sets $B \subset \mathbf{R}^\times_{>0}$ of finite measure, defines a white noise on $(0\,,\infty) \times \mathbf{R}^\times_{>0}$.

Since $h(x) = \mathrm{e}^x$ is a topological isomorphism from $\mathbf{R}$ onto $\mathbf{R}^\times_{>0}$, every Lévy process $X := \{X_t\}_{t\geqslant 0}$ on $\mathbf{R}^\times_{>0}$ can be written as $X_t = \exp(Y_t)$, where $Y := \{Y_t\}_{t\geqslant 0}$ is a Lévy process on $\mathbf{R}$. An interesting special case is $Y_t = B_t + \delta t$, where $B := \{B_t\}_{t\geqslant 0}$ denotes 1-dimensional Brownian motion on $\mathbf{R}$ and $\delta \in \mathbf{R}$ is a parameter. Thus,
$$t \mapsto X_t := \mathrm{e}^{B_t + \delta t} \tag{4.12}$$

defines a continuous Lévy process on $\mathbf{R}^{\times}_{>0}$. The best-known example is the case that $\delta = -1/2$, in which case $X$ is the exponential martingale.

An application of Itô's formula [or an appeal to classical generator computations] shows that if $f \in C^\infty(\mathbf{R})$, then for all $x > 0$,

$$\mathrm{E}f(xX_t) = f(x) + \frac{t}{2}x^2 f''(x) + \frac{t(1+2\delta)}{2} x f'(x) + o(t) \qquad \text{as } t \downarrow 0. \tag{4.13}$$

This shows that the generator of the exponential martingale $X$, viewed as a Lévy process on $\mathbf{R}^{\times}_{>0}$, is

$$(\mathscr{L} f)(x) = \frac{x^2}{2} f''(x) + \left(\delta + \frac{1}{2}\right) x f'(x) \qquad \text{for all } x > 0. \tag{4.14}$$

Thanks to (4.11), we can understand our stochastic heat equation (SHE) is, in this context, as the following euclidean SPDE:

$$\frac{\partial u_t(x)}{\partial t} = \frac{x^2}{2}\frac{\partial^2 u_t(x)}{\partial x^2} + \left(\delta + \frac{1}{2}\right) x \frac{\partial u_t(x)}{\partial x} + \sigma(u_t(x))\xi; \tag{4.15}$$

for $t, x > 0$. Such SPDEs appear to be new. Therefore, let us expend a few lines and make the following amusing observation as an aside: From the perspective of these SPDEs, the most natural case is the drift-free case where $\delta = -1/2$. In that case, the underlying Lévy process $X$ is the exponential martingale, as was noted earlier. The exponential martingale is one of the archetypal classical examples of an intermittent process [45]. Moreover, $X$ is centered when $\delta = -1/2$ in the sense that $\mathrm{E}X_t$ is the group identity. Interestingly enough, the exponential martingale is natural in other sense as well: (1) The process $X$ is Gaussian [it is the image of a real-valued Gaussian process under the exponential map]; and (2) $X$ has quadratic variation $t$; i.e.,

$$\lim_{n\to\infty} \sum_{0 \leqslant k \leqslant 2^n t} \left[ X_{(k+1)/2^n} X^{-1}_{k/2^n} \right]^2 = t \quad \text{almost surely for all } t \geqslant 0. \tag{4.16}$$

This can be verified by direct elementary means.

# 5 Lévy processes

Let us recall some basic facts about Lévy process on LCA groups. For more details, see Berg and Forst [2] and Port and Stone [39, 40]. Bertoin [4], Jacob [28], and Spitzer [42] are masterly accounts of the probabilistic and analytic aspects of the theory of Lévy process on $\mathbf{R}^n$ and $\mathbf{Z}^n$.

Throughout, $(\Omega, \mathscr{F}, \mathrm{P})$ is a fixed probability space.

Let $G$ denote an LCA group, and suppose $Y := \{Y_t\}_{t \geqslant 0}$ is a stochastic process on $(\Omega, \mathscr{F}, \mathrm{P})$ with values in $G$. [We always opt to write $Y_t$ in place of $Y(t)$, as is customary in the theory of stochastic processes.]

We say that $Y$ is a *Lévy process* on $G$ if:

1. $Y_0 = \mathrm{e}_G$, the identity element of $G$;
2. $Y_{t+s}Y_s^{-1}$ is independent of $\{Y_u\}_{u\in[0,s]}$ and has the same distribution as $Y_t$, for all $s, t \geqslant 0$; and
3. The random function $t \mapsto Y_t$ is right continuous and has left limits everywhere with probability one.

This is slightly more stringent-seeming than the standard definition, but is equivalent to the standard definition, for instance, when $G$ is metrizable.

Let $\mu_t := \mathrm{P} \circ Y_t^{-1}$ denote the distribution of the random variable $Y_t$. Then, $\{P_t\}_{t\geqslant 0}$ is a convolution semigroup, where

$$(P_t f)(x) := \mathrm{E} f(xY_t) := \int_G f(xy)\, \mu_t(\mathrm{d}y). \tag{5.1}$$

We can always write the Fourier transform of the probability measure $\mu_t$ as follows:

$$\hat{\mu}_t(\chi) = \mathrm{E}(\chi\,, Y_t) = \mathrm{e}^{-t\Psi(\chi)} \qquad \text{for all } t \geqslant 0 \text{ and } \chi \in G^*, \tag{5.2}$$

where $\Psi : G^* \to \mathbf{C}$ is continuous and $\Psi(\mathrm{e}_{G^*}) = 0$. It is easy to see that Dalang's Condition (D) always implies the following:

$$\int_{G^*} \mathrm{e}^{-t\mathrm{Re}\Psi(\chi)}\, m_{G^*}(\mathrm{d}\chi) < \infty \qquad \text{for all } t > 0. \tag{5.3}$$

See, for example, [23, Lemma 8.1]. In this case, the following is well defined

$$p_t(x) = \int_{G^*} (x^{-1}, \chi)\mathrm{e}^{-t\Psi(\chi)}\, m_{G^*}(\mathrm{d}\chi) \qquad \text{for all } t > 0 \text{ and } x \in G. \tag{5.4}$$

The following is a consequence of Fubini's theorem.

**Lemma 5.1.** *The function $(t\,, x) \mapsto p_t(x)$ is well defined and bounded as well as uniformly continuous for $(t\,, x) \in [\delta\,, \infty) \times G$ for every fixed $\delta > 0$. Moreover, we can describe the semigroup via*

$$(P_t f)(x) = \int f(xy)p_t(y)\, m(\mathrm{d}y) \qquad \textit{for all } t > 0,\ x \in \Gamma,\ f \in L^1(G). \tag{5.5}$$

*Consequently, $p_t(x) \geqslant 0$ for all $t > 0$ and $x \in G$.*

We omit the proof, as it is elementary. Let us mention, however, that the preceding lemma guarantees that the Chapman–Kolmogorov equation holds pointwise. That is,

$$p_{t+s}(x) = (p_t * p_s)(x) \qquad \text{for all } s, t > 0 \text{ and } x \in G, \tag{5.6}$$

where "$*$" denotes the usual convolution on $L^1(G)$; that is,

$$(f * g)(x) := \int_G f(y)g(xy^{-1})\, m_G(\mathrm{d}y). \tag{5.7}$$

Define, for all $t > 0$ and $x \in G$,

$$\bar{p}_t(x) := (P_t p_t)(x) = \int_G p_t(xy) p_t(y)\, m_G(\mathrm{d}y). \tag{5.8}$$

Then, the Chapman–Kolmogorov equation ensures that

$$\bar{p}_t(0) = \|p_t\|^2_{L^2(G)} \qquad \text{for all } t > 0. \tag{5.9}$$

Furthermore, it can be shown that the following inversion theorem holds for all $t > 0$ and $x \in G$:

$$\bar{p}_t(x) = \int_{G^*} (x^{-1},\, \chi) \mathrm{e}^{-2t\mathrm{Re}\Psi(\chi)}\, m_{G^*}(\mathrm{d}\chi). \tag{5.10}$$

Thus, we find that

$$\Upsilon(\beta) := \int_0^\infty \mathrm{e}^{-\beta t} \|p_t\|^2_{L^2(G)}\, \mathrm{d}t \tag{5.11}$$

satisfies

$$\Upsilon(\beta) := \int_0^\infty \mathrm{e}^{-\beta t} \|p_t\|^2_{L^2(G)}\, \mathrm{d}t = \int_{G^*} \frac{m_{G^*}(\mathrm{d}\chi)}{\beta + 2\mathrm{Re}\Psi(\chi)}. \tag{5.12}$$

Consequently, Dalang's Condition (D) can be recast equivalently and succinctly as the condition that $\Upsilon : [0\,,\infty) \to [0\,,\infty]$ is finite on $(0\,,\infty)$.

Since $t \mapsto \int_0^t \bar{p}_s(0)\, \mathrm{d}s$ is nondecreasing, Lemma 3.3 of [23] implies the following abelian/tauberian bound:

$$\mathrm{e}^{-1}\Upsilon(1/t) \leqslant \int_0^t \bar{p}_s(0)\, \mathrm{d}s \leqslant \mathrm{e}\Upsilon(1/t) \qquad \text{for all } t > 0. \tag{5.13}$$

Finally, by *the generator* of $\{X_t\}_{t\geqslant 0}$ we mean the linear operator $\mathscr{L}$ with domain

$$\mathrm{Dom}[\mathscr{L}] := \left\{ f \in L^2(G) :\ \mathscr{L} f := \lim_{t\downarrow 0} t^{-1}(P_t f - f) \text{ in } L^2(G) \right\}. \tag{5.14}$$

This defines $\mathscr{L}$ as an $L^2$-generator, which is a slightly different operator than the one that is usually obtained from the Hille-Yosida theorem. The $L^2$-theory makes good sense here for a number of reasons; chief among them is the fact that $G$ need not be second countable and hence the standard form of the Hille–Yosida theorem is not applicable. The $L^2$-theory has the added advantage that the domain is more or less explicit, as will be seen shortly.

Recall that each $P_t$ is a contraction on $L^2(G)$, and observe that

$$\widehat{P_t f}(\chi) = \hat{f}(\chi) \exp\left\{ -t\overline{\Psi(\chi)} \right\} \qquad \text{for all } t \geqslant 0 \text{ and } \chi \in G^*. \tag{5.15}$$

Therefore, for all $f, g \in L^2(G)$,

$$\int_G g(P_t f - f)\, \mathrm{d}m_G = -\int_{G^*} \hat{f}(\chi)\, \overline{\hat{g}(\chi)} \left( 1 - \mathrm{e}^{-t\overline{\Psi(\chi)}} \right) m_{G^*}(\mathrm{d}\chi). \tag{5.16}$$

It follows fairly readily from this relation that $\mathscr{L}:\text{Dom}[\mathscr{L}]\to L^2(G)$,

$$\text{Dom}[\mathscr{L}]=\left\{f\in L^2(G):\ \int_{G^*}|\hat f(\chi)|^2\,|\Psi(\chi)|^2\,m_{G^*}(\mathrm{d}\chi)<\infty\right\},\tag{5.17}$$

and for all $f\in\text{Dom}[\mathscr{L}]$ and $g\in L^2(G)$,

$$\int_G g\mathscr{L}f\,\mathrm{d}m_G=-\int_{G^*}\hat f(\chi)\overline{\hat g(\chi)\Psi(\chi)}\,m_{G^*}(\mathrm{d}\chi).\tag{5.18}$$

The latter identity is another way to write

$$\widehat{\mathscr{L}f}(\chi)=-\hat f(\chi)\overline{\Psi(\chi)}\qquad\text{for all } f\in\text{Dom}[\mathscr{L}]\text{ and }\chi\in G^*.\tag{5.19}$$

In other words, $\mathscr{L}$ is a psuedo-differential operator on $L^2(G)$ with Fourier multiplier ["symbol"] $-\overline{\Psi}$.

# 6 Stochastic convolutions

Throughout this paper, $\xi$ will denote space-time white noise on $\mathbf{R}_+\times G$. That is, $\xi$ is a set-valued Gaussian random field, indexed by Borel subsets of $\mathbf{R}_+\times G$ that have finite measure $\text{Leb}\times m_G$ [product of Lebesgue and Haar measures, respectively on $\mathbf{R}_+$ and $G$]. Moreover, $\mathrm{E}\xi(A\times T)=0$ for all measurable $A\subset\mathbf{R}_+$ and $T\subset G$ of finite measure [resp. Lebesgue and Haar], and

$$\text{Cov}\left(\xi(B\times T)\,,\xi(A\times S)\right)=\text{Leb}(B\cap A)\cdot m_G(T\cap S),\tag{6.1}$$

for all Borel sets $A,B\subset\mathbf{R}_+$ that have finite Lebesgue measure and all Borel sets $S,T\subseteq G$ that have finite Haar measure. It is easy to see that $\xi$ is then a vector-valued measure with values in $L^2(\mathrm{P})$.

The principal goal of this section is to introduce and study stochastic convolutions of the form

$$\left(K\circledast Z\right)_t(x):=\int_{(0,t)\times G}K_{t-s}(y-x)Z_s(y)\,\xi(\mathrm{d}s\,\mathrm{d}y),\tag{6.2}$$

where $Z$ is a suitable space-time random field and $K$ is a nice non-random space-time function from $(0\,,\infty)\times G$ to $\mathbf{R}$.

If $Z$ is a predictable random field, in the sense of Walsh [44] and Dalang [13], and satisfies

$$\sup_{t\in[0,T]}\sup_{x\in G}\mathrm{E}(|Z_t(x)|^2)<\infty,\quad\int_0^T\mathrm{d}s\int_G m_G(\mathrm{d}y)\ [K_s(y)]^2<\infty,$$

for all $T>0$, then the stochastic convolution $K\circledast Z$ is the same stochastic integral that has been obtained in Walsh [44] and, in particular, Dalang [13].

One of the essential properties of the resulting stochastic integral is the following $L^2$ isometry:

$$\mathrm{E}\left(\left|(K \circledast Z)_t\,(x)\right|^2\right) = \int_0^t \mathrm{d}s \int_G m_G(\mathrm{d}y)\ \ [K_{t-s}(y-x)]^2\, \mathrm{E}\left(|Z_s(y)|^2\right). \tag{6.3}$$

In this section we briefly describe an extension of the Walsh–Dalang stochastic integral that has the property that $t \mapsto (K \circledast Z)_t$ is a stochastic process with values in the group algebra $L^2(G)$. Thus, the resulting stochastic convolution need not be, and in general is not, a random field in the modern sense of the word. Rather, we can realize the stochastic convolution process $t \mapsto (K \circledast Z)_t$ as a Hilbert-space-valued stochastic process, where the Hilbert space is $L^2(G)$.

Our construction has a similar flavor as some other recent constructions; see, in particular, Da Prato and Zabczyk [12] and Dalang and Quer–Sardanyons [16]. However, our construction also has some novel aspects.

Let us set forth some notation first. As always, let $(\Omega\,, \mathscr{F}, \mathrm{P})$ denote a probability space.

**Definition 6.1.** Let $Z := \{Z_t(x)\}_{t\in I, x\in G}$ be a two-parameter [space-time] real-valued stochastic process indexed by $I \times G$, where $I$ is a measurable subset of $\mathbf{R}_+$. We say that $Z$ is a *random field* when the function $Z : (\omega\,, t\,, x) \mapsto Z_t(x)(\omega)$ is product measurable from $\Omega \times I \times G$ to $\mathbf{R}$.

The preceding definition is somewhat unconventional; our random fields are frequently referred to as "universally measurable random fields." Because we will never have need for any other random fields than universally measurable ones, we feel justified in abbreviating the terminology.

**Definition 6.2.** For every random field $Z := \{Z_t(x)\}_{t\geqslant 0, x\in G}$ and $\beta \geqslant 0$, let us define

$$\mathcal{N}_\beta(Z\,; G) := \sup_{t\geqslant 0} \left\{ \mathrm{e}^{-2\beta t} \mathrm{E}\left( \|Z_t\|_{L^2(G)}^2 \right) \right\}^{1/2}. \tag{6.4}$$

We may sometimes only write $\mathcal{N}_\beta(Z)$ when it is clear which underlying group we are referring to.

Each $\mathcal{N}_\beta$ defines a norm on space-time random fields, provided that we identify a random field with all of its versions.

**Definition 6.3.** For every $\beta \geqslant 0$, we define $\mathcal{L}^2_\beta(G)$ be the $L^2$-space of all measurable functions $\Phi : (0\,, \infty) \times G \to \mathbf{R}$ with $\|\Phi\|_{\mathcal{L}^2_\beta(G)} < \infty$, where

$$\|\Phi\|^2_{\mathcal{L}^2_\beta(G)} := \int_0^\infty \mathrm{e}^{-2\beta s} \|\Phi_s\|^2_{L^2(G)}\, \mathrm{d}s. \tag{6.5}$$

We emphasize that the elements of $\mathcal{L}^2_\beta(G)$ are non random.

Define, for every $\varphi \in L^2(G)$ and $t \geqslant 0$,

$$B_t(\varphi) := \int_{(0,t)\times G} \varphi(y)\, \xi(\mathrm{d}s\, \mathrm{d}y). \tag{6.6}$$

The preceding is understood as a Wiener integral, and it is easy to see that $\{B_t(\varphi)\}_{t\geqslant 0}$ is Brownian motion scaled to have variance $\|\varphi\|_{L^2(G)}$ at time one. Let $\mathscr{F}_t$ denote the sigma-algebra generated by all random variables of the form $B_s(\varphi)$, as $s$ ranges within $[0\,,t]$ and $\varphi$ ranges within $L^2(G)$. Then, $\{\mathscr{F}_t\}_{t\geqslant 0}$ is the [raw] filtration of the white noise $\xi$. Without changing the notation, we will complete [P] every sigma-algebra $\mathscr{F}_t$ and also make $\{\mathscr{F}_t\}_{t\geqslant 0}$ right continuous in the usual way. In this way, we may apply the martingale-measure machinary of Walsh [44] whenever we need to.

A space-time stochastic process $Z := \{Z_t(x)\}_{t\geqslant 0, x\in G}$ is called an *elementary random field* [44] if we can write $Z_t(x) = X\mathbf{1}_{[a,b)}(t)\psi(x)$, where $0 < a < b$, $\psi \in C_c(G)$ [the usual space of real-valued continuous functions on $G$], and $X \in L^2(\mathrm{P})$ is $\mathscr{F}_a$-measurable. Clearly, elementary random fields are random fields in the sense mentioned earlier.

A space-time stochastic process is a *simple random field* [44] if it is a finite non-random sum of elementary random fields.

**Definition 6.4.** For every $\beta \geqslant 0$, we define $\mathcal{P}^2_\beta(G)$ to be the completion of the collection of simple random fields in the norm $\mathcal{N}_\beta$. We may observe that: (i) Every $\mathcal{P}^2_\beta(G)$ is a Banach space, once endowed with norm $\mathcal{N}_\beta$; and (ii) If $\alpha < \beta$, then $\mathcal{P}^2_\alpha(G) \subseteq \mathcal{P}^2_\beta(G)$.

We can think of an element of $\mathcal{P}^2_\beta(G)$ as a "predictable random field" in some extended sense.

Let us observe that if $K \in \mathcal{L}^2_\beta(G)$, then $\int_0^T \mathrm{d}s \int_G m_G(\mathrm{d}y)\ [K_s(y)]^2 < \infty$ all $T > 0$. Indeed,

$$\int_0^T \mathrm{d}s \int_G m_G(\mathrm{d}y)\ [K_s(y)]^2 \leqslant \mathrm{e}^{2\beta T}\|K\|^2_{\mathcal{L}^2_\beta(G)}. \tag{6.7}$$

Therefore, we can define the stochastic convolution $K \circledast Z$ for all simple random fields $Z$ and all $K \in \mathcal{L}^2_\beta(G)$ as in Walsh [44]. The following yields further information on this stochastic convolution. For other versions of such stochastic Young inequalities see Foondun and Khoshnevisan [22], and especially Conus and Khoshnevisan [9].

**Lemma 6.5** (Stochastic Young inequality)**.** *Suppose that $Z$ is a simple random field and $K \in \mathcal{L}^2_\beta(G)$ for some $\beta \geqslant 0$. Then, $K \circledast Z \in \mathcal{P}^2_\beta(G)$, and*

$$\mathcal{N}_\beta\left(K \circledast Z\right) \leqslant \mathcal{N}_\beta(Z) \cdot \|K\|_{\mathcal{L}^2_\beta(G)}. \tag{6.8}$$

If $K \in \mathcal{L}^2_\beta(G)$, then Walsh's theory [44] produces a space-time stochastic process $(t\,,x) \mapsto (K \circledast Z)_t(x)$; that is, a collection of random variables $(K \circledast Z)_t(x)$, one for every $(t\,,x) \in (0\,,\infty) \times G$. Thus, the stochastic convolution in Lemma 6.5 is well defined.

Lemma 6.5 implies that the stochastic convolution operator $K \circledast \bullet$ is a bounded linear map from $Z \in \mathcal{P}^2_\beta(G)$ to $K \circledast Z \in \mathcal{P}^2_\beta(G)$ with operator norm being at most $\|K\|_{\mathcal{L}^2_\beta(G)}$. In particular, it follows readily from this lemma that $K \circledast Z$ is a random field, since it is an element of $\mathcal{P}^2_\beta(G)$.

*Proof.* It suffices to consider the case that $Z$ is an elementary random field.

Let us say that a function $K : (0\,,\infty) \times G \to \mathbf{R}$ is *elementary* [in the sense of Lebesgue] if we can write $K_s(y) = A\mathbf{1}_{[c,d)}(s)\phi(y)$ where $A \in \mathbf{R}$, $0 \leqslant c < d$, and $\phi \in C_c(G)$ [the usual space of continuous real-valued functions on $G$ that have compact support]. Let us say also that $K$ is a *simple* function [also in the sense of Lebesgue] if it is a finite sum of elementary functions. These are small variations on the usual definitions of the Lebesgue theory of integration. But they produce the same theory as that of Lebesgue. Here, these variations are particularly handy.

From now on, let us choose and fix some constant $\beta \geqslant 0$, and let us observe that if $K$ were an elementary function, then $K \in \mathcal{L}^2_\beta(G)$ for every $\beta \geqslant 0$.

Suppose we could establish (6.8) in the case that $K$ is an elementary function. Then of course (6.8) also holds when $K$ is a simple function. Because $C_c(G)$ is dense in $L^1(m_G)$ [41, E8, p. 267], the usual form of Lebesgue's theory ensures that simple functions are dense in $\mathcal{L}^2_\beta(G)$. Therefore by density, if we could prove that "$K \circledast Z \in \mathcal{P}^2_\beta(G)$" and (6.8) both hold in the case that $K$ is elementary, then we can deduce "$K \circledast Z \in \mathcal{P}^2_\beta(G)$" and (6.8) for all $K \in \mathcal{L}^2_\beta(G)$. This reduces our entire problem to the case where $Z$ is an elementary random field and $K$ is an elementary function, properties that we assume to be valid throughout the remainder of this proof. Thus, from now on we consider

$$K_s(y) = A \cdot \mathbf{1}_{[c,d)}(s)\phi(y) \quad \text{and} \quad Z_t(x) = X \cdot \mathbf{1}_{[a,b)}(t)\psi(x), \tag{6.9}$$

where $A \in \mathbf{R}$, $0 \leqslant c < d$, $0 < a < b$, $X \in L^2(\mathrm{P})$ is $\mathscr{F}_a$-measurable, $\psi \in C_c(G)$, and $\phi \in C_c(G)$. The remainder of the proof works is divided naturally into three steps.

**Step 1 (measurability).** We first show that $K \circledast Z$ is a random field in the sense of this paper.

According to the Walsh theory [44],

$$(K \circledast Z)_t(x) = AX \cdot \int_{\mathcal{T}(t)\times G} \phi(y-x)\psi(y)\,\xi(\mathrm{d}s\,\mathrm{d}y), \tag{6.10}$$

where $\mathcal{T}(t) := (0\,,t) \cap [a\,,b) \cap [t-d\,,t-c)$, and the stochastic integral can be understood as a Wiener integral, since the integrand is non random and square integrable $[\mathrm{d}s \times m_G(\mathrm{d}y)]$. In particular, we may observe that for all $x, w \in G$ and $t \geqslant 0$,

$$\begin{aligned}
&\mathrm{E}\left(|(K \circledast Z)_t(x) - (K \circledast Z)_t(w)|^2\right)\\
&= A^2\mathrm{E}(X^2)|\mathcal{T}(t)| \cdot \int_G m_G(\mathrm{d}y)\ [\psi(y)]^2\,|\phi(y-x) - \phi(y-w)|^2\\
&\leqslant \mathrm{const} \cdot \int_G |\phi(y-w+x) - \phi(y)|^2\, m_G(\mathrm{d}y),
\end{aligned} \tag{6.11}$$

where $|\mathcal{T}(t)|$ denotes the Lebesgue measure of $\mathcal{T}(t)$, and the implied constant does not depend on $(t\,,x\,,w)$. Similarly, for every $0 \leqslant t \leqslant \tau$ and $x \in G$,

$$\mathrm{E}\left(\left|(K \circledast Z)_t(x) - (K \circledast Z)_\tau(x)\right|^2\right) \leqslant \mathrm{const}\cdot(\tau - t), \tag{6.12}$$

where the implied constant does not depend on $(t\,,x,\,w)$. Consequently,

$$\lim_{\substack{x\to w\\ t\to\tau}} \mathrm{E}\left(\left|(K \circledast Z)_t\,(x) - (K \circledast Z)_\tau\,(w)\right|^2\right) = 0, \tag{6.13}$$

uniformly for all $\tau \geqslant 0$ and $w \in G$. In light of a separability theorem of Doob [18, Ch. 2], the preceding implies that $(\Omega\,,(0\,,\infty)\,,G) \ni (\omega\,,t\,,x) \mapsto (K \circledast Z)_t(x)(\omega)$ has a product-measurable version.[2]

**Step 2 (extended predictability).** Next we prove that $K \circledast Z \in \mathcal{P}^2_\beta(G)$.

Let us define another elementary function $\bar{K}_s(y) := A\mathbf{1}_{[c,d)}(s)\bar{\phi}(y)$ where $A$ and $(c\,,d)$ are the same as they were in the construction of $K$, but $\bar{\phi} \in C_c(G)$ is not necessarily the same as $\phi$. It is easy to see that

$$\begin{aligned}
&\mathrm{E}\left(\left|(K \circledast Z)_t(x) - (\bar{K} \circledast Z)_t(x)\right|^2\right)\\
&= A^2\mathrm{E}(X^2)|\mathcal{T}(t)|\cdot\int_G [\psi(y)]^2\left|\phi(y-x) - \bar{\phi}(y-x)\right|^2 m_G(\mathrm{d}y)\\
&\leqslant \mathrm{const}\cdot\left\|\phi - \bar{\phi}\right\|^2_{L^2(G)},
\end{aligned} \tag{6.14}$$

where the implied constant does not depend on $(t\,,x\,,\phi\,,\bar{\phi})$. The definition of the stochastic convolution shows that

$$\mathrm{supp}\left((K \circledast Z)_t\right) \subseteq \mathrm{supp}(\psi) \oplus \mathrm{supp}(\phi), \tag{6.15}$$

almost surely for all $t \geqslant 0$, where "supp" denotes "support." Since $K \circledast Z$ and $\bar{K} \circledast Z$ are both random fields (Step 1), we can integrate both sides of (6.14) $[\exp(-2\beta t)\,\mathrm{d}t \times m_G(\mathrm{d}x)]$ in order to find that

$$\left[\mathcal{N}_\beta(K \circledast Z - \bar{K} \circledast Z)\right]^2 \leqslant \mathrm{const}\cdot\left\|\phi - \bar{\phi}\right\|^2_{L^2(G)}\cdot m_G\left(\mathrm{supp}(\psi) \oplus S\right), \tag{6.16}$$

where $S$ is any compact set that contains both the supports of both $\phi$ and $\bar{\phi}$. Of course, $\mathrm{supp}(\psi) \oplus S$ has finite $m_G$-measure since it is a compact set.

We now use the preceding computations as follows: Let us choose in place of $\bar{\phi}$ a sequence of functions $\phi^1, \phi^2, \ldots$, all in $C_c(G)$ and all supported in one fixed compact set $S \supset \mathrm{supp}(\phi)$, such that: (i) Each $\phi^j$ can be written as $\phi^j(x) := \sum_{i=1}^{n_j} a_{i,j}\mathbf{1}_{E_j}(x)$ for some constants $a_{i,j}$'s and compact sets $E_j \subset G$; and (ii) $\|\phi - \phi^j\|_{L^2(G)} \to 0$ as $j \to \infty$. The resulting kernel can be written as $K^j$ [in place of $\bar{K}$]. Thanks to (6.16),

$$\lim_{j\to\infty} \mathcal{N}_\beta\left(K \circledast Z - K^j \circledast Z\right) = 0. \tag{6.17}$$

[2] As written, Doob's theorem is applicable to the case of stochastic processes that are indexed by euclidean spaces. But the very same proof will work for processes that are indexed by $\mathbf{R}_+ \times G$.

A direct computation shows that $K^j \circledast Z$ is an elementary random field, and hence it is in $\mathcal{P}^2_\beta$. Thanks to the preceding display, $K \circledast Z$ is also in $\mathcal{P}^2_\beta$. This completes the proof of Step 2.

**Step 3 (proof of** (6.8)**).** Since

$$\mathrm{E}\left(\left|(K \circledast Z)_t(x)\right|^2\right) = \int_0^t \mathrm{d}s \int_G m_G(\mathrm{d}y)\ [K_{t-s}(y-x)]^2 \mathrm{E}\left(|Z_s(y)|^2\right), \quad (6.18)$$

we integrate both sides $[\mathrm{d}m]$ in order to obtain

$$\begin{aligned}\mathrm{E}\left(\|(K \circledast Z)_t\|^2_{L^2(G)}\right) &= \int_0^t \|K_{t-s}\|^2_{L^2(G)}\, \mathrm{E}\left(\|Z_s\|^2_{L^2(G)}\right)\, \mathrm{d}s \\ &\leqslant \mathrm{e}^{2\beta t}[\mathcal{N}_\beta(Z)]^2 \int_0^t \mathrm{e}^{-2\beta(t-s)}\|K_{t-s}\|^2_{L^2(G)}\,\mathrm{d}s \\ &\leqslant \mathrm{e}^{2\beta t}[\mathcal{N}_\beta(Z)]^2\|K\|^2_{\mathcal{L}^2_\beta}.\end{aligned} \quad (6.19)$$

The interchange of integrals and expectation is justified by Tonelli's theorem, thanks to Step 1. Divide by $\exp(-2\beta t)$ and optimize over $t \geqslant 0$ to deduce (6.8) whence the lemma. □

Now we extend the definition of the stochastic convolution as follows: Suppose $K \in \mathcal{L}^2_\beta$ and $Z \in \mathcal{P}^2_\beta$ for some $\beta \geqslant 0$. Then we can find simple random fields $Z^1, Z^2, \ldots$ such that $\lim_{n\to\infty} \mathcal{N}_\beta(Z^n - Z) = 0$. Lemma 6.5 ensures that $\lim_{n\to\infty} \mathcal{N}_\beta(K^n \circledast Z - K \circledast Z) = 0$, and hence the following result holds.

**Theorem 6.6.** *If $K \in \mathcal{L}^2_\beta(G)$ and $Z \in \mathcal{P}^2_\beta(G)$ for some $\beta \geqslant 0$, then there exists $K \circledast Z \in \mathcal{P}^2_\beta(G)$ such that $(K, Z) \mapsto K \circledast Z$ is a.s. a bilinear map that satisfies* (6.8)*. This stochastic convolution $K \circledast Z$ agrees with the Walsh stochastic convolution when $Z$ is a simple random field.*

The random field $K \circledast Z$ is the *stochastic convolution* of $K$ and $Z$. Let us emphasize, however, that this construction of $K \circledast Z$ produces a stochastic process $t \mapsto (K \circledast Z)_t$ with values in $L^2(G)$.

# 7 Proof of Theorem 2.1: Part 1

The proof of Theorem 2.1 is divided naturally in two parts: First we study the case that $\sigma(0) = 0$; after that we visit the case that $G$ is compact. The two cases are handled by different methods. Throughout this section, we address only the first case, and hence we assume that

$$\sigma(0) = 0, \quad \text{whence} \quad |\sigma(z)| \leqslant \mathrm{L}_\sigma |z| \text{ for all } z \in \mathbf{R}; \quad (7.1)$$

see (2.9).

Our derivation follows ideas of Walsh [44] and Dalang [13], but has novel features as well, since our stochastic convolutions are not defined as classical [everywhere defined] random fields but rather as elements of the space $\cup_{\beta\geqslant 0}\mathcal{P}^2_\beta(G)$. Therefore, we hash out some of the details of the proof of Theorem 2.1. Throughout, we write $u_t(x)$ in place of $u(t\,,x)$, as is customary in the theory of stochastic processes. Thus, let us emphasize that we *never* write $u_t$ in place of $\partial u/\partial t$.

Let us follow [essentially] the treatment of Walsh [44], and say that a stochastic process $u := \{u_t\}_{t\geqslant 0}$ with values in $L^2(G)$ is a *mild solution* to (SHE) with initial function $u_0 \in L^2(G)$, when $u$ satisfies

$$u_t = P_t u_0 + \lambda \left(p \circledast \sigma(u)\right)_t \qquad \text{a.s. for all } t > 0, \tag{7.2}$$

viewed as a random dynamical system on $L^2(G)$.[3] Somewhat more precisely, we wish to find a $\beta \geqslant 0$, sufficiently large, and solve the preceding as a stochastic integration equation for processes in $\mathcal{P}^2_\beta(G)$, using that value of $\beta$. Since the spaces $\{\mathcal{P}^2_\beta(G)\}_{\beta\geqslant 0}$ are nested, there is no unique choice. But as it turns out there is a minimal acceptable choice for $\beta$, which we also will identify for later purposes.

The proof proceeds, as usual, by an appeal to Picard iteration. Let $u^{(0)}_t(x) := u_0(x)$ and define iteratively

$$u^{(n+1)}_t := P_t u_0 + \lambda \left(p \circledast \sigma\left(u^{(n)}\right)\right)_t, \tag{7.3}$$

for all $n \geqslant 1$. Since

$$\mathcal{N}_\beta\left(P_t u_0\right) \leqslant \sup_{t\geqslant 0} \|P_t u_0\|_{L^2(G)} = \|u_0\|_{L^2(G)} \qquad \text{for all } \beta \geqslant 0, \tag{7.4}$$

and because $\|p\|^2_{\mathcal{L}^2_\beta} = \Upsilon(2\beta)$, it follows that

$$\begin{aligned}\mathcal{N}_\beta\left(u^{(n+1)}\right) &\leqslant \|u_0\|_{L^2(G)} + \lambda\mathcal{N}_\beta\left(\sigma\circ u^{(n)}\right)\left(\int_0^\infty \mathrm{e}^{-2\beta s}\|p_s\|^2_{L^2(G)}\,\mathrm{d}s\right)^{1/2}\\ &= \|u_0\|_{L^2(G)} + \lambda\mathcal{N}_\beta\left(\sigma\circ u^{(n)}\right)\sqrt{\Upsilon(2\beta)},\end{aligned} \tag{7.5}$$

for all $n \geqslant 1$ and $\beta \geqslant 0$. Next we apply the Lipschitz condition of $\sigma$ together with the fact that $\sigma(0) = 0$ in order to deduce the iterative bound

$$\mathcal{N}_\beta\left(u^{(n+1)}\right) \leqslant \|u_0\|_{L^2(G)} + \mathcal{N}_\beta\left(u^{(n)}\right)\lambda \mathrm{L}_\sigma\sqrt{\Upsilon(2\beta)}. \tag{7.6}$$

Now we choose $\beta$ somewhat carefully. Let us choose and fix some $\varepsilon \in (0\,,1)$, and then define

$$\beta := \frac{1}{2}\Upsilon^{-1}\left(\frac{1}{(1+\varepsilon)^2\lambda^2\mathrm{L}^2_\sigma}\right), \tag{7.7}$$

[3] In statements such as this, we sometimes omit writing "a.s.," particularly when the "almost sure" assertion is implied clearly.

which leads to the identity $\lambda \mathrm{L}_\sigma \sqrt{\Upsilon(2\beta)} = (1+\varepsilon)^{-1}$, whence

$$\mathcal{N}_\beta\left(u^{(n+1)}\right) \leqslant \|u_0\|_{L^2(G)} + \frac{1}{(1+\varepsilon)} \mathcal{N}_\beta\left(u^{(n)}\right). \tag{7.8}$$

Since $\mathcal{N}_\beta(u_0) = \|u_0\|_{L^2(G)}$, it follows that

$$\sup_{n \geqslant 0} \mathcal{N}_\beta\left(u^{(n)}\right) \leqslant \frac{2+\varepsilon}{\varepsilon} \|u_0\|_{L^2(G)}. \tag{7.9}$$

The same value of $\beta$ can be applied in a similar way in order to deduce that

$$\mathcal{N}_\beta\left(u^{(n+1)} - u^{(n)}\right) \leqslant \frac{1}{1+\varepsilon} \mathcal{N}_\beta\left(u^{(n)} - u^{(n-1)}\right). \tag{7.10}$$

This shows, in particular, that $\sum_{n=0}^{\infty} \mathcal{N}_\beta(u^{(n+1)} - u^{(n)}) < \infty$, whence there exists $u$ such that $\lim_{n\to\infty} \mathcal{N}_\beta(u^{(n)} - u) = 0$. Since

$$\begin{aligned} &\mathcal{N}_\beta\left(p \circledast \left[\sigma\left(u^{(n)}\right) - \sigma(u)\right]\right) \\ &\leqslant \lambda \mathcal{N}_\beta\left(\sigma\left(u^{(n)}\right) - \sigma(u)\right) \cdot \left(\int_0^\infty \mathrm{e}^{-2\beta s} \|p_s\|_{L^2(G)}^2 \, \mathrm{d}s\right)^{1/2} \\ &\leqslant \lambda \mathrm{L}_\sigma \cdot \mathcal{N}_\beta\left(u^{(n)} - u\right) \sqrt{\Upsilon(2\beta)}, \end{aligned} \tag{7.11}$$

it follows that the stochastic convolution $p \circledast \sigma(u^{(n)})$ converges in norm $\mathcal{N}_\beta$ to the stochastic convolution $p \circledast \sigma(u)$. Thus, it follows that $u$ solves the stochastic heat equation and the $L^2$ moment bound on $u$ is a consequence of the fact that $\mathcal{N}_\beta(u) \leqslant (2+\varepsilon)\varepsilon^{-1}\|u_0\|_{L^2(G)}$, for the present choice of $\beta$. The preceding can be unscrambled as follows:

$$\mathrm{E}\left(\|u_t\|_{L^2(G)}^2\right) \leqslant \frac{(2+\varepsilon)^2}{\varepsilon^2} \|u_0\|_{L^2(G)}^2 \exp\left\{\frac{t}{2} \Upsilon^{-1}\left(\frac{1}{(1+\varepsilon)^2 \lambda^2 \mathrm{L}_\sigma^2}\right)\right\}, \tag{7.12}$$

for all $\varepsilon \in (0\,,1)$ and $t \geqslant 0$. Of course, (2.2) is a ready consequence. This proves the existence of the right sort of mild solution to (SHE).

The proof of uniqueness follows the ideas of Dalang [13] but computes norms in $L^2(G)$ rather than pointwise norms. To be more specific, suppose $v$ is another solution that satisfies (2.2) for some finite constant $c \geqslant 0$. Then of course $v$ satisfies (2.2) also when $c$ is replaced by any other larger constant. Therefore, there exists $\beta \geqslant c \geqslant 0$ such that $u, v \in \mathcal{P}_\beta^2$ [for the same $\beta$]. A calculation, very much similar to those we made earlier for Picard's iteration, shows that

$$\mathcal{N}_\beta(u - v) \leqslant \lambda \mathrm{L}_\sigma \cdot \mathcal{N}_\beta(u - v) \cdot \sqrt{\Upsilon(2\beta)}, \tag{7.13}$$

whence it follows that the $L^2(G)$-valued stochastic processes $\{u_t\}_{t \geqslant 0}$ and $\{v_t\}_{t \geqslant 0}$ are modifications of one another. This completes the proof. □

# 8 Proof of Theorem 2.1: Part 2

It remains to prove Theorem in the case that $G$ is compact. The proof requires a number of small technical steps.

Recall the norms $\mathcal{N}_\beta$. We now introduce a slightly different family of norms that were introduced earlier in Foondun and Khoshnevisan [22].

**Definition 8.1.** For all $\beta \geqslant 0$ and all every-where-defined random fields $Z := \{Z_t(x)\}_{t\geqslant 0, x\in G}$ we define

$$\mathcal{M}_\beta(Z) := \sup_{t\geqslant 0}\sup_{x\in G}\left\{\mathrm{e}^{-2\beta t}\mathrm{E}\left(|Z_t(x)|^2\right)\right\}^{1/2}. \tag{8.1}$$

We can define predictable random fields $\mathcal{P}^\infty_\beta(G)$ with respect to the preceding norms, just as we defined spaces $\mathcal{P}^2_\beta(G)$ of predictable random fields for $\mathcal{N}_\beta$ in Definition 6.4.

**Definition 8.2.** For every $\beta \geqslant 0$, we define $\mathcal{P}^\infty_\beta(G)$ to be the completion of the collection of simple random fields in the norm $\mathcal{M}_\beta$. We may observe that: (i) Every $\mathcal{P}^\infty_\beta(G)$ is a Banach space, once endowed with norm $\mathcal{M}_\beta$; and (ii) If $\alpha < \beta$, then $\mathcal{P}^\infty_\alpha(G) \subseteq \mathcal{P}^\infty_\beta(G)$.

The stochastic convolution $K \circledast Z$ can be defined for $Z \in \mathcal{P}^\infty_\beta(G)$ as well, just as one does it for $Z \in \mathcal{P}^2_\beta(G)$ [Theorem 6.6]. The end result is the following.

**Theorem 8.3.** *If $K \in \mathcal{L}^2_\beta(G)$ and $Z \in \mathcal{P}^\infty_\beta(G)$ for some $\beta \geqslant 0$, then there exists $K \circledast Z \in \mathcal{P}^\infty_\beta(G)$ such that $(K, Z) \mapsto K \circledast Z$ is a.s. a bilinear map that satisfies the stochastic Young inequality,*

$$\mathcal{M}_\beta(K \circledast Z) \leqslant \mathcal{M}_\beta(Z) \cdot \|K\|_{\mathcal{L}^2_\beta(G)}. \tag{8.2}$$

*This stochastic convolution $K \circledast Z$ agrees with the Walsh stochastic convolution when $Z$ is a simple random field.*

The proof of Theorem 6.6 follows the same general pattern of the proof of Theorem 6.6 but one has to make a few adjustments that, we feel, are routine. Therefore, we omit the details. However, we would like to emphasize that this stochastic convolution is not always the same as the one that was constructed in the previous sections. In particular, let us note that if $K \in \mathcal{L}^2_\beta(G)$ and $Z \in \mathcal{P}^\infty_\beta(G)$ for some $\beta \geqslant 0$, then $(K \circledast Z)_t(x)$ is a well-defined uniquely-defined random variable for all $t > 0$ and $x \in G$. This should be compared to the fact that $(K \circledast Z)_t$ is defined only as an element of $L^2(G)$ when $Z \in \mathcal{P}^2_\beta(G)$.

The next result shows that (SHE) has a a.s.-unique mild pointwise solution $u$ whenever $u_0 \in L^\infty(G)$, in the sense that $u$ is the a.s.-unique solution to the equation

$$u_t(x) = (P_t u_0)(x) + (p \circledast u)_t(x), \tag{8.3}$$

valid a.s. for every $x \in G$ and $t > 0$. The preceding stochastic convolution is understood to be the one that we just constructed in this section. Among other things, the following tacitly ensures that the said stochastic comvolution is well defined.

**Theorem 8.4.** *Let $G$ be an LCA group, and $\{X_t\}_{t\geqslant 0}$ be a Lévy process on $G$. If $u_0 \in L^\infty(G)$, then for every $\lambda > 0$, the stochastic heat equation* (SHE) *has a mild pointwise solution $u$ that satisfies the following: There exists a finite constant $b \geqslant 1$ that yields the energy inequality*

$$\sup_{x\in G} \mathrm{E}\left(|u_t(x)|^2\right) \leqslant b\mathrm{e}^{bt} \qquad \textit{for every } t \geqslant 0. \tag{8.4}$$

*Moreover, if $v$ is any mild solution that satisfies* (2.2) *as well as $v_0 = u_0$, then* $\mathrm{P}\{u_t(x) = v_t(x)\} = 1$ *for all $t \geqslant 0$.*

The proof of Theorem 8.4 is modeled after the already-proved portion of Theorem 2.1 [that is, in the case that $\sigma(0) = 0$], but uses the norm $\mathcal{M}_\beta$ in place of $\mathcal{N}_\beta$. When $G = \mathbf{R}$, this theorem is also contained within the theory of Dalang [13]. For these reasons, we omit the proof. But let us emphasize that since $u$ is a random field in the sense of the present paper, (8.4) and Fubini's theorem together imply that if $u_0 \in L^\infty(G)$, then

$$\mathrm{E}\left(\|u_t\|^2_{L^2(G)}\right) \leqslant b\mathrm{e}^{bt} m_G(G). \tag{8.5}$$

Now we begin our proof of Theorem 2.1 in the case that $G$ is compact, an assumption which we assume for the remainder of the section.

In the present compact case, $m_G(G) = 1$, and hence we find that if $u_0 \in L^\infty(G)$, then (SHE) has a random field $L^2(G) \cap L^\infty(G)$-valued solution that satisfies

$$\mathrm{E}\left(\|u_t\|^2_{L^2(G)}\right) \leqslant b\mathrm{e}^{bt}. \tag{8.6}$$

We also find, *a priori*, that $u \in \mathcal{P}^2_\beta(G)$ for all sufficiently large $\beta$. This proves the theorem when $G$ is compact and $u_0 \in L^\infty(G)$.

In fact, we can now use the *a priori* existence bounds that we just developed in order to argue, somewhat as in the Walsh theory, and see that [in this case where $u_0 \in L^\infty(G)$] for all $t > 0$ and $x \in G$,

$$\begin{aligned} &\mathrm{E}\left(|u_t(x)|^2\right) \\ &= |(P_t u_0)(x)|^2 + \lambda^2 \int_0^t \mathrm{d}s \int_G m_G(\mathrm{d}y)\ [p_{t-s}(y-x)]^2 \mathrm{E}\left(|\sigma(u_s(y))|^2\right). \end{aligned} \tag{8.7}$$

But we will not need this formula at this time. Instead, let us observe the following variation: If $v$ solves (SHE)—for the same white noise $\xi$—with $v_0 \in L^\infty(G)$, then

$$\begin{aligned} &\mathrm{E}\left(|u_t(x) - v_t(x)|^2\right) \\ &= |(P_t u_0)(x) - (P_t v_0)(x)|^2 \\ &\qquad + \lambda^2 \int_0^t \mathrm{d}s \int_G m_G(\mathrm{d}y)\ [p_{t-s}(y-x)]^2 \mathrm{E}\left(|\sigma(u_s(y)) - \sigma(v_s(y))|^2\right) \\ &\leqslant |(P_t u_0)(x) - (P_t v_0)(x)|^2 \\ &\qquad + \lambda^2 \mathrm{L}^2_\sigma \cdot \int_0^t \mathrm{d}s \int_G m_G(\mathrm{d}y)\ [p_{t-s}(y-x)]^2 \mathrm{E}\left(|u_s(y) - v_s(y)|^2\right). \end{aligned} \tag{8.8}$$

Since each $P_t$ is linear and a contraction on $L^2(G)$, we may integrate both sides of the preceding inequality to deduce the following from Fubini's theorem: For every $\beta \geqslant 0$,

$$\begin{aligned}
&\mathrm{E}\left(\|u_t - v_t\|_{L^2(G)}^2\right) \\
&\leqslant \|u_0 - v_0\|_{L^2(G)}^2 + \lambda^2 \mathrm{L}_\sigma^2 \cdot \int_0^t \|p_{t-s}\|_{L^2(G)}^2 \mathrm{E}\left(\|u_s - v_s\|_{L^2(G)}^2\right) \\
&\leqslant \|u_0 - v_0\|_{L^2(G)}^2 + \lambda^2 \mathrm{L}_\sigma^2 \mathrm{e}^{2\beta t} \left[\mathcal{N}_\beta(u-v)\right]^2 \cdot \Upsilon(2\beta).
\end{aligned} \tag{8.9}$$

In particular,

$$[\mathcal{N}_\beta(u-v)]^2 \leqslant \|u_0 - v_0\|_{L^2(G)}^2 + \lambda^2 \mathrm{L}_\sigma^2 \left[\mathcal{N}_\beta(u-v)\right]^2 \Upsilon(2\beta). \tag{8.10}$$

Owing to (8.6), we know that $\mathcal{N}_\beta(u-v) < \infty$ if $\beta$ is sufficiently large. By the dominated convergence theorem, $\lim_{\beta\uparrow\infty} \Upsilon(2\beta) = 0$, whence we have

$$\lambda^2 \mathrm{L}_\sigma^2 \Upsilon(2\beta) \leqslant 1/2 \qquad \text{for all } \beta \text{ large enough.} \tag{8.11}$$

This shows that

$$\mathcal{N}_\beta(u-v) \leqslant \mathrm{const} \cdot \|u_0 - v_0\|_{L^2(G)}, \tag{8.12}$$

for all $u_0, v_0 \in L^\infty(G)$ and an implied constant that is finite and depends only on $(\lambda, \mathrm{L}_\sigma, \Upsilon)$.

Now that we have proved (8.12), we can complete the proof of Theorem 2.1 [in the case that $G$ is compact] as follows: Suppose $u_0 \in L^2(G)$. Since $C_c(G)$ is dense in $L^2(G)$, we can find $u_0^{(1)}, u_0^{(2)}, \cdots \in C_c(G)$ such that $u_0^{(n)} \to u_0$ in $L^2(G)$ as $n \to \infty$. Let $u^{(n)} := \{u_t^{(n)}(x)\}_{t \geqslant 0, x \in G}$ denote the solution to (SHE) starting at $u_0^{(n)}$. Eq. (8.12) shows that $\{u^{(n)}\}_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{P}_\beta^2(G)$, provided that $\beta$ is chosen to be sufficiently large [but fixed!]. Therefore, $w := \lim_{n\to\infty} u^{(n)}$ exists in $\mathcal{P}_\beta^2(G)$. Lemma 6.5 ensures that $p \circledast u^{(n)}$ converges to $p \circledast w$, and hence $w$ solves (SHE) starting at $u_0$. This proves existence. Uniqueness is proved by similar approximation arguments.

# 9 Proof of Proposition 2.2

First, consider the case that $u_0 \in L^\infty(G)$. In that case, we may apply (8.7) in order to see that the solution $u$ is defined pointwise and satisfies

$$\mathrm{E}\left(|u_t(x)|^2\right) \leqslant |(P_t u_0)(x)|^2 + \lambda^2 \|\sigma\|_{L^\infty(\mathbf{R})}^2 \int_0^t \|p_s\|_{L^2(G)}^2 \, \mathrm{d}s. \tag{9.1}$$

Since $\int_0^t \|p_s\|_{L^2(G)}^2 \, \mathrm{d}s = \int_0^t \bar{p}_s(0) \, \mathrm{d}s \leqslant \mathrm{e}\Upsilon(1/t) < \infty$ [(5.13)] and $G$ is compact, the $L^2(G)$-contractive property of $P_t$ yields

$$[\mathscr{E}_t(\lambda)]^2 = \mathrm{E}\left(\|u_t\|_{L^2(G)}^2\right) \leqslant \|u_0\|_{L^2(G)}^2 + \mathrm{e}\lambda^2 \|\sigma\|_{L^\infty(\mathbf{R})}^2 \Upsilon(1/t). \tag{9.2}$$

If $u$ is known to be only in $L^2(G)$, then by density we can find for every $\varepsilon > 0$ a function $v \in L^\infty(G)$ such that $\|u_0 - v_0\|_{L^2(G)} \leqslant \varepsilon$. The preceding paragraph and (8.12) together yield

$$\begin{aligned}[\mathscr{E}_t(\lambda)]^2 &\leqslant 2\mathrm{e}^{2\beta t}\left[\mathcal{N}_\beta(u-v)\right]^2 + 2\left(\|v_0\|_{L^2(G)}^2 + \mathrm{e}\lambda^2\|\sigma\|_{L^\infty(\mathbf{R})}^2\Upsilon(1/t)\right)\\ &\leqslant 2\mathrm{e}^{2\beta t}\varepsilon^2 + 2\left(2\|u_0\|_{L^2(G)}^2 + 2\varepsilon^2 + \mathrm{e}\lambda^2\|\sigma\|_{L^\infty(\mathbf{R})}^2\Upsilon(1/t)\right).\end{aligned} \tag{9.3}$$

This is more than enough to show that $\mathscr{E}_t(\lambda) = O(\lambda)$ for all $t > 0$. In fact, it yields also the quantitative bound,

$$\mathscr{E}_t(\lambda) \leqslant \text{const}\cdot\left(\|u_0\|_{L^2(G)} + \lambda\|\sigma\|_{L^\infty(\mathbf{R})}\sqrt{\Upsilon(1/t)}\right), \tag{9.4}$$

for a finite universal constant. This completes the first portion of the proof.

If $|\sigma|$ is bounded uniformly from below, then we reduce the problem to the case that $u_0 \in L^\infty(G)$ just as we did in the first half, using (8.12), and then apply (8.7) in order to see that [in the case that $u_0 \in L^\infty(G)$],

$$\mathrm{E}\left(|u_t(x)|^2\right) \geqslant \inf_{z\in G}|u_0(z)|^2 + \lambda^2\inf_{z\in\mathbf{R}}|\sigma(z)|^2\cdot\int_0^t \|p_s\|_{L^2(G)}^2\,\mathrm{d}s. \tag{9.5}$$

We will skip the remaining details on how to go from $u_0 \in L^\infty(G)$ to $u_0 \in L^2(G)$, since this issue has been dealt with already in the first half. Instead, let us conclude the proof by observing that the preceding is consistent, since $\int_0^t \|p_s\|_{L^2(G)}^2\,\mathrm{d}s > 0$, for if this integral were zero for all $t$ then the proof would fail. But because $G$ is compact and $m_G$ is a probability measure on $G$, Jensen's inequality reveals that $\|p_s\|_{L^2(G)}^2 \geqslant \|p_s\|_{L^1(G)}^2 = 1$. Therefore, $\int_0^t \|p_s\|_{L^2(G)}^2\,\mathrm{d}s \geqslant t$ is positive when $t$ is positive, as was advertized.

# 10 Condition (D) and local times

Dalang's Condition (D) is connected intimately to the theory of local times for Lévy processes. This connection was pointed out in Foondun, Khoshnevisan, and Nualart [23] when $G = \mathbf{R}$; see also Eisenbaum et al [20]. Here we describe how one can extend that connection to the present, more general, setting where $G$ is an LCA group.

Let $Y := \{Y_t\}_{t\geqslant 0}$ be an independent copy of $X$, and consider the stochastic process

$$S_t := X_t Y_t^{-1} \qquad (t \geqslant 0). \tag{10.1}$$

It is easy to see that $S := \{S_t\}_{t\geqslant 0}$ is a Lévy process with characteristic function

$$\mathrm{E}(S_t\,,\chi) = \mathrm{e}^{-2t\mathrm{Re}\Psi(\chi)} \qquad \text{for all } t \geqslant 0 \text{ and } \chi \in G^*. \tag{10.2}$$

The process $S$ is called the Lévy symmetrization of $X$; the nomenclature is motivated by the fact that each $S_t$ is a *symmetric* random variable in the sense that $S_t$ and $S_t^{-1}$ have the same distribution for all $t \geqslant 0$.

Let $J$ denote the *weighted occupation measure* of $S$; that is,

$$J(A) := 2\int_0^\infty \mathbf{1}_A(S_s)\mathrm{e}^{-2s}\,\mathrm{d}s, \tag{10.3}$$

for all Borel sets $A \subset G$. It is easy to see that

$$\hat{J}(\chi) := \int_G (x\,,\chi)\,J(\mathrm{d}x) = 2\int_0^\infty (S_s\,,\chi)\mathrm{e}^{-2s}\,\mathrm{d}s \qquad (\chi \in G^*), \tag{10.4}$$

whence

$$\mathrm{E}\left(|\hat{J}(\chi)|^2\right) = 8\int_0^\infty \mathrm{e}^{-2t}\,\mathrm{d}t \int_0^t \mathrm{e}^{-2s}\,\mathrm{d}s\;\mathrm{E}\left[(S_s\,,\chi)\overline{(S_t\,,\chi)}\right]. \tag{10.5}$$

For every $s,t \geqslant 0$ and for all characters $\chi \in G^*$,

$$(S_s\,,\chi)\overline{(S_t\,,\chi)} = \chi(S_s)\chi(S_t^{-1}) = \chi(S_sS_t^{-1}) = \left(S_sS_t^{-1}\right)(\chi). \tag{10.6}$$

Since the distribution of $S_sS_t^{-1}$ is the same as that of $S_{t-s}$ for $t \geqslant s \geqslant 0$, it follows that for every $\chi \in G^*$,

$$\begin{aligned}\mathrm{E}\left(|\hat{J}(\chi)|^2\right) &= 8\int_0^\infty \mathrm{e}^{-2t}\,\mathrm{d}t\int_0^t \mathrm{e}^{-2s}\,\mathrm{d}s\;\mathrm{E}\left[(S_{t-s}\,,\chi)\right]\\ &= 8\int_0^\infty \mathrm{e}^{-2s}\,\mathrm{d}s\int_s^\infty \mathrm{e}^{-2t}\,\mathrm{d}t\;\mathrm{e}^{-2(t-s)\mathrm{Re}\Psi(\chi)}\\ &= \frac{2}{1+\mathrm{Re}\Psi(\chi)}.\end{aligned} \tag{10.7}$$

Therefore,

$$\mathrm{E}\left(\|\hat{J}\|_{L^2(G^*)}^2\right) = 2\int_{G^*}\left(\frac{1}{1+\mathrm{Re}\Psi(\chi)}\right) m_{G^*}(\mathrm{d}\chi) = 2\Upsilon(1). \tag{10.8}$$

In particular, we have proved that Dalang's Condition (D) is equivalent to the condition that

$$\ell(x) := \frac{\mathrm{d}J}{\mathrm{d}m_G}(x) \text{ exists and is in } L^2(\mathrm{P}\times m_G), \tag{10.9}$$

and in this case,

$$\mathrm{E}\left(\|\ell\|_{L^2(G)}^2\right) = \mathrm{E}\left(\|\hat{J}\|_{L^2(G^*)}^2\right) = 2\Upsilon(1), \tag{10.10}$$

thanks to Plancherel's theorem. For real-valued Lévy processes, this observation is due essentially to Hawkes [27].

The random field $\ell$ is called the *local times* of $\{S_t\}_{t\geqslant 0}$; $\ell$ has, by its very definition, the property that it is a random probability function on $G$ such that

$$\int_G f\ell\,\mathrm{d}m_G = 2\int_0^\infty f(S_t)\mathrm{e}^{-2t}\,\mathrm{d}t \qquad \text{a.s.}, \tag{10.11}$$

for all non-random functions $f \in L^2(G)$.

Let us now return to the following remark that was made in the Introduction.

**Lemma 10.1.** *Dalang's Condition (D) holds whenever $G$ is discrete.*

This lemma was shown to hold as a consequence of Pontryagin–van Kampen duality. We can now understand this lemma, probabilistically.

*A probabilistic proof of Lemma 10.1.* When $G$ is discrete, local times always exist and are described via

$$\ell(x) := 2\int_0^\infty \mathbf{1}_{\{x\}}(S_t)\mathrm{e}^{-2t}\,\mathrm{d}t \qquad (x\in G). \tag{10.12}$$

In light of (10.10), it remains to check only that $\ell\in L^2(\mathrm{P}\times m_G)$, since it is evident that $\ell = \mathrm{d}J/\mathrm{d}m_G$ in this case. But since $m_G$ is the counting measure on $G$,

$$\begin{aligned}\Upsilon(1) &= \frac12\|\ell\|^2_{L^2(\mathrm{P}\times m_G)}\\ &= 4\sum_{x\in G}\int_0^\infty \mathrm{e}^{-2s}\,\mathrm{d}s\int_s^\infty \mathrm{e}^{-2t}\,\mathrm{d}t\ \mathrm{P}\{S_s=x\,,S_t=x\}\\ &= 4\int_0^\infty \mathrm{e}^{-2s}\,\mathrm{d}s\int_s^\infty \mathrm{e}^{-2t}\,\mathrm{d}t\ \mathrm{P}\{S_{t-s}=\mathrm{e}_G\},\end{aligned} \tag{10.13}$$

where $\mathrm{e}_G$ denotes the identity element in $G$. Since $\mathrm{P}\{S_{t-s}=\mathrm{e}_G\}\leqslant 1$, it follows readily that $\Upsilon(1)<\infty$, whence follows Condition (D). □

# 11 Group invariance of the excitation indices

The principal aim of this section is to prove that the noise excitation indices $\overline{\mathfrak{e}}(t)$ and $\underline{\mathfrak{e}}(t)$ are "group invariants." In order to do this we need to apply some care, but it is easy to describe informally what group invariance means: If we apply a topological isomorphism to $G$ then we do not change the values of $\overline{\mathfrak{e}}(t)$ and $\underline{\mathfrak{e}}(t)$.

**Definition 11.1.** Recall that two LCA groups $G$ and $\Gamma$ are *isomorphic* [as topological groups] if there exists a homeomorphic homomorphism $h: G\to\Gamma$. We will denote by $\mathrm{Iso}(G\,,\Gamma)$ the collection of all such topological isomorphisms, and write "$G\simeq\Gamma$" when $\mathrm{Iso}(G\,,\Gamma)\neq\varnothing$; that is precisely when $G$ and $\Gamma$ are isomorphic to one another.

Throughout this section, we consider two LCA groups $G\simeq\Gamma$.

It is easy to see that if $h\in\mathrm{Iso}(G\,,\Gamma)$, then $m_\Gamma\circ h$ is a translation-invariance Borel measure on $G$ whose total mass agrees with the total mass of $m_G$. Therefore, we can find a constant $\mu(h)\in(0\,,\infty)$ such that

$$m_\Gamma\circ h = \mu(h)m_G \qquad \text{for all } h\in\mathrm{Iso}(G\,,\Gamma). \tag{11.1}$$

**Definition 11.2.** We refer to $\mu:\mathrm{Iso}(G\,,\Gamma)\to(0\,,\infty)$ as the *modulus function*, and $\mu(h)$ as the *modulus of an isomorphism* $h\in\mathrm{Iso}(G\,,\Gamma)$.

This definition is motivated by the following: Since $G \simeq G$, the collection $\mathrm{Aut}(G) := \mathrm{Iso}(G\,,G)$ of all automorphisms of $G$ is never empty. Recall that $\mathrm{Aut}(G)$ is in general a non-abelian group endowed with group product $h \circ g$ [composition] and group inversion $h^{-1}$ [functional inversion]. It is then easy to see that $\mu$ is a homomorphism from $\mathrm{Aut}(G)$ into the multiplicative positive reals $\mathbf{R}^\times_{>0}$; i.e., that $\mu(h\circ g) = \mu(h)\mu(g)$ and $\mu(h^{-1}) = 1/\mu(h)$ for every $h, g \in \mathrm{Aut}(G)$.

The following simple lemma is an immediate consequence of our standard normalization of Haar measures and states that compact and/or discrete LCA groups are unimodular. But it is worth recording.

**Lemma 11.3.** *Every element of* $\mathrm{Iso}(G\,,\Gamma)$ *is measure preserving when* $G$ *is either compact or discrete. In other words, if* $G$ *is compact or discrete, then so is* $\Gamma$*, and* $\mu(h) = 1$ *for every* $h \in \mathrm{Iso}(G\,,\Gamma)$.

Next, let $\xi$ denote a space time white noise on $\mathbf{R}_+ \times G$. Given a function $h \in \mathrm{Iso}(G\,,\Gamma)$, we may define a random set function $\xi_h$ on $\Gamma$ as follows:

$$\xi_h(A \times B) := \sqrt{\mu(h)}\, \xi(A \times h^{-1}(B)), \tag{11.2}$$

for all Borel sets $A \subset \mathbf{R}_+$ and $B \subset \Gamma$ with finite respective measures $m_{\mathbf{R}}(A)$ and $m_G(B)$. In this way, we find that $\xi_h$ is a scattered Gaussian random measure on $\mathbf{R}_+ \times \Gamma$ with control measure $m_{\mathbf{R}} \times m_\Gamma$. Moreover,

$$\begin{aligned} \mathrm{E}\left(|\xi_h(A \times B)|^2\right) &= \mu(h) m_{\mathbf{R}}(A)(m_G \circ h^{-1})(B) \\ &= m_{\mathbf{R}}(A) m_\Gamma(B). \end{aligned} \tag{11.3}$$

In other words, we have verified the following simple fact.

**Lemma 11.4.** *Let* $\xi$ *denote a space-time white noise on* $\mathbf{R}_+ \times G$*. Then,* $\xi_h$ *is a white noise on* $\mathbf{R}_+ \times \Gamma$ *for every* $h \in \mathrm{Iso}(G\,,\Gamma)$.

Note, in particular, that we can solve SPDEs on $(0\,,\infty) \times \Gamma$ using the space-time white noise $\xi_h$. We will return to this matter shortly.

If $f \in L^2(G)$ and $h \in \mathrm{Iso}(G\,,\Gamma)$, then $f \circ h^{-1}$ can be defined uniquely as an element of $L^2(\Gamma)$ as well as pointwise. Here is how: First let us consider $f \in C_c(G)$, in which case $f \circ h^{-1} : \Gamma \to \mathbf{R}$ is defined pointwise and is in $C_c(\Gamma)$. Next we observe that

$$\begin{aligned} \|f \circ h^{-1}\|^2_{L^2(\Gamma)} &= \int_\Gamma |f(h^{-1}(x))|^2\, m_\Gamma(\mathrm{d}x) \\ &= \int_G |f(y)|^2\, (m_\Gamma \circ h)(\mathrm{d}y) \\ &= \mu(h) \|f\|^2_{L^2(G)}. \end{aligned} \tag{11.4}$$

Since $C_c(G)$ is dense in $L^2(G)$, the preceding constructs uniquely $f \circ h^{-1} \in L^2(\Gamma)$ for every topological isomorphism $h : G \to \Gamma$. Moreover, it follows that (11.4) is valid for all $f \in L^2(G)$. This construction has a handy consequence which we describe next.

For the sake of notational simplicity, if $Z$ is a random field, then we write $Z \circ h^{-1}$ for the random field $Z_t(h^{-1}(x))$, whenever $h$ is such that this definition makes sense. Of course, if $Z$ is non-random, then we may use the very same notation; thus, $K \circ h^{-1}$ makes sense equally well in what follows.

**Lemma 11.5.** *Let $\beta \geqslant 0$ and $h \in \mathrm{Iso}(G\,,\Gamma)$. If $Z \in \mathcal{P}^2_\beta(G)$, then $Z \circ h^{-1} \in \mathcal{P}^2_\beta(\Gamma)$, where*

$$(Z \circ h^{-1})_t(x) := Z_t(h^{-1}(x)) \qquad \text{for all } t > 0 \text{ and } x \in G. \tag{11.5}$$

*Moreover,*

$$\mathcal{N}_\beta(Z \circ h^{-1}\,;\Gamma) = \sqrt{\mu(h)}\,\mathcal{N}_\beta(Z\,;G). \tag{11.6}$$

*Proof.* It suffices to prove the lemma when $Z$ is an elementary random field. But then the result follows immediately from first principles, thanks to (11.4). □

Our next result is a change or variables formula for Wiener integrals.

**Lemma 11.6.** *If $F \in L^2(\mathbf{R}_+ \times \Gamma)$ and $h \in \mathrm{Iso}(G\,,\Gamma)$, then*

$$\int_{\mathbf{R}_+\times G} (F \circ h)\,\mathrm{d}\xi = \frac{1}{\sqrt{\mu(h)}} \int_{\mathbf{R}_+\times\Gamma} F\,\mathrm{d}\xi_h \qquad a.s. \tag{11.7}$$

*Proof.* Thanks to the very construction of Wiener integrals, it suffices to prove the lemma in the case that $F_t(x) = A\mathbf{1}_{[c,d]}(t)\mathbf{1}_Q(x)$ for some $A \in \mathbf{R}$, $0 \leqslant c < d$, and Borel-measurable set $Q \subset \Gamma$ with $m_\Gamma(Q) < \infty$. In this special case, $(F \circ h)_t(x) = A\mathbf{1}_{[c,d)}(t)\mathbf{1}_{h^{-1}(Q)}(x)$, whence we have:

$$\int_{\mathbf{R}_+\times G} (F \circ h)\,\mathrm{d}\xi = A\xi\left([c\,,d) \times h^{-1}(Q)\right) \tag{11.8}$$

which is $[\mu(h)]^{-1/2}$ times $A\xi_h\left([a\,,d) \times Q\right) = \int_{\mathbf{R}_+\times\Gamma} F\,\mathrm{d}\xi_h$, by default. □

**Lemma 11.7.** *Let $\circledast$ denote stochastic convolution with respect to the white noise $\xi$ on $\mathbf{R}_+ \times G$, as before. For every $h \in \mathrm{Iso}(G\,,\Gamma)$, let $\circledast_h$ denote stochastic convolution with respect to the white noise $\xi_h$ on $\mathbf{R}_+ \times \Gamma$. Choose and fix some $\beta \geqslant 0$. Then, for all $K \in \mathcal{L}^2_\beta(\Gamma)$ and $Z \in \mathcal{P}^2_\beta(\Gamma)$,*

$$(K \circ h) \circledast (Z \circ h) = \frac{1}{\sqrt{\mu(h)}}\,(K \circledast_h Z) \circ h^{-1}, \tag{11.9}$$

*almost surely.*

*Proof.* Lemma 11.4 shows that $\xi_h$ is indeed a white noise on $\mathbf{R}_+ \times \Gamma$; and Lemma 11.5 guarantees that $Z \circ h \in \mathcal{P}^2_\beta(G)$. In order for $(K \circ h) \circledast (Z \circ h)$ to be a well-defined stochastic convolution, we need $K \circ h$ to be in $\mathcal{L}^2_\beta(G)$ [Theorem 6.6]. But (11.4) tells us that

$$\|K_t \circ h\|^2_{L^2(G)} = \frac{1}{\mu(h)}\|K_t\|^2_{L^2(\Gamma)} \quad \text{for all } t > 0, \tag{11.10}$$

and hence

$$\|K \circ h\|_{\mathcal{L}^2_\beta(G)}^2 = \frac{1}{\mu(h)} \|K\|_{\mathcal{L}^2_\beta(\Gamma)}^2 < \infty. \tag{11.11}$$

This shows that $(K \circ h) \circledast (Z \circ h)$ is a properly-defined stochastic convolution.

In order to verify (11.9), which is the main content of the lemma, it suffices to consider the case that $K$ and $Z$ are both elementary; see Lemma 6.5 and our construction of stochastic convolutions. In other words, it remains to consider the case that $K$ and $Z$ have the form described in (6.9): That is, in the present context: (i) $K_s(y) = A\mathbf{1}_{(c,d]}(s)\phi(y)$ where $A \in \mathbf{R}$, $0 \leqslant c < d$, and $\phi \in C_c(\Gamma)$; and (ii) $Z_t(x) = X\mathbf{1}_{[a,b)}(t)\psi(x)$ for $0 < a < b$, $X \in L^2(\mathrm{P})$ is $\mathscr{F}_a$-measurable, and $\psi \in C_c(\Gamma)$. In this case,

$$(K \circ h)_s(y) = A\mathbf{1}_{(c,d]}(s)\phi(h(y)), \quad (Z \circ h)_t(x) = X\mathbf{1}_{(a,b]}(t)\psi(h(x)). \tag{11.12}$$

Therefore,

$$\begin{aligned}&\left[(K \circ h) \circledast (Z \circ h)\right]_t(x)\\ &= AX \int_{(0,t)\times G} \mathbf{1}_{(c,d]}(s)\mathbf{1}_{(a,b]}(t-s)\phi(h(y-x))\psi(h(y))\,\xi(\mathrm{d}s\,\mathrm{d}y).\end{aligned} \tag{11.13}$$

The preceding integral is a Wiener integral, and the above quantity is a.s. equal to

$$\begin{aligned}&\frac{AX}{\sqrt{\mu(h)}} \int_{(0,t)\times\Gamma} \mathbf{1}_{(c,d]}(s)\mathbf{1}_{(a,b]}(t-s)\phi\left(y - h^{-1}(x)\right)\psi(y)\,\xi_h(\mathrm{d}s\,\mathrm{d}y) \\ &\qquad = \frac{1}{\sqrt{\mu(h)}}(K \circledast_h Z)_t\left(h^{-1}(x)\right),\end{aligned} \tag{11.14}$$

thanks to Lemma 11.6. □

Finally, if $X := \{X_t\}_{t\geqslant 0}$ is a Lévy process on $G$, then $Y_t := h(X_t)$ defines a Lévy process $Y := h\circ X$ on $\Gamma$. In order to identify better the process $Y := h\circ X$, let us first recall [36, Ch. 4] that since $\Gamma = h(G)$, every character $\zeta \in \Gamma^*$ is of the form $\chi \circ h^{-1}$ for some $\chi \in G^*$ and vice versa. In particular, we can understand the dynamics of $Y = h \circ X$ via the following computation:

$$\begin{aligned}\mathrm{E}(\zeta\,, Y_t) &= \mathrm{E}\left(\chi \circ h^{-1}\,, Y_t\right) = \mathrm{E}\left[\chi\left(h^{-1}(Y_t)\right)\right]\\ &= \mathrm{E}\left[\chi(X_t)\right] = \mathrm{E}\left(\chi\,, X_t\right) = \mathrm{E}\left(\zeta \circ h\,, X_t\right),\end{aligned} \tag{11.15}$$

for every $t \geqslant 0$ and $\zeta = \chi\circ h^{-1} \in \Gamma^*$. Let $\Psi_W$ denote the characteristic exponent of every Lévy process $W$. Then, it follows that

$$\Psi_{h\circ X}(\zeta) = \Psi_X(\zeta \circ h) \qquad \text{for all } \zeta \in \Gamma^*. \tag{11.16}$$

In particular, we can evaluate the $\Upsilon$-function for $Y := h \circ X$ as follows:

$$\begin{aligned}&\int_{\Gamma^*} \left(\frac{1}{1 + \mathrm{Re}\Psi_{h\circ X}(\zeta)}\right) m_{\Gamma^*}(\mathrm{d}\zeta)\\ &\qquad = \int_{\Gamma^*} \left(\frac{1}{1 + \mathrm{Re}\Psi_X(\zeta \circ h)}\right) m_{\Gamma^*}(\mathrm{d}\zeta).\end{aligned} \tag{11.17}$$

Since $\zeta \circ h$ is identified with $\chi$ through the Pontryagin-van Kampen duality pairing, we find the familiar fact that $\Gamma^* \simeq G^*$ [36, Ch. 4], whence we may deduce the following:

$$\begin{aligned}
\int_{\Gamma^*} \left( \frac{1}{1+\mathrm{Re}\Psi_{h\circ X}(\zeta)} \right) m_{\Gamma^*}(\mathrm{d}\zeta) \\
&= \int_{G^*} \left( \frac{1}{1+\mathrm{Re}\Psi_X(\chi)} \right) \left( m_{\Gamma^*} \circ h^{-1} \right)(\mathrm{d}\chi) \qquad (11.18) \\
&= \mu(h) \cdot \int_{G^*} \left( \frac{1}{1+\mathrm{Re}\Psi_X(\chi)} \right) m_{G^*}(\mathrm{d}\chi)
\end{aligned}$$

This $\mu(h)$ the same as the constant in (11.1), because our normalization of Haar measures makes the Fourier transform an $L^2$-isometry.

In other words, we have established the following.

**Lemma 11.8.** *Let $X := \{X_t\}_{t\geqslant 0}$ denote a Lévy process on $G$, and choose and fix $h \in \mathrm{Iso}(G\,,\Gamma)$. Then, the $G$-valued process $X$ satisfies Dalang's Condition (D) if and only if the $\Gamma$-valued process $Y := h \circ X$ satisfies Dalang's Condition (D).*

Let us make another simple computation, this time about the invariance properties of semigroups and their $L^2$-generators.

**Lemma 11.9.** *Let $X := \{X_t\}_{t\geqslant 0}$ denote a Lévy process on $G$, with semigroup $\{P_t^X\}_{t\geqslant 0}$ and generator $\mathscr{L}^X$, and choose and fix $h \in \mathrm{Iso}(G\,,\Gamma)$. Then, the semigroup and generator of $Y := h \circ X$ are*

$$\left(P_t^{h\circ X} f\right)(y) = \left(P_t^X (f\circ h)\right)(h^{-1}(y)), \qquad (11.19)$$

*and*

$$\left(\mathscr{L}^{h\circ X} f\right)(y) = \left(\mathscr{L}^X (f\circ h)\right)\left(h^{-1}(y)\right), \qquad (11.20)$$

*respectively, where $t \geqslant 0$, $y \in \Gamma$, and $f \in L^2(\Gamma)$.*

*Proof.* If $t \geqslant 0$ and $y \in \Gamma$, then $yh(X_t) = h\left(h^{-1}(y)X_t\right)$, whence it follows that for all $f \in C_c(\Gamma)$,

$$\left(P_t^{h\circ X} f\right)(y) = \mathrm{E}\left[f(yh(X_t))\right] = \mathrm{E}\left[(f\circ h)\left(h^{-1}(y)X_t\right)\right]. \qquad (11.21)$$

This yields the semigroup of $h\circ X$ by the density of $C_c(G)$ in $L^2(G)$. Differentiate with respect to $t$ to compute the generator. □

As a ready consequence of Lemma 11.9, we find that if $X := \{X_t\}_{t\geqslant 0}$ denotes a Lévy process on $G$ with transition densities $p^X$ [with respect to $m_G$], and if $h \in \mathrm{Iso}(G\,,\Gamma)$, then $h \circ X$ is a Lévy process on $\Gamma$ with transition densities $p^{h\circ X}$ [with respect to $m_\Gamma$] that are given by

$$p^{h\circ X} := \frac{p^X \circ h^{-1}}{\mu(h)}. \qquad (11.22)$$

Indeed, Lemma 11.9 and the definition of $\mu(h)$ together imply that

$$\int \psi p_t^{h\circ X}\, \mathrm{d}m_\Gamma = \mathrm{E}[\psi(h(X_t))], \tag{11.23}$$

for all $t > 0$ and $\psi \in C_c(G)$. Therefore, $p^{h\circ X}$ is a version of the transition density of $h \circ X$. Lemma 5.1 ensures that $p^{h\circ X}$ is in fact the unique continuous version of any such transition density.

We are ready to present and prove the main result of this section. Throughout, $X := \{X_t\}_{t\geqslant 0}$ denotes a Lévy process on $G$ that satisfies Dalang's Condition (D), and recall our convention that either $G$ is compact or $\sigma(0) = 0$. In this way, we see that (SHE) has a unique solution for every non-random initial function in $L^2(G)$.

**Theorem 11.10** (Group invariance of SPDEs)**.** *Suppose $u_0 \in L^2(G)$ is non random, and let $u$ denote the solution to* (SHE)*—viewed as an SPDE on* $(0\,,\infty)\times G$*—whose existence and uniqueness is guaranteed by Theorem 2.1. Choose and fix $h \in \mathrm{Iso}(G\,,\Gamma)$. Then, $v_t := u_t \circ h^{-1}$ defines the unique solution to the stochastic heat equation*

$$\left|\begin{aligned} &\frac{\partial v_t(x)}{\partial t} = \left(\mathscr{L}^{h\circ X} v_t\right)(x) + \lambda\sqrt{\mu(h)}\,\sigma\left(v_t(x)\right)\xi_h,\\ &v_0 = u_0 \circ h^{-1}, \end{aligned}\right. \tag{11.24}$$

*viewed as an SPDE on $\Gamma = h(G)$, for $x \in \Gamma$ and $t > 0$.*

*Proof.* With the groundwork under way, the proof is quite simple. Let $v$ be the solution to (11.24); its existence is guaranteed thanks to Lemma 11.8 and Theorem 2.1.

Let $v^{(n)}$ and $u^{(n)}$ respectively denote the Picard iterates of (11.24) and $u$. That is, $u^{(n)}$'s are defined iteratively by (7.3), and $v$'s are defined similarly as

$$v_t^{(n+1)} := P_t^{h\circ X} v_0 + \lambda\sqrt{\mu(h)}\left(p^{h\circ X} \circledast_h \sigma\left(v^{(n)}\right)\right)_t. \tag{11.25}$$

We first claim that for all $t > 0$,

$$v_t^{(n)} = u_t^{(n)} \circ h^{-1} \qquad \text{a.s. for all } n \geqslant 0. \tag{11.26}$$

This is a tautology when $n = 0$, by construction. Suppose $v_t^{(n)} = u_t^{(n)} \circ h^{-1}$ a.s. for every $t > 0$, where $n \geqslant 0$ is an arbitrary fixed integer. We next verify that $v_t^{(n+1)} = u_t^{(n+1)} \circ h^{-1}$ a.s. for all $t > 0$, as well. This and a relabeling $[n \leftrightarrow n+1]$ will establish (11.26).

Thanks to the induction hypothesis, Lemma 11.9, and (11.22),

$$v_t^{(n+1)} := \left(P_t^X u_0\right)\circ h^{-1} + \frac{\lambda}{\sqrt{\mu(h)}}\left(\left(p^X \circ h^{-1}\right) \circledast_h \sigma\left(u^{(n)} \circ h^{-1}\right)\right)_t, \tag{11.27}$$

almost surely. Therefore, Lemma 11.7 implies that

$$v_t^{(n+1)} := \left(P_t^X u_0\right)\circ h^{-1} + \lambda\left(p^X \circledast \sigma\left(u^{(n)}\right)\right)_t \circ h^{-1}, \tag{11.28}$$

almost surely. We now merely recognize the right-hand side as $u_t^{(n+1)}$; see (7.3). In this way we have proved (11.26).

Since we now know that $v^{(n)} = u^{(n)} \circ h^{-1}$, two appeals to Theorem 2.1 [via Lemma 11.5] show that if $\beta$ is sufficiently large, then $v^{(n)}$ converges in $\mathcal{P}^2_\beta(\Gamma)$ to $v$ and $u^{(n)} \to u$ in $\mathcal{P}^2_\beta(G)$, as $n \to \infty$. Thus it follows from a second application of Lemma 11.5 that $v = u \circ h^{-1}$. $\square$

The following is a ready corollary of Theorem 11.10; its main content is in the last line where it shows that our noise excitation indices are "invariant under group isomorphisms."

**Corollary 11.11.** *In the context of Theorem 11.10, we have the following energy identity*

$$\mathrm{E}\left(\|u_t\|^2_{L^2(G)}\right) = \frac{1}{\mu(h)}\mathrm{E}\left(\|v_t\|^2_{L^2(\Gamma)}\right), \tag{11.29}$$

*valid for all $t \geqslant 0$. In particular, $u$ and $v$ have the same noise excitation indices.*

*Proof.* Since $v_t(x) = u_t(h^{-1}(x))$, it follows from Theorem 11.10 and (11.4) that

$$\|u_t\|^2_{L^2(G)} = \frac{1}{\mu(h)}\|v_t\|^2_{L^2(\Gamma)} \qquad \text{a.s.}, \tag{11.30}$$

which is more than enough to imply (11.29). The upper noise-excitation index of $u$ at time $t \geqslant 0$ is

$$\overline{\mathfrak{e}}(t) = \limsup_{\lambda\uparrow\infty} \frac{1}{\log\lambda} \log\log\sqrt{\mathrm{E}\left(\|u_t\|^2_{L^2(G)}\right)}, \tag{11.31}$$

whereas the upper noise excitation index of $v$ at time $t$ is

$$\limsup_{\lambda\uparrow\infty} \frac{1}{\log\left[\lambda\sqrt{\mu(h)}\right]} \log\log\sqrt{\mathrm{E}\left(\|v_t\|^2_{L^2(\Gamma)}\right)}, \tag{11.32}$$

which is equal to $\overline{\mathfrak{e}}(t)$, thanks to (11.29) and the fact that $\log[\lambda\sqrt{\mu(h)}] \sim \log\lambda$ as $\lambda \uparrow \infty$. This proves that the upper excitation indices of $u$ and $v$ are the same. The very same proof shows also that the lower excitation indices are shared as well. $\square$

# 12 Projections

Throughout this section, we let $G$ denote an LCA group and $K$ a compact abelian group. Then, it is well known, and easy to see directly, that $G \times K$ is an LCA group with dual group $(G \times K)^* = G^* \times K^*$ [36, Ch. 4].

Let $\pi : G \times K \to G$ denote the canonical projection map. Since $\pi$ is a continuous homomorphism, it follows that if $X := \{X_t\}_{t\geqslant 0}$ is a Lévy process on

$G \times K$, then $(\pi \circ X)_t := \pi(X_t)$ defines a Lévy process on $G$. If $\chi \in G^*$, then $\chi \circ \pi \in (G \times K)^*$, and hence

$$\mathrm{E}\,(\chi\,,\pi(X_t)) = \mathrm{E}\,[(\chi \circ \pi\,, X_t)] = \mathrm{e}^{-t\Psi_X(\chi\circ\pi)}, \tag{12.1}$$

for all $t \geqslant 0$ and $\chi \in G^*$. In other words, we can write

$$\Psi_{\pi\circ X}(\chi) = \Psi_X(\chi \circ \pi) \qquad \text{for all } \chi \in G^*. \tag{12.2}$$

**Proposition 12.1.** *If $X$ satisfies Dalang's Condition (D) on $G \times K$, then the Lévy process $\pi \circ X$ satisfies Dalang's Condition (D) on $G$. In fact,*

$$\Upsilon_{\pi\circ X}(\beta) \leqslant \Upsilon_X(\beta) \qquad \textit{for all } \beta \geqslant 0, \tag{12.3}$$

*where $\Upsilon_W$ is the function defined in* (5.11) *and/or* (5.12) *for every Lévy process $W$ that has transition densities.*

*Proof.* First of all, note that the product measure $m_G \times m_K$ is a translation-invariant Borel measure on $G \times K$, whence $m_{G\times K} = cm_G \times m_K$ for some constant $c$. It is easy to see that $c \in (0\,,\infty)$; let us argue next that $c = 1$. If $f \in L^2(G)$ and $g \in L^2(K)$ satisfy $m_G\{f > 0\} > 0$ and $m_K\{g > 0\} > 0$, then $(f \otimes g)(x \times y) := f(x)g(y)$ satisfies $f \otimes g \in L^2(G \times K)$, and

$$\|f \otimes g\|_{L^2(G\times K)} = \|f\|_{L^2(G)}\|g\|_{L^2(K)} = \|f \otimes g\|_{L^2(m_G\times m_K)}. \tag{12.4}$$

Since the left-most term is $c$ times the right-most term, it follows that $c = 1$.

Let $p^W$ denote the transition densities of $W$ for every Lévy process $W$ that possesses transition densities. It is a simple fact about "marginal probability densities" that since $X$ has nice transition densities $p^X$ [see Lemma 5.1], so does $\pi \circ X$. In fact, because $m_{G\times K} = m_G \times m_K$—as was proved in the previous paragraph—we may deduce that

$$p_t^{\pi\circ X}(x) = \int_K p_t^X(x \times y)\, m_G(\mathrm{d}y) \qquad \text{for all } t > 0 \text{ and } x \in G. \tag{12.5}$$

Now we simply compute: Because $K$ is compact, $m_K$ is a probability measure, and hence the Cauchy–Schwarz inequality yields

$$\begin{aligned} \|p_t^X\|^2_{L^2(G\times K)} &= \int_G m_G(\mathrm{d}x) \int_K m_K(\mathrm{d}y)\ |p_t^X(x \times y)|^2 \\ &\geqslant \int_G m_G(\mathrm{d}x) \left| \int_K m_K(\mathrm{d}y)\ p_t^X(x \times y) \right|^2 = \left\|p_t^{\pi\circ X}\right\|^2_{L^2(G)}. \end{aligned} \tag{12.6}$$

In particular,

$$\int_0^\infty \mathrm{e}^{-\beta s}\|p_s^{\pi\circ X}\|^2_{L^2(G)}\,\mathrm{d}s \leqslant \int_0^\infty \mathrm{e}^{-\beta s}\|p_s^X\|^2_{L^2(G\times K)}\,\mathrm{d}s, \tag{12.7}$$

for all $\beta \geqslant 0$, and the result follows. □

# 13 An abstract lower bound

The main result of this section is an abstract lower estimate for the energy of the solution in terms of the function $\Upsilon$ that was defined in (5.11); see also (5.12).

**Proposition 13.1.** *If $u_0 \in L^2(G)$, $\|u_0\|_{L^2(G)} > 0$, and (2.8) holds, then there exists a finite constant $c \geqslant 1$ such that*

$$\mathscr{E}_t(\lambda) \geqslant c^{-1} \exp(-ct) \cdot \sqrt{1 + \sum_{j=1}^{\infty} \left( \frac{\ell_\sigma^2 \lambda^2}{\mathrm{e}} \cdot \Upsilon(j/t) \right)^j}, \tag{13.1}$$

*for all $t \geqslant 0$. The constant $c$ depends on $u_0$ as well as the underlying Lévy process $X$.*

*Proof.* Consider first the case that

$$u_0 \in L^\infty(G) \cap L^2(G). \tag{13.2}$$

Thanks to (13.2), we may apply (8.7); upon integration $[m_G(\mathrm{d}x)]$, this and Fubini's theorem together yield the following formula:

$$\begin{aligned} \mathrm{E}\left(\|u_t\|^2_{L^2(G)}\right) &= \|P_t u_0\|^2_{L^2(G)} + \lambda^2 \int_0^t \|p_{t-s}\|^2_{L^2(G)} \mathrm{E}\left(\|\sigma \circ u_s\|^2_{L^2(G)}\right) \mathrm{d}s \\ &\geqslant \|P_t u_0\|^2_{L^2(G)} + \ell_\sigma^2 \lambda^2 \int_0^t \|p_{t-s}\|^2_{L^2(G)} \mathrm{E}\left(\|u_s\|^2_{L^2(G)}\right) \mathrm{d}s \\ &= \|P_t u_0\|^2_{L^2(G)} + \ell_\sigma^2 \lambda^2 \int_0^t \bar{p}_{t-s}(0) \mathrm{E}\left(\|u_s\|^2_{L^2(G)}\right) \mathrm{d}s. \end{aligned} \tag{13.3}$$

Appeals to Fubini's theorem are indeed justified, since Theorem 2.1 contains implicitly the desired measurability statements about $u$.

Next we prove that (13.3) holds for every $u_0 \in L^2(G)$ and not just those that satisfy (13.2). With this aim in mind, let us appeal to density in order to find $u_0^{(1)}, u_0^{(2)}, \ldots \in L^\infty(G) \cap L^2(G)$ such that

$$\lim_{n\to\infty} \left\| u_0^{(n)} - u_0 \right\|_{L^2(G)} = 0. \tag{13.4}$$

Then, (8.12) assures us that there exists $\beta > 0$, sufficiently large, such that

$$\lim_{n\to\infty} \mathcal{N}_\beta\left(u^{(n)} - u\right) = 0, \tag{13.5}$$

where $u_t^{(n)}(x)$ denotes the solution to (SHE) with initial value $u_0^{(n)}$. Eq. (13.5) implies readily that

$$\lim_{n\to\infty} \mathrm{E}\left(\left\|u_t^{(n)}\right\|^2_{L^2(G)}\right) = \mathrm{E}\left(\|u_t\|^2_{L^2(G)}\right) \qquad \text{for all } t \geqslant 0. \tag{13.6}$$

And because $P_t$ is contractive on $L^2(G)$,

$$\lim_{n\to\infty}\left\|P_t u_0^{(n)}\right\|_{L^2(G)} = \|P_t u_0\|_{L^2(G)} \qquad \text{for all } t \geqslant 0. \tag{13.7}$$

Therefore, our claim that (13.3) holds is verified once we show that, for all $t > 0$,

$$\lim_{n\to\infty}\int_0^t \bar{p}_{t-s}(0)\mathrm{E}\left(\left\|u_s^{(n)} - u_s\right\|^2_{L^2(G)}\right)\mathrm{d}s = 0. \tag{13.8}$$

This is so because of (13.5) and the fact that the preceding integral is bounded above by

$$\begin{aligned}\left[\mathcal{N}_\beta\left(u^{(n)} - u\right)\right]^2 \cdot \int_0^t \mathrm{e}^{-2\beta(t-s)}\bar{p}_{t-s}(0)\,\mathrm{d}s \\ \leqslant \left[\mathcal{N}_\beta\left(u^{(n)} - u\right)\right]^2 \cdot \Upsilon(2\beta);\end{aligned} \tag{13.9}$$

see also (5.12). Thus, we have established (13.3) in all cases of interest. We can now proceed to prove the main part of the proposition.

Let us define, for all $t > 0$,[4]

$$\mathcal{P}(t) := \ell_\sigma^2\lambda^2\bar{p}_t(0),\ \mathcal{I}(t) := \|P_t u_0\|^2_{L^2(G)},\ \mathcal{E}(t) := \mathrm{E}\left(\|u_t\|^2_{L^2(G)}\right). \tag{13.10}$$

Thanks to (13.3), we obtain the pointwise convolution inequality,

$$\begin{aligned}\mathcal{E} &\geqslant \mathcal{I} + (\mathcal{P} * \mathcal{E}) \\ &\geqslant \mathcal{I} + (\mathcal{P} * \mathcal{I}) + (\mathcal{P} * \mathcal{P} * \mathcal{E}) \\ &\ \vdots \\ &\geqslant \mathcal{I} + (\mathcal{P} * \mathcal{I}) + (\mathcal{P} * \mathcal{P} * \mathcal{I}) + (\mathcal{P} * \mathcal{P} * \mathcal{P} * \mathcal{I}) + \cdots,\end{aligned} \tag{13.11}$$

where $(\psi * \phi)(t) := \int_0^t \psi(s)\phi(t-s)\,\mathrm{d}s$. In particular, we may note that the final quantity depends only on the function $\mathcal{I}$, which is related only to the initial function $u_0$.

A direct computation shows us that the Fourier transform of $P_t u_0$, evaluated at $\chi \in G^*$, is $\exp\{-t\Psi(\chi^{-1})\}\hat{u}_0(\chi)$; see (5.4). Therefore, we may apply the Plancherel theorem to see that

$$\mathcal{I}(t) = \int_{G^*} \mathrm{e}^{-2t\mathrm{Re}\Psi(\chi)}\left|\hat{u}_0(\chi)\right|^2 m_{G^*}(\mathrm{d}\chi) \quad \text{for all } t > 0. \tag{13.12}$$

Since $u_0 \in L^2(G)$, we can find a compact neighborhood $K$ of the identity of $G^*$ such that

$$\int_K |\hat{u}_0(\chi)|^2\, m_{G^*}(\mathrm{d}\chi) \geqslant \frac{1}{2}\int_{G^*} |\hat{u}_0(\chi)|^2\, m_{G^*}(\mathrm{d}\chi) = \frac{1}{2}\|u_0\|^2_{L^2(G)}, \tag{13.13}$$

[4]In terms of the energy of the solution, $\mathcal{E}(t) = [\mathscr{E}_t(\lambda)]^2$.

thanks to Plancherel's theorem. In this way, we find that

$$\mathcal{I}(t) \geqslant \frac{\|u_0\|^2_{L^2(G)}}{2} \mathrm{e}^{-ct} \qquad \text{for all } t > 0. \tag{13.14}$$

where

$$c := 2 \sup_{\chi \in K} \mathrm{Re}\Psi(\chi). \tag{13.15}$$

We will require the fact that $0 \leqslant c < \infty$; this holds simply because $\Psi$ is continuous and $\mathrm{Re}\Psi$ is non negative. In this way, (13.14) yields an estimate for the first term on the right-hand side of (13.11).

As for the other terms, let us write $\mathcal{P}^{*(n)}$ in place of the $n$-fold convolution, $\mathcal{P} * \cdots * \mathcal{P}$, where $\mathcal{P}^{*(1)} := \mathcal{P}$. Then, it is easy to deduce from (13.14) that

$$\left(\mathcal{P}^{*(n)} * \mathcal{I}\right)(t) \geqslant \frac{\|u_0\|^2_{L^2(G)}}{2} \mathrm{e}^{-ct} \left(\mathcal{P}^{*(n)} * \mathbf{1}\right)(t) \quad \text{for all } t > 0, \tag{13.16}$$

where $\mathbf{1}(t) := 1$ for all $t > 0$. Thus, we conclude from (13.11) that

$$\mathcal{E}(t) \geqslant \frac{\|u_0\|^2_{L^2(G)}}{2} \mathrm{e}^{-ct} \cdot \sum_{n=0}^{\infty} \left(\mathcal{P}^{*(n)} * \mathbf{1}\right)(t), \tag{13.17}$$

where $\mathcal{P}^{*(0)} * \mathbf{1} := 1$.

Now,

$$(\mathcal{P} * \mathbf{1})(t) = \ell_\sigma^2 \lambda^2 \cdot \int_0^t \bar{p}_s(0)\,\mathrm{d}s. \tag{13.18}$$

Consequently,

$$\begin{aligned}
(\mathcal{P} * \mathcal{P} * \mathbf{1})(t) &= \ell_\sigma^4 \lambda^4 \cdot \int_0^t \bar{p}_{s_2}(0)\,\mathrm{d}s_2 \int_0^{t-s_2} \bar{p}_{s_1}(0)\,\mathrm{d}s_1, \qquad (13.19)\\
(\mathcal{P} * \mathcal{P} * \mathcal{P} * \mathbf{1})(t) &= \ell_\sigma^8 \lambda^8 \cdot \int_0^t \bar{p}_{s_3}(0)\,\mathrm{d}s_3 \int_0^{t-s_3} \bar{p}_{s_2}(0)\,\mathrm{d}s_2 \int_0^{t-s_3-s_2} \bar{p}_{s_1}(0)\,\mathrm{d}s_1,\\
&\ \ \vdots
\end{aligned}$$

For all real $t \geqslant 0$ and integers $n \geqslant 1$,

$$\begin{aligned}
\left(\mathcal{P}^{*(n)} * \mathbf{1}\right)(t) &\geqslant \ell_\sigma^{2n} \lambda^{2n} \left(\int_0^{t/n} \bar{p}_s(0)\,\mathrm{d}s\right)^n \\
&\geqslant \left(\frac{\ell_\sigma^2 \lambda^2}{\mathrm{e}} \cdot \Upsilon(n/t)\right)^n.
\end{aligned} \tag{13.20}$$

The first bound follows from an application of induction to the variable $n$, and the second follows from (5.13). Since $(\mathcal{P}^{*(0)} * \mathbf{1})(t) = 1$, the proposition follows from (13.17). □

## 14 Proofs of the main theorems

We have set in place all but one essential ingredients of our proofs. The remaining part is the following simple real-variable result. We prove the result in detail, since we will need the following quantitative form of the ensuing estimates.

**Lemma 14.1.** *For all integers $a \geqslant 0$ and real numbers $\rho > 0$, there exists a positive and finite constant $c_{a,\rho}$ such that*

$$\sum_{j=a}^{\infty} \left(\frac{b}{j^{\rho}}\right)^{j} \geqslant c_{a,\rho} \exp\left((\rho/\mathrm{e})b^{1/\rho}\right) \quad \text{for all } b \geqslant 1. \tag{14.1}$$

*Proof.* It is an elementary fact that $(j/\mathrm{e})^j \leqslant j!$ for every integer $j \geqslant 1$. Therefore, whenever $n$, $m$, and $jm/n$ are positive integers,

$$\left(\frac{jm}{\mathrm{e}n}\right)^{jm/n} \leqslant \left(\frac{jm}{n}\right)!. \tag{14.2}$$

In particular, for all $b > 0$,

$$\sum_{j=a}^{\infty} \left(\frac{b}{j^{m/n}}\right)^{j} \geqslant \sum_{\substack{j \geqslant a \\ jm \in n\mathbf{Z}_+}} \frac{b^j (m/\mathrm{e}n)^{jm/n}}{(jm/n)!} \geqslant \sum_{\substack{k \geqslant am/n \\ k \in \mathbf{Z}_+}} \frac{c^k}{k!}, \tag{14.3}$$

where $c := b^{n/m}m/(\mathrm{e}n)$. Since

$$\sum_{\substack{k < am/n \\ k \in \mathbf{Z}_+}} \frac{c^k}{k!} \leqslant \max(b^a\,, 1) \sum_{k=0}^{\infty} \frac{(m/\mathrm{e}n)^k}{k!} = \exp\left\{\frac{m}{\mathrm{e}n}\right\} \cdot \max(b^a\,, 1)\,, \tag{14.4}$$

we immediately obtain the inequality

$$\begin{aligned} \sum_{j=a}^{\infty} \left(\frac{b}{j^{m/n}}\right)^{j} &\geqslant \mathrm{e}^c - \exp\left\{\frac{m}{\mathrm{e}n}\right\} \cdot \max(b^a\,, 1) \\ &= \exp\left\{\frac{b^{n/m}m}{\mathrm{e}n}\right\} - \exp\left\{\frac{m}{\mathrm{e}n}\right\} \cdot \max(b^a\,, 1)\,. \end{aligned} \tag{14.5}$$

The preceding bound is valid for all integers $n$ and $m$ that are strictly positive. We can choose now a sequence $n_k$ and $m_k$ of positive integers such that $\lim_{k\to\infty}(m_k/n_k) = \rho$. Apply the preceding with $(m\,, n)$ replaced by $(m_k\,, n_k)$ and then let $k \to \infty$ to deduce the following bound:

$$\sum_{j=a}^{\infty} \left(\frac{b}{j^{\rho}}\right)^{j} \geqslant \exp\left((\rho/\mathrm{e})b^{1/\rho}\right) - \exp(\rho/\mathrm{e}) \cdot \max(b^a\,, 1)\,. \tag{14.6}$$

Since is valid for all $b > 0$, the lemma follows readily. □

With the preceding under way, we conclude the paper by proving Theorems 2.4, 2.5, and 2.6 in this order.

*Proof of Theorem 2.4.* Since $\mathrm{Re}\Psi$ is non negative,

$$\Upsilon(\beta) \leqslant \beta^{-1} \qquad \text{for all } \beta > 0, \tag{14.7}$$

and hence for every $\varepsilon \in (0\,,1)$,

$$\Upsilon^{-1}\left(\frac{1}{(1+\varepsilon)^2\lambda^2\mathrm{L}_\sigma^2}\right) \leqslant \mathrm{const}\cdot\lambda^2 \qquad \text{for all } \lambda > 1, \tag{14.8}$$

where the implied constant is independent of $\lambda$. Now we merely apply (7.12) in order to see that there exist finite constants $a$ and $b$ such that $\mathscr{E}_t(\lambda) \leqslant a\exp(b\lambda^2)$ for all $\lambda > 1$. This proves that $\bar{\mathfrak{e}}(t) \leqslant 2$.

For the converse bound we recall that $m_{G^*}$ has total mass one because $G^*$ is compact. Since $\Psi$ is continuous, it follows that $\mathrm{Re}\Psi$ is uniformly bounded on $G^*$ and hence for all $\beta_0 > 0$ there exists a positive constant such that

$$\Upsilon(\beta) = \int_{G^*}\left(\frac{1}{\beta + \mathrm{Re}\Psi(\chi)}\right) m_{G^*}(\mathrm{d}\chi) \geqslant \frac{\mathrm{const}}{\beta} \qquad \text{for all } \beta > \beta_0. \tag{14.9}$$

Proposition 13.1 then ensures that

$$\mathscr{E}_t(\lambda) \geqslant \mathrm{const}\cdot\sqrt{1 + \sum_{j=1}^\infty\left(\frac{t\ell_\sigma^2\lambda^2}{\mathrm{e}j}\right)^j} \geqslant a\exp\left(b\lambda^2\right), \tag{14.10}$$

for some finite $a$ and $b$ that depend only on $t$, and in particular are independent of $\lambda > 1$. For the last inequality, we have appealed to Lemma 14.1—with $\rho := 1$— and the bound $\ell_\sigma > 0$ which is a part of the assumptions of the theorem. This proves that $\underline{\mathfrak{e}}(t) \geqslant 2$ when $\ell_\sigma > 0$, and concludes our proof of Theorem 2.4. $\square$

*Proof of Theorem 2.5.* First we consider the case that $G$ is non compact.

According to the structure theory of LCA groups [36, Ch. 6], since $G$ is connected we can find $n \geqslant 0$ and a compact abelian group $K$ such that

$$G \simeq \mathbf{R}^n \times K. \tag{14.11}$$

Because $G$ is not compact, we must have $n \geqslant 1$. Now we put forth the following claim:

$$n = 1. \tag{14.12}$$

In order to prove (14.12), let us define $\pi$ denote the canonical projection from $G \simeq \mathbf{R}^n \times K$ to $\mathbf{R}^n$. Because Condition (D) holds for the Lévy process $X$ on $G \simeq \mathbf{R}^n \times K$, Proposition 12.1 assures us that the Lévy process $\pi\circ X$ on $\mathbf{R}^n$

also satisfies Condition (D). That is, $\Upsilon_{\pi\circ X}(\beta) < \infty$ for one, hence all, $\beta > 0$. Recall from (5.12) that

$$\Upsilon_{\pi\circ X}(\beta) = \text{const}\cdot\int_{\mathbf{R}^n}\left(\frac{1}{\beta+\text{Re}\Psi_{\pi\circ X}(z)}\right)\mathrm{d}z \quad \text{for all } \beta>0, \tag{14.13}$$

where "const" accounts for a suitable normalization of Haar measure on $\mathbf{R}^n$. Since $\pi\circ X$ is a Lévy process on $\mathbf{R}^n$, a theorem of Bochner [7, see (3.4.14) on page 67] ensures that there exists $A \in (0\,,\infty)$ such that

$$\text{Re}\Psi_{\pi\circ X}(z) \leqslant A(1+\|z\|^2) \quad \text{for all } z\in\mathbf{R}^n. \tag{14.14}$$

Because $\Upsilon_{\pi\circ X}(\beta) < \infty$, by assumption, it follows that $\int_{\mathbf{R}^n}(\beta+\|z\|^2)^{-1}\,\mathrm{d}z < \infty$ and hence $n=1$. This proves our earlier assertion (14.12).

Now that we have (14.12), we know that $G \simeq \mathbf{R}\times K$ for a compact abelian group $K$. Because of Theorem 11.10 we may assume, without loss of generality, that our LCA group $G$ is in fact equal to $\mathbf{R}\times K$. Thus, thanks to Propositions 12.1 and 13.1,

$$\begin{aligned}\mathrm{E}\left(\|u_t\|^2_{L^2(\mathbf{R}\times K)}\right) &\geqslant \text{const}\cdot\left\{1+\sum_{j=1}^\infty\left(\frac{\ell_\sigma^2\lambda^2}{\mathrm{e}}\cdot\Upsilon_X(j/t)\right)^j\right\}\\ &\geqslant \text{const}\cdot\left\{1+\sum_{j=1}^\infty\left(\frac{\ell_\sigma^2\lambda^2}{\mathrm{e}}\cdot\Upsilon_{\pi\circ X}(j/t)\right)^j\right\}.\end{aligned} \tag{14.15}$$

In accord with Bochner's estimate (14.14),

$$\Upsilon_{\pi\circ X}(\beta) \geqslant \text{const}\cdot\int_0^\infty\frac{\mathrm{d}x}{\beta+x^2} \geqslant \frac{\text{const}}{\sqrt{\beta}}, \tag{14.16}$$

uniformly for all $\beta \geqslant \beta_0$, for every fixed $\beta_0 > 0$. Thus, we may appeal to Lemma 14.1—with $\rho := 1/2$—in order to see that $\mathrm{E}(\|u_t\|^2_{L^2(\mathbf{R}\times K)}) \geqslant a\exp(b\lambda^4)$, simultaneously for all $\lambda > 1$. This proves that $\underline{\mathfrak{e}}(t) \geqslant 4$ when $G$ is non compact [as well as connected].

We complete the proof of the theorem by proving it when $G$ is compact, connected, metrizable, and has at least 2 elements.

A theorem of Pontryagin [36, Theorem 33, p. 106] states that if $G$ is a locally connected LCA group that is also metrizable then

$$G \simeq \mathbf{R}^n\times\mathbf{T}^m\times D, \tag{14.17}$$

where $0\leqslant n<\infty$, $0\leqslant m\leqslant\infty$, and $D$ is discrete. Of course, $\mathbf{T}^\infty := \mathbf{T}\times\mathbf{T}\times\cdots$ denotes the countable direct product of the torus $\mathbf{T}$ with itself, as is customary.

Since $G$ is compact and connected, we can deduce readily that $n=0$ and $D$ is trivial; that is, $G\simeq\mathbf{T}^m$ for some $0\leqslant m\leqslant\infty$. Because, in addition, $G$ contains at least 2 elements, we can see that $m\neq 0$; thus,

$$G\simeq\mathbf{T}^m \text{ for some } 1\leqslant m\leqslant\infty. \tag{14.18}$$

As a matter of fact, the forthcoming argument can be refined to prove that $m = 1$; see our earlier proof of (14.12) for a model of such a proof. But since we will not need this fact, we will not prove explicitly that $m = 1$. Suffices it to say that, since $m \geqslant 1$,

$$G \simeq \mathbf{T} \times K, \tag{14.19}$$

for a *compact* Hausdorff abelian group $K$; this follows directly from Tychonoff's theorem. Theorem 11.10 reduces our problem to the case that $G = \mathbf{T} \times K$, owing to projection.

Let now $\pi$ denote the canonical projection from $\mathbf{T} \times K$ to $K$, and argue as in the non-compact case to see that

$$\mathrm{E}\left(\|u_t\|^2_{L^2(\mathbf{T}\times K)}\right) \geqslant \text{const} \cdot \left\{ 1 + \sum_{j=1}^{\infty} \left( \frac{\ell_\sigma^2 \lambda^2}{\mathrm{e}} \cdot \Upsilon_{\pi\circ X}(j/t) \right)^j \right\}. \tag{14.20}$$

Bochner's estimate (14.14) has the following analogue for the Lévy process $\pi \circ X$ on $\mathbf{T}$: There exists $A \in (0\,,\infty)$ such that

$$\mathrm{Re}\Psi_{\pi\circ X}(n) \leqslant A(1+n^2) \qquad \text{for all } n \in \mathbf{Z}. \tag{14.21}$$

[The proof of this bound is essentially the same as the proof of (14.14).] Since the dual to $\mathbf{T}$ is $\mathbf{Z}$, it follows that

$$\Upsilon_{\pi\circ X}(\beta) = \text{const} \cdot \sum_{n=-\infty}^{\infty} \frac{1}{\beta + \mathrm{Re}\Psi(n)} \geqslant \frac{\text{const}}{\sqrt{\beta}}, \tag{14.22}$$

uniformly for all $\beta \geqslant \beta_0$, for every fixed $\beta_0 > 0$. A final appeal to Lemma 14.1—with $\rho := 1/2$—finishes the proof. □

*Proof of Theorem 2.6.* Consider the special case that $G = \mathbf{R}$ and $X$ is a symmetric stable Lévy process with index $\alpha \in (0\,,2]$; that is, $\Psi(\xi) = |\xi|^\alpha$. Condition (D) holds if and only if $\alpha \in (1\,,2]$, a condition which we now assume. The generator of $X$ is the fractional Laplacian $\mathscr{L} := -(-\Delta)^{\alpha/2}$ on $\mathbf{R}$. A direct computation reveals that

$$\Upsilon(\beta) = \text{const} \cdot \int_0^\infty \frac{\mathrm{d}x}{\beta + x^\alpha} = \text{const} \cdot \beta^{-(\alpha-1)/\alpha}. \tag{14.23}$$

In particular, for every $\varepsilon \in (0\,,1)$,

$$\Upsilon^{-1}\left( \frac{1}{(1+\varepsilon)^2 \lambda^2 \mathrm{L}_\sigma^2} \right) \leqslant \text{const} \cdot \lambda^{2\alpha/(\alpha-1)} \qquad \text{for all } \lambda > 1, \tag{14.24}$$

This yields

$$\bar{\mathfrak{e}}(t) \leqslant \frac{2\alpha}{\alpha - 1}, \tag{14.25}$$

in this case; see the proof of the first portion of Theorem 2.4, for more details. And an appeal to Lemma 14.1 yields

$$\underline{\mathfrak{e}}(t) \geqslant \frac{2\alpha}{\alpha - 1}. \tag{14.26}$$

See the proof of Theorem 2.5 for some details.

Thus, for every $\alpha \in (1\,,2]$, we have found a model whose noise excitation index is

$$\mathfrak{e} = \frac{2\alpha}{\alpha - 1}. \tag{14.27}$$

Since $\theta := 2\alpha/(\alpha - 1)$ can take any value in $[4\,,\infty)$, as $\alpha$ varies in $(1\,,2]$, this proves the theorem. □

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