

On the Stochastic Heat Equation with Spatially-Colored Random Forcing

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Contents

Chapter 1. Introduction and Statements of Main Results	1
Chapter 2. Lévy Processes	13
2.1. Preliminaries	13
2.2. The Generator	15
2.3. The Replica Semigroup and Associated Sobolev Spaces	17
2.4. On the Heat Equation and Transition Functions	18
2.5. On a Family of Isotropic Lévy Processes	21
Chapter 3. Positive-Definite Functions, Fourier Analysis, and Probabilistic Potential Theory	25
3.1. Fourier Analysis	25
3.2. Positive-Definite Functions	26
3.3. A Preliminary Maximum Principle	27
3.4. Proof of the Maximum Principle	30
3.5. Probabilistic Potential Theory	34
3.6. A Final Observation	41
Chapter 4. The Linear Equation	43
4.1. Existence and uniqueness	43
4.2. Spatial regularity: Examples	48
Chapter 5. The Nonlinear Equation	55
5.1. Existence and Uniqueness	55
5.2. Intermittency	62
5.3. The Massive and Dissipative Operators	69
Chapter 6. Temperate Solutions to Parabolic SPDEs	75
6.1. Stochastic Convolutions	79
6.2. A Priori Estimates	83
6.3. Proof of Existence	86
6.4. More A Priori Estimates	89

6.5. An Upper Bound for Growth	92
Bibliography	95
Index	103

Abstract

We consider the stochastic heat equation of the following form

$$\frac{\partial}{\partial t} u_t(x) = (\mathcal{L}u_t)(x) + b(u_t(x)) + \sigma(u_t(x))\dot{F}_t(x) \quad \text{for } t > 0, x \in \mathbf{R}^d,$$

where \mathcal{L} is the generator of a Lévy process and \dot{F} is a spatially-colored, temporally white, gaussian noise. We will be concerned mainly with the long-term behavior of the mild solution to this stochastic PDE.

For the most part, we work under the assumptions that the initial data u_0 is a bounded and measurable function and σ is nonconstant and Lipschitz continuous. In this case, we find conditions under which the preceding stochastic PDE admits a unique solution which is also *weakly intermittent*. In addition, we study the same equation in the case that $\mathcal{L}u$ is replaced by its massive/dispersive analogue $\mathcal{L}u - \lambda u$ where $\lambda \in \mathbf{R}$. And we describe accurately the effect of the parameter λ on the intermittence of the solution in the case that $\sigma(u)$ is proportional to u [the “parabolic Anderson model”].

Furthermore, we extend our analysis to the case that the initial data u_0 is a measure rather than a function. As it turns out, the stochastic PDE in question does not have a mild solution in this case. We circumvent this problem by introducing a new concept of a solution that we call a *temperate solution*, and proceed to investigate the existence and uniqueness of a temperate solution. We are able to also give partial insight into the long-time behavior of the temperate solution when it exists and is unique.

Finally, we look at the linearized version of our stochastic PDE, that is the case when σ is identically equal to one [any other constant works also]. In this case, we study not only the existence and uniqueness of a solution, but also the regularity of the solution when it exists and is unique.

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CHAPTER 1

Introduction and Statements of Main Results

The principle aim of this paper is to describe the asymptotic large-time behavior of the mild solution $\mathbf{u} := \{u_t(x)\}_{t \geq 0, x \in \mathbf{R}^d}$ of the stochastic heat equation,

$$(1.1) \quad \frac{\partial}{\partial t} u_t(x) = (\mathcal{L}u_t)(x) + b(u_t(x)) + \sigma(u_t(x))\dot{F}_t(x),$$

where $t > 0$ and $x \in \mathbf{R}^d$, and the preceding stochastic PDE can be understood in the sense of Walsh [Wal86].

For the most part, we consider the case that the initial data u_0 is a nonrandom, as well as bounded and measurable, function. But we will also consider the physically-interesting case that u_0 is a nonrandom finite Borel measure on \mathbf{R}^d . The latter case will be the subject of Chapter 6.

Throughout we consider only functions $\sigma, b : \mathbf{R} \rightarrow \mathbf{R}$ that are nonrandom and Lipschitz continuous. Also, we let \mathcal{L} be the L^2 -generator of a d -dimensional Lévy process $X := \{X_t\}_{t \geq 0}$, and assume that X has transition functions.

In the above discussion, we have used the standard notation of probability theory: Namely, g_t denotes the evaluation of a [random or nonrandom] function g at time t , and *never* the time derivative of g . This notation will be used throughout the rest of the paper.

As regards the forcing term \dot{F} in (1.1), we assume that \dot{F} is a generalized Gaussian random field [GV77, Chapter 2, §2.4] whose covariance kernel is $\delta_0(s-t)f(x-y)$, where the “correlation function” f is a nonnegative definite, symmetric, and nonnegative function that is not identically zero.¹ Alternatively, one can use the following

$$(1.2) \quad \dot{F}_t(x) := \frac{\partial^{d+1}}{\partial t \partial x_1 \cdots \partial x_d} F(t, x),$$

¹The symbol “ f ” is reserved for this correlation function here and throughout. We *never* refer to any other function as f .

in the sense of generalized random fields, where F is a centered generalized Gaussian random field with covariance kernel

$$(1.3) \quad \begin{aligned} \text{Cov} \left(\int_{\mathbf{R}_+ \times \mathbf{R}^d} \phi \, dF, \int_{\mathbf{R}_+ \times \mathbf{R}^d} \zeta \, dF \right) \\ = \int_0^\infty ds \int_{\mathbf{R}^d} dx \int_{\mathbf{R}^d} dy \phi_s(x) \zeta_s(y) f(x-y), \end{aligned}$$

where $\int \phi \, dF$ and $\int \zeta \, dF$ are Wiener integrals of $\mathbf{R}_+ \times \mathbf{R}^d \ni (s, x) \mapsto \phi_s(x)$ and $\mathbf{R}_+ \times \mathbf{R}^d \ni (s, x) \mapsto \zeta_s(x)$ with respect to F , and ϕ and ζ are nonnegative measurable functions for which the right-most multiple integral in (1.3) is absolutely convergent.

According to the Bochner–Schwartz theorem [GV77, Theorem 3, p. 157]:

- (a) The Fourier transform \hat{f} of f is a [nonnegative Borel] tempered measure on \mathbf{R}^d ; and
- (b) Conversely, every tempered measure \hat{f} on \mathbf{R}^d is the Fourier transform of one such correlation function f .

The measure \hat{f} is known as the “spectral measure” of the noise F . Throughout, we assume without further mention that F “has a spectral density.” That is,

$$(1.4) \quad \hat{f} \text{ is a measurable function.}$$

This implies that \hat{f} is locally integrable on \mathbf{R}^d as well. Strictly speaking, these conditions are not always needed in our work, but we assume them for the sake of simplicity.

By enlarging the underlying probability space, if need be, we introduce an independent copy $X^* := \{X_t^*\}_{t \geq 0}$ of the dual process $-X$. We can then use X^* to define a symmetric Lévy process $\bar{X} := \{\bar{X}_t\}_{t \geq 0}$ on \mathbf{R}^d via the assignment

$$(1.5) \quad \bar{X}_t := X_t + X_t^* \quad \text{for all } t \geq 0.$$

Motivated by the works of Kardar, Parisi, and Zhang [KPZ86] and Kardar [Kar87], we may refer to \bar{X} as the *replica Lévy process* corresponding to X and will therefore call the resolvent $\{\bar{R}_\alpha\}_{\alpha > 0}$ of \bar{X} , the *replica resolvent*.² We will consider the condition that the correlation function f has finite α -potential at zero for all $\alpha > 0$. That is, we consider the following:

²These quantities are defined in more detail in Chapter 2.

CONDITION 1.1. $(\bar{R}_\alpha f)(0) < \infty$ for all $\alpha > 0$

The above condition will imply an existence and uniqueness result for the stochastic heat equation (1.1). Moreover, our proof of existence and uniqueness is closely linked to the large-time behavior of the solution itself [via *a priori* estimates]. We describe these results next. But first, let us define two important quantities: The first denotes the *upper* $L^p(\mathbb{P})$ -*Liapounov exponent* of the solution $\mathbf{u} := \{u_t(x)\}_{t>0, x \in \mathbf{R}^d}$ to (1.1) at the spatial point $x \in \mathbf{R}^d$:

$$(1.6) \quad \bar{\gamma}_x(p) := \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{E} (|u_t(x)|^p);$$

and the second the *upper maximum* $L^p(\mathbb{P})$ -*Liapounov exponent*:

$$(1.7) \quad \bar{\gamma}_*(p) := \limsup_{t \rightarrow \infty} \frac{1}{t} \sup_{x \in \mathbf{R}^d} \ln \mathbb{E} (|u_t(x)|^p).$$

The above two quantities are variants of the well known *Liapounov exponent*.

We are now ready to state the first main contribution of this paper.

THEOREM 1.2. *Assume that Condition 1.1 holds, and suppose $u_0 : \mathbf{R}^d \rightarrow \mathbf{R}$ is bounded and measurable. Then, (1.1) has an a.s.-unique mild solution which satisfies the following: For all even integers $p \geq 2$,*

$$(1.8) \quad \bar{\gamma}_*(p) \leq \inf \{ \beta > 0 : Q(p, \beta) < 1 \},$$

where

$$(1.9) \quad Q(p, \beta) := \frac{p \text{Lip}_b}{\beta} + z_p \text{Lip}_\sigma \sqrt{(\bar{R}_{2\beta/p} f)(0)},$$

and z_p denotes the largest positive zero of the Hermite polynomial He_p .

Let us make two remarks before we continue with our presentation of the main results of this paper. The first one is consequence of the above result.

REMARK 1.3. It is possible to deduce from Condition 1.1 and the monotone convergence theorem that $\lim_{\alpha \rightarrow \infty} (\bar{R}_\alpha f)(0) = 0$. This, in turn, implies that

$$(1.10) \quad \lim_{\beta \rightarrow \infty} Q(p, \beta) = 0.$$

Consequently, Theorem 1.2 implies among other things that $\bar{\gamma}_*(p) < \infty$ for all $p \in (0, \infty)$. \square

REMARK 1.4 (Borrowed from [FK09, Remark 2.2]). It might help to recall that

$$(1.11) \quad He_k(x) = \frac{1}{2^{k/2}} H_k\left(\frac{x}{\sqrt{2}}\right) \quad \text{for all integers } k \geq 0 \text{ and } x \in \mathbf{R},$$

where $\{H_k\}_{k=0}^\infty$ is defined uniquely via the following:

$$(1.12) \quad e^{-2xt-t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x) \quad \text{for all } t > 0 \text{ and } x \in \mathbf{R}.$$

It is not hard to verify that

$$(1.13) \quad z_2 = 1 \quad \text{and} \quad z_4 = \sqrt{3 + \sqrt{6}} \approx 2.334.$$

This is valid simply because $He_2(x) = x^2 - 1$ and $He_4(x) = x^4 - 6x^2 + 3$. In addition,

$$(1.14) \quad z_p \sim 2\sqrt{p} \quad \text{as } p \rightarrow \infty, \text{ and} \quad \sup_{p \geq 1} \left(\frac{z_p}{\sqrt{p}}\right) = 2;$$

see Carlen and Kree [CK91, Appendix]. □

Next we put Theorem 1.2 in the context of the existing literature on the stochastic heat equation. With this in mind, define for all $\beta \geq 0$,

$$(1.15) \quad \Upsilon(\beta) := \frac{1}{(2\pi)^d} \int \frac{\hat{f}(\xi)}{\beta + 2\text{Re}\Psi(\xi)} d\xi,$$

where Ψ is the characteristic exponent of the Lévy process X . Dalang [Dal99] has established a very general result which guarantees the existence and uniqueness of solutions to large families of SPDEs. If we apply Dalang's result to the present parabolic problem (1.1), then we find the following: If u_0 is a constant, then the condition

$$(1.16) \quad \Upsilon(1) < \infty$$

insures the existence and uniqueness of a [mild] solution to (1.1). Moreover, Dalang's result shows that (1.16) is necessary and sufficient for existence and uniqueness in the case that (1.1) is a linear SPDE; that is, when $\sigma(u) = 1$ and $b(u) = 0$. For closely-related results [that also include hyperbolic equations] see Carmona and Molchanov [CM94], Conus and Dalang [CD08], Dalang and Frangos [DF98], Dalang and Mueller [DM03], Dalang, Mueller, and Tribe [DMT08], Dalang and Sanz-Solé [DSS09], and Peszat and Zabczyk [PZ00].

Our next result implies among many other things that Dalang's condition (1.16) is generically equivalent to the potential-theoretic Condition 1.1.

THEOREM 1.5 (A Maximum Principle). *For all $\beta > 0$,*

$$(1.17) \quad (\bar{R}_\beta f)(0) = \sup_{x \in \mathbf{R}^d} (\bar{R}_\beta f)(x) = \Upsilon(\beta).$$

Thus, Condition 1.1 holds if and only if (1.16) holds. Furthermore, if Condition 1.1 [and/or (1.16)] holds and f is lower semicontinuous, then for all $\beta > 0$ there exists $\pi_\beta \in C_0(\mathbf{R}^d)$ such that $\bar{R}_\beta f = \pi_\beta$ almost everywhere.

In light of Theorem 1.5 and Dalang's theorem [Dal99], the novel contributions of our Theorem 1.2 are:

- (a) The condition that u_0 is a constant can be improved to one about the boundedness of u_0 ; and more significantly
- (b) As an *a priori* consequence of existence, we obtain a uniform upper bound for the maximum $L^p(\mathbf{P})$ -Liapounov exponent of the solution to (1.1).

This second contribution leads to the weak intermittence of solutions, which is a notion that is rooted in the literature of statistical mechanics. With this in mind, let us recall the following [FK09]:

DEFINITION 1.6. Suppose that there exists an a.s.-unique solution $\mathbf{u} := \{u_t(x)\}_{t>0, x \in \mathbf{R}^d}$ to (1.1). We say that \mathbf{u} is *weakly intermittent* if

$$(1.18) \quad \bar{\gamma}_*(p) < \infty \quad \text{for all } p \in [2, \infty), \quad \text{and} \quad \inf_{x \in \mathbf{R}^d} \bar{\gamma}_x(2) > 0.$$

The same reasoning that was employed in [FK09] can be used to deduce that if the solution to (1.1) is nonnegative for all $t > 0$, then weak intermittence implies the much better-known property of *intermittency* [CM94, Mol91, ZRS90]; that is, the property that

$$(1.19) \quad p \mapsto \frac{\bar{\gamma}_x(p)}{p} \quad \text{is strictly increasing on } [2, \infty) \text{ for all } x \in \mathbf{R}^d.$$

There is a large literature which shows that, under further mild hypotheses on \mathcal{L} and/or f , if u_0 is nonnegative then the solution to (1.1) is nonnegative at all times; see, for example the papers by Assing and Manthey [AM95], Carmona and Molchanov [CM94], Donati-Martin and Pardoux [DMP93], Hausmann and Pardoux [HP89], Kotelenz [Kot92], Manthey [Man86],

Manthey and Stiewe [MS92, MS91], Mueller [Mue91], Nualart and Pardoux [NP92], and Shiga [Shi97, Shi94].³

Thus, we can draw the conclusion that, in all such cases, weak intermittence actually implies intermittency.

A quick calculation, using only Hölder's inequality, shows that $p \mapsto \bar{\gamma}_x(p)/p$ is always nondecreasing on $[2, \infty)$. However, the mentioned strict monotonicity does not always hold. When it does hold, then it has some physical significance; see Zeldovitch, Ruzmaikin, and Sokoloff [ZRS90] for a physical discussion of intermittency. And Molchanov [Mol91] for a mathematical explanation of that physical phenomenon.

Our next main goal is to find nontrivial conditions that guarantee the weak intermittence of the solution to (1.1). In light of Theorem 1.2, we aim to derive a positive lower bound on $\inf_{x \in \mathbf{R}^d} \bar{\gamma}_x(2)$. Unfortunately, it is quite hard to do this at the level of generality of the conditions of Theorem 1.2. In fact, informal arguments suggest that the solution to (1.1) might not always be weakly intermittent. Thus, we seek to find reasonable restrictions of the various parameters of (1.1) which guarantee that the solution to (1.1) is weakly intermittent.

Let \hat{g} denotes the Fourier transform of a locally-integrable function g , and consider the following:

CONDITION 1.7. Suppose:

- (1) $\hat{f}(\xi)$ depends on $\xi \in \mathbf{R}^d$ only through $|\xi_1|, \dots, |\xi_d|$;
- (2) $|\xi_j| \mapsto \hat{f}(\xi)$ is nonincreasing for every $j = 1, \dots, d$; and
- (3) $\text{Re}\Psi(\xi)$ depends on $\xi \in \mathbf{R}^d$ only through $|\xi_1|, \dots, |\xi_d|$.

These are relatively mild provisions on the spectral density \hat{f} and the process \bar{X} . Our conditions on the spectral density can be applied to all of the examples that we would like to cover. It is possible to show that they include the following choices for f :

- (i) *Ornstein–Uhlenbeck-type kernels.*

$$f(x) = c_1 e^{-c_2 \|x\|^\alpha} \left[\hat{f}(\xi) = \frac{c_1}{(2\pi)^d} \int_{\mathbf{R}^d} e^{-i\xi \cdot x - c_2 \|x\|^\alpha} dx \right],$$

for constants $c_1, c_2 \in (0, \infty)$ and $\alpha \in (0, 2]$;

³In connection to matters of positivity and regularity, we mention also a closely-related and fundamental paper by Dawson, Iscoe, and Perkins [DIP89], where (1.1) with $\sigma(u) = \text{const} \cdot \sqrt{u}$ is considered. And positivity of the solution is shown to follow from many-particle approximations to the underlying SPDE.

(ii) *Poisson kernels.*

$$f(x) = \frac{c_1}{(\|x\|^2 + c_2)^{(d+1)/2}} \quad \left[\hat{f}(\xi) = \text{const} \cdot e^{-\text{const}\|\xi\|} \right],$$

for $c_1, c_2 \in (0, \infty)$.

(iii) *Cauchy kernels.*

$$f(x) = \frac{c_1}{\prod_{j=1}^d (c_2 + x_j^2)} \quad \left[\hat{f}(\xi) = \text{const} \cdot e^{-\text{const} \sum_{j=1}^d |\xi_j|} \right],$$

for $c_1, c_2 \in (0, \infty)$; and

(iv) *Riesz kernels.*

$$f(x) = \frac{c}{\|x\|^\alpha} \quad \left[\hat{f}(\xi) = \frac{\text{const}}{\|\xi\|^{d-\alpha}} \right],$$

for $c \in (0, \infty)$ and $\alpha \in (0, d)$.

And one can construct a great number of other permissible examples as well.

Having introduced Condition 1.7, we can now present the third main result of this paper.

THEOREM 1.8. *Suppose $b \equiv 0$ and Conditions 1.1 and 1.7 hold. Suppose, in addition, that $\eta := \inf_{x \in \mathbf{R}^d} u_0(x) > 0$ and there exists $L_\sigma \in (0, \infty)$ such that $\sigma(z) \geq L_\sigma |z|$ for all $z \in \mathbf{R}$. Then,*

$$(1.20) \quad \inf_{x \in \mathbf{R}^d} \bar{\gamma}_x(2) \geq \sup \left\{ \beta > 0 : (\bar{R}_\beta f)(0) \geq \frac{2^{d-1}}{L_\sigma^2} \right\},$$

where $\sup \emptyset := 0$.

Before we pause to make a few remarks, let us briefly study an example. Consider the case that $(\bar{R}_0 f)(0) = \infty$. In that case, $(\bar{R}_\beta f)(0) \geq 2^{d-1}/L_\sigma^2$ for all $\beta > 0$ sufficiently small. Hence, in this case, the hypotheses of Theorem 1.8 guarantee weak intermittence of the solution to (1.1) without further restrictions.

REMARK 1.9. (1) In the case that \dot{F} is space-time white noise, the condition “ $\sigma(z) \geq L_\sigma |z|$ ” can be replaced with the slightly-better condition “ $|\sigma(z)| \geq L_\sigma |z|$ ” [**FK09**].

(2) We will see later on that, when $d = 1$, the lower bound (1.20) and the upper bound (1.8) can sometimes match. However, the two bounds can never agree when $d \geq 2$. This phenomenon is due to the fact that level sets of $\beta \mapsto (\bar{R}_\beta f)(0)$ cannot describe the growth of \mathbf{u} exactly. The correct gauge appears to be a much more

complicated function, except in the cases that \dot{F} is space-time white noise and when $d = 1$; compare with [FK09] for results on the case that \dot{F} denotes space-time white noise. \square

We are aware of a few variants of Theorem 1.8, but the next one is perhaps the most striking since it assumes only that the nonlinearity term σ is asymptotically sublinear. Thus, the local behavior of σ is shown to not have an effect on weak intermittence, provided that the initial data u_0 is sufficiently large. A significant drawback of this result is that its proof does not provide any information about how large “sufficiently large” should be. We introduce the following condition.

CONDITION 1.10. $(\bar{R}_0 f)(0) = \infty$.

The following result is the mentioned variant of the Theorem 1.8.

THEOREM 1.11. *Suppose $b \equiv 0$ and Conditions 1.1, 1.7, and 1.10 hold. Suppose, in addition, that $\sigma \geq 0$ pointwise, and $q := \liminf_{|z| \rightarrow \infty} \sigma(z)/|z| > 0$. If $u_0(x) > 0$ and $\mathbb{P}\{u_t(x) > 0\} = 1$ for all $t > 0$ and $x \in \mathbf{R}^d$, then*

$$(1.21) \quad \bar{\gamma}_x(2) > 0 \quad \text{for every } x \in \mathbf{R}^d,$$

provided that $\eta := \inf_{x \in \mathbf{R}^d} u_0(x)$ is sufficiently large.

The preceding results describe our main contributions to the analysis of the stochastic heat equation (1.1) in the case that σ is not a constant and that u_0 is a bounded and measurable function. But we also study the linearization of (1.1); this is the case when σ is identically equal to one. In addition to studying existence-and-uniqueness issues, we use the theory of Gaussian processes to study continuity properties of the solutions. Moreover, we produce a class of interesting examples which we briefly describe next.

Consider the linear stochastic heat equation

$$(1.22) \quad \frac{\partial}{\partial t} u_t(x) = (\Delta u_t)(x) + \dot{F}_t(x),$$

where $u_0 \equiv 0$, $x \in \mathbf{R}^3$, $t > 0$, and the Laplacian acts on the x variable only. Then, we construct families of noises \dot{F} which ensure that (1.22) has a solution $\mathbf{u} := \{u_t(x)\}_{t \geq 0, x \in \mathbf{R}^d}$ that is a square-integrable random field. But that random field is discontinuous densely with probability one. In fact, outside of a single null set [of realizations of the process \mathbf{u}],

$$(1.23) \quad \sup_{(t,x) \in V} u_t(x) = - \inf_{(t,x) \in V} u_t(x) = \infty,$$

for all open balls $V \subset \mathbf{R}_+ \times \mathbf{R}^d$ with rational centers and radii!⁴ We know of only a few examples of SPDEs with well-defined random-field solutions that have unbounded oscillations densely; see Dalang and L ev eque [DL06, DL04b, DL04a], Mytnik and Perkins [MP03], and Foondun, Khoshnevisan, and Nualart [FKN09]. The preceding (1.23) yields a quite-simple example of an otherwise physically-natural stochastic PDE [the operator is the Laplacian in \mathbf{R}^3 and the noise is white in time] which has a very badly-behaved solution.

This paper was influenced greatly by the theoretical physics literature on the “parabolic Anderson model” (see, for example, Kardar, Parisi, and Zhang [KPZ86], Krug and Spohn [KS91, §5], Medina, Hwa, Kardar, and Zhang [MHKZ89], and the book by Zeldovitch, Ruzmaikin, and Sokoloff [ZRS90]), as well as the mathematical physics literature on the very same topic (see, for example, Bertini and Cancrini [BC98, BC95], Bertini, Cancrini, and Jona-Lasinio [BCJL94], Bertini and Giacomin [BG99, BG97], Carmona, Korolov, and Molchanov [CKM01], Carmona and Molchanov [CM95, CM94], Carmona and Viens [CV98], Cranston and Molchanov [CM07b, CM07a], Cranston, Mountford, and Shiga [CMS05, CMS02], Florescu and Viens [FV06], G artner and den Hollander [GdH06], G artner and K onig [GK05], Hofsted, K onig, and M orters [vdHKM06], K onig, Lacoine, and M orters [KLMS09], Lieb and Liniger [LL63], Molchanov [Mol91], and Woyczy nski [Woy98] for a partial listing). Furthermore, there are interesting variations of the parabolic Anderson model that correspond to continuous directed-polymer measures; see Comets and Yoshida [CY05] and Comets, Shiga, and Yoshida [CSY04, CSY03].

In a nutshell, the parabolic Anderson model is equation (1.1) where $\sigma(u)$ is proportional to u . There are many good reasons why that equation has been studied intensively; see for instance the Introduction of Carmona and Molchanov [CM94]. Two such reasons are that the parabolic Anderson model it is exactly solvable in the two cases where $u_0 \equiv \text{constant}$ and $u_0 = \delta_0$; and it is related deeply to the stochastic Burgers equation as well as the KPZ equation of statistical mechanics.

And perhaps not surprisingly, the results of our Theorems 1.2, 1.8, and 1.11 are sharpest for the parabolic Anderson model, particularly when $d = 1$.

⁴It would be wonderful to construct versions of these examples that apply to fully nonlinear problems; but we do not know how to proceed in a fully nonlinear [or even semilinear] setting.

However, an inspection of Theorems 1.2 and 1.8 reveals an inconsistency: Our upper bound on the Liapounov exponent [Theorem 1.2] does not require the drift b to be zero; whereas our lower bound [Theorem 1.8] does.

David Nualart has asked us whether we know how the drift b can affect the weak intermittence of the solution to (1.1). This seems to be a hard question to answer rigorously when the drift b is a general Lipschitz-continuous function. But it is intuitively clear that a sufficiently-strong drift ought to destroy the natural tendency of the solution to be weakly intermittent.

Although we are not aware of general theorems of this type, we are able to give a partial answer to Nualart's question; and the striking nature of that partial answer confirms our initial suspicion that it might be rather difficult to answer D. Nualart's question in good generality.

Here is an instance where we can rigorously prove weak intermittency: Consider the one-dimensional parabolic Anderson model for the relativistic [or massive/dissipative] Laplacian; i.e., the stochastic PDE

$$(1.24) \quad \frac{\partial}{\partial t} u_t(x) = (\Delta u_t)(x) + \frac{\lambda}{2} u_t(x) + \kappa u_t(x) \dot{F}_t(x),$$

where $t > 0$ and $x \in \mathbf{R}$, $\kappa \neq 0$, $\lambda \in \mathbf{R}$, and $u_0 : \mathbf{R} \rightarrow \mathbf{R}$ is a measurable function that is bounded uniformly away from zero and infinity. Let us consider the special case that the correlation function of the noise is of Riesz type; that is,

$$(1.25) \quad f(z) := \|z\|^{-1+b} \quad \text{for all } z \in \mathbf{R},$$

where $b \in (0, 1)$. Then, Example 5.8 on page 71 implies that weak intermittence holds if and only if

$$(1.26) \quad \lambda > -|\kappa|^{4/(1+b)} 8^{-(1-b)/(1+b)} \left[\frac{\Gamma(b/2)\Gamma((b+1)/2)}{\sqrt{\pi}} \right]^{2/(1+b)}.$$

This completes our investigation of the solution to (1.1) when the initial data u_0 is a bounded and measurable function.

We conclude this paper by considering (1.1) in the other physically-interesting cases where u_0 is a finite Borel measure on \mathbf{R}^d . This condition on u_0 is very natural, as in many physical applications u_0 denotes the initial distribution of particles in a particle system in a disordered medium [KPZ86, MHKZ89, KS91, Mol91, CM94, BC95]. From a purely mathematical point of view, this problem is interesting since it leads us to a different notion of a solution, which we call *temperate*. Our proposed

temperate solutions differ from the much better-known notion of “mild solutions” [Wal86, Chapter 3]. They also lead us to an extension of the notion of a stochastic convolution, which defines stochastic integrals at almost every time, rather than pointwise. We believe that the notion of temperate solutions has other uses in describing otherwise hard-to-define SPDEs. In order to describe our results on temperate solutions—the final main contributions of this paper—we would have to develop some machinery. The details can be found in Chapter 6.

A brief outline of the paper follows: In Chapter 2 we review, very briefly, some analytical facts about Lévy processes and their generators, and also construct examples that will be used in subsequent chapters.

Chapter 3 is concerned with positive-definite functions and their connections to potential theory and harmonic analysis. And Theorem 1.5 is shown to be a consequence of these connections. Chapter 3 also contains a probabilistic characterization of the analytic condition (1.10) and Condition 1.1 in terms of continuous additive functionals of the replica process \bar{X} . Also, a family of useful correlation functions is constructed in that chapter; that construction uses the results of Chapter 2 on probabilistic potential theory.

In Chapter 4 we study the linearization of (1.1), and derive necessary and sufficient conditions for the existence and spatial continuity of the solution. We also consider various examples that include (1.22) above.

Chapter 5 contains the proofs of Theorems 1.2 and 1.8. In that chapter we consider also the relativistic version of (1.1), thereby constructing examples that include the mentioned analysis of (1.24).

Finally, we conclude with Chapter 6, where a new notion of “temperate solution” is introduced. And then that notion is used to produce solutions to (1.1) in the case that $b \equiv 0$ and u_0 belongs to a suitable family of finite Borel measures on \mathbf{R}^d .

Let us conclude the present chapter by introducing some notation that will be used throughout the paper. For all integers $k \geq 1$,

$$(1.27) \quad \|x\| := (x_1^2 + \cdots + x_k^2)^{1/2} \quad \text{for every } x \in \mathbf{R}^k.$$

And if $g : \mathbf{R}^k \rightarrow \mathbf{R}$ is a function, then

$$(1.28) \quad \text{Lip}_g := \sup_{\substack{x, y \in \mathbf{R}^k \\ x \neq y}} \frac{|g(x) - g(y)|}{\|x - y\|}.$$

This so-called Lipschitz constant of g is well defined, but might be infinity.

Throughout this paper, “ $\hat{\cdot}$ ” denotes the Fourier transform in the sense of L. Schwartz; our Fourier transform is normalized so that

$$(1.29) \quad \hat{g}(\xi) := \int_{\mathbf{R}^d} e^{ix \cdot \xi} g(x) \, dx \quad \text{for all } \xi \in \mathbf{R}^d \text{ and } g \in L^1(\mathbf{R}^d).$$

Finally, if $g : \mathbf{R}^d \rightarrow \mathbf{R}$ is a function, then we define

$$(1.30) \quad \tilde{g}(x) := g(-x) \quad \text{for all } x \in \mathbf{R}^d.$$

And similarly, if μ is a Borel measure on \mathbf{R}^d , then for all Borel sets $A \subseteq \mathbf{R}^d$,

$$(1.31) \quad \tilde{\mu}(A) := \mu(-A), \quad \text{where } -A := \{-a : a \in A\}.$$

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CHAPTER 2

Lévy Processes

2.1. Preliminaries

We begin this chapter with the definition of a Lévy process which throughout this paper, will be denoted by $X := \{X_t\}_{t \geq 0}$.

DEFINITION 2.1. We say that $X := \{X_t\}_{t \geq 0}$ is a *Lévy process* if:

- (1) $X_{t+s} - X_s$ is independent of the sigma-algebra generated by $\{X_r\}_{r \in [0, s]}$ for every $s, t \geq 0$;
- (2) $X_{t+s} - X_s$ has the same distribution as X_t for every $s, t \geq 0$;
- (3) $t \mapsto X_t$ is continuous in probability; that is, X_s converges to X_t in probability as $s \rightarrow t$; and
- (4) $X_0 = 0$.

By adopting a suitable modification of the paths, we can and will always assume, without loss of generality, that the trajectories of X are cadlag; i.e., $t \mapsto X_t$ is almost surely right-continuous with left limits. Comprehensive treatments can be found in the books by Bertoin [Ber96], Jacob [Jac05], Kyprianou [Kyp06], and Sato [Sat99].

Let m_t denote the distribution of X_t for every $t \geq 0$; that is,

$$(2.1) \quad m_t(A) := \mathbb{P}\{X_t \in A\} \quad \text{for all } t \geq 0 \text{ and Borel sets } A \subseteq \mathbf{R}^d.$$

Let us recall that throughout this paper we are assuming that the process X has transition functions; that is,

$$(2.2) \quad m_t(dx) \ll dx \quad \text{for all } t > 0.$$

According to Theorem 2.2 of Hawkes [Haw79], we can always select a version of these transition functions that has the following *regularity features*:

- (1) $\int_A p_t(z) dz = m_t(A)$ for all $t > 0$ and Borel sets $A \subseteq \mathbf{R}^d$;
- (2) $(0, \infty) \times \mathbf{R}^d \ni (t, x) \mapsto p_t(x) \in \mathbf{R}_+$ is Borel measurable;
- (3) $x \mapsto p_t(x)$ is lower semicontinuous for all $t > 0$;
- (4) $p_{t+s}(x) = (p_t * p_s)(x)$ for all $s, t > 0$ and $x \in \mathbf{R}^d$,

where “ $*$ ” denotes the convolution operator, defined in the sense of L. Schwartz. We work only with such a version of these transition functions. Note that for all $t \geq 0$ and $x \in \mathbf{R}^d$, and for every Borel-measurable function $\phi : \mathbf{R}^d \rightarrow \mathbf{R}_+$,

$$(2.3) \quad \begin{aligned} \mathbb{E}\phi(x + X_t) &= \int_{\mathbf{R}^d} \phi(z)p_t(z - x) dz \\ &= (\phi * \tilde{p}_t)(x), \end{aligned}$$

where we recall $\tilde{p}_t(x) := p_t(-x)$.

Alternatively, one can work with the semigroup $\{P_t\}_{t \geq 0}$ of X , which is defined via

$$(2.4) \quad (P_t\phi)(x) := \mathbb{E}\phi(x + X_t).$$

It is easy to verify that $\{P_t\}_{t \geq 0}$ is a Feller semigroup; i.e.,

$$(2.5) \quad P_t : C_0(\mathbf{R}^d) \rightarrow C_0(\mathbf{R}^d),$$

where $C_0(\mathbf{R}^d)$ denotes the collection of all continuous functions $g : \mathbf{R}^d \rightarrow \mathbf{R}$ that vanish at infinity. In fact, under the present conditions, $\{P_t\}_{t \geq 0}$ is strong Feller in the sense of Girsanov [Gir60]; see Hawkes [Haw79].

Let us emphasize that

$$(2.6) \quad P_t\phi = \phi * \tilde{p}_t,$$

valid for all $t \geq 0$, and for instance for every nonnegative Borel-measurable functions $\phi : \mathbf{R}^d \rightarrow \mathbf{R}$.

Let $\{R_\alpha\}_{\alpha \geq 0}$ denote the resolvent of $\{P_t\}_{t \geq 0}$; i.e.,

$$(2.7) \quad R_\alpha := \int_0^\infty e^{-\alpha s} P_s ds.$$

It follows that if $\phi : \mathbf{R}^d \rightarrow \mathbf{R}_+$ is Borel measurable, then

$$(2.8) \quad \begin{aligned} (R_\alpha\phi)(x) &= \int_0^\infty \phi(z)r_\alpha(x - z) dz \\ &= (\phi * \tilde{r}_\alpha)(x), \end{aligned}$$

where

$$(2.9) \quad r_\alpha(x) := \int_0^\infty e^{-\alpha t} p_t(x) dt \quad \text{for } \alpha \geq 0 \text{ and } x \in \mathbf{R}^d.$$

Each “ α -potential density” $r_\alpha(x)$ is well defined, but could well be infinity at some [in fact, even all, when $\alpha = 0$] $x \in \mathbf{R}^d$. Nevertheless, the regularity properties of the transition functions imply that every r_α is lower

semicontinuous. Furthermore,

$$(2.10) \quad R_\alpha : C_0(\mathbf{R}^d) \rightarrow C_0(\mathbf{R}^d) \quad \text{for every } \alpha > 0.$$

In fact, $R_\alpha(C_0(\mathbf{R}^d))$ is uniformly dense in $C_0(\mathbf{R}^d)$ when $\alpha > 0$; see Blumenthal and Gettoor [BG68, Exercise (9.13), p. 51].

The *characteristic exponent* of the process X is a function $\Psi : \mathbf{R}^d \rightarrow \mathbf{C}$ that is defined uniquely via

$$(2.11) \quad \mathbb{E}e^{i\xi \cdot X_t} = e^{-t\Psi(\xi)} \quad \text{for all } \xi \in \mathbf{R}^d \text{ and } t \geq 0.$$

The Lévy–Khintchine formula [Ber96, Theorem 1.2, p. 13], and a theorem of Schoenberg [Sch38a, Sch38b] together imply that the family of all Lévy processes is in one-to-one correspondence with the family of all “negative-definite functions.”

2.2. The Generator

We will be working with the L^2 -theory of generators, as developed, for instance, in the book by Fukushima, Ōshima, and Takeda [FŌT94] for more general Markov processes. We outline the details in the present special case; matters are greatly simplified and in some cases generalized because of harmonic analysis.

Define

$$(2.12) \quad \text{Dom}[\mathcal{L}] = \left\{ \phi \in L^2(\mathbf{R}^d) : \Psi\hat{\phi} \in L^2(\mathbf{R}^d) \right\}.$$

Plancherel’s theorem guarantees that $\phi \in \text{Dom}[\mathcal{L}]$ if and only if $\phi : \mathbf{R}^d \rightarrow \mathbf{R}$ is Borel-measurable, locally integrable, and

$$(2.13) \quad \int_{\mathbf{R}^d} (1 + |\Psi(\xi)|^2) |\hat{\phi}(\xi)|^2 d\xi < \infty.$$

It is well known that the following holds:

$$(2.14) \quad \limsup_{\|\xi\| \rightarrow \infty} \frac{|\Psi(\xi)|}{\|\xi\|^2} < \infty.$$

This can be derived directly from the Lévy–Khintchine formula; see the book by Bochner [Boc55, (3.4.14), p. 67]. In the “Notes and References” section for Chapter 3 [Boc55, p. 169] of his influential book, Bochner ascribes (2.14) in part to Kolmogorov and Lévy.

Recall that

$$(2.15) \quad W^{1,2}(\mathbf{R}^d) := \left\{ \phi \in L^2(\mathbf{R}^d) : \nabla\phi \in L^2(\mathbf{R}^d) \right\}.$$

Because of Plancherel's theorem,

$$(2.16) \quad \|\phi\|_{L^2(\mathbf{R}^d)}^2 + \|\nabla\phi\|_{L^2(\mathbf{R}^d)}^2 = \int_{\mathbf{R}^d} (1 + \|\xi\|^2) |\hat{\phi}(\xi)|^2 d\xi.$$

Therefore, we can see from (2.14) that

$$(2.17) \quad W^{1,2}(\mathbf{R}^d) \subseteq \text{Dom}[\mathcal{L}] \subseteq L^2(\mathbf{R}^d).$$

Of course, \mathcal{S} is dense in $W^{1,2}(\mathbf{R}^d)$, when the latter is endowed with the usual Sobolev norm,

$$(2.18) \quad \|\phi\|_{W^{1,2}(\mathbf{R}^d)}^2 := \|\phi\|_{L^2(\mathbf{R}^d)}^2 + \|\nabla\phi\|_{L^2(\mathbf{R}^d)}^2.$$

According to Plancherel's theorem,

$$(2.19) \quad \begin{aligned} (\psi, P_t\phi)_{L^2(\mathbf{R}^d)} &= (\psi * m_t, \phi)_{L^2(\mathbf{R}^d)} \\ &= \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \overline{\hat{\phi}(\xi)} \hat{\psi}(\xi) e^{-t\Psi(\xi)} d\xi, \end{aligned}$$

for all $t \geq 0$ and $\psi, \phi \in L^2(\mathbf{R}^d)$. Moreover,

$$(2.20) \quad \left(\psi, \frac{P_t\phi - \phi}{t} \right)_{L^2(\mathbf{R}^d)} = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \overline{\hat{\phi}(\xi)} \hat{\psi}(\xi) \left[\frac{e^{-t\Psi(\xi)} - 1}{t} \right] d\xi.$$

It follows easily from this and (2.12) that

$$(2.21) \quad \mathcal{L}\phi := \lim_{t \downarrow 0} \frac{P_t\phi - \phi}{t}$$

exists in $L^2(\mathbf{R}^d)$ if and only if $\phi \in \text{Dom}[\mathcal{L}]$. Indeed, the sufficiency follows from the elementary bound

$$(2.22) \quad \left| e^{-t\Psi(\xi)} - 1 \right| \leq t|\Psi(\xi)|,$$

and the Cauchy–Schwarz inequality. And the necessity follows from Fatou's lemma, upon setting $\psi := (P_t\phi - \phi)/t$.

Thus, we have the so-called *generator* [L^2 -generator, in fact] \mathcal{L} , defined on its *domain* $\text{Dom}[\mathcal{L}]$. In addition, \mathcal{L} can be thought of as a convolution [or pseudo-differential] operator with multiplier [or symbol] $\hat{\mathcal{L}} = -\Psi$. More precisely,

$$(2.23) \quad \widehat{\mathcal{L}\phi}(\xi) = -\Psi(\xi)\hat{\phi}(\xi) \quad \text{for all } \phi \in \text{Dom}[\mathcal{L}] \text{ and } \xi \in \mathbf{R}^d.$$

Let us note that for all $t \geq 0$, $\xi \in \mathbf{R}^d$, and $\phi \in L^1(\mathbf{R}^d)$,

$$(2.24) \quad \left| \widehat{P_t\phi}(\xi) \right|^2 = e^{-2t\text{Re}\Psi(\xi)} \cdot |\hat{\phi}(\xi)|^2.$$

Therefore, the well-known nonnegativity of $\operatorname{Re}\Psi(\xi)$ —which we prove at the beginning of the following section—implies the following.

LEMMA 2.2. *P_t is a contraction on $W^{1,2}(\mathbf{R}^d)$ for all $t \geq 0$. Hence, αR_α is also a contraction on $W^{1,2}(\mathbf{R}^d)$ for all $\alpha > 0$.*

2.3. The Replica Semigroup and Associated Sobolev Spaces

Let X^* denote an independent copy of the Lévy process $-X$ and, following Lévy [Lév37], define

$$(2.25) \quad \bar{X}_t := X_t + X_t^* \quad \text{for all } t \geq 0.$$

It is easy to see that $X^* := \{X_t^*\}_{t \geq 0}$ is the dual process to X , and $\bar{X} := \{\bar{X}_t\}_{t \geq 0}$ is a symmetric Lévy process on \mathbf{R}^d . And if we denote the distribution of \bar{X}_t by \bar{m}_t , then

$$(2.26) \quad \bar{m}_t(A) = (m_t * \tilde{m}_t)(A) \quad \text{for all Borel sets } A \subseteq \mathbf{R}^d,$$

where $\tilde{m}_t(A) := m_t(-A)$. Note that the Fourier transform of \bar{m}_t is

$$(2.27) \quad \begin{aligned} \widehat{\bar{m}_t}(\xi) &= |\widehat{m}_t(\xi)|^2 \\ &= e^{-2t \operatorname{Re}\Psi(\xi)}. \end{aligned}$$

Among other things, this implies the classical fact that

$$(2.28) \quad \operatorname{Re}\Psi(\xi) \geq 0 \quad \text{for all } \xi \in \mathbf{R}^d.$$

The absolute-continuity condition (2.2) implies that every \bar{m}_t is absolutely continuous with respect to the Lebesgue measure on \mathbf{R}^d [$t > 0$]. We denote the resulting transition density by \bar{p}_t . Every \bar{p}_t is a symmetric function on \mathbf{R}^d [$t > 0$].

We can always choose a version of \bar{p} that has good regularity features [of the type mentioned earlier for p]. In fact, the following version works:

$$(2.29) \quad \begin{aligned} \bar{p}_t(x) &:= (p_t * \tilde{p}_t)(x) \\ &= \int_{\mathbf{R}^d} p_t(x+z)p_t(z) \, dz \quad \text{for } x \in \mathbf{R}^d \text{ and } t > 0. \end{aligned}$$

Equivalently, if $\bar{P} := \{\bar{P}_t\}_{t \geq 0}$ denotes the semigroup of \bar{X} , then

$$(2.30) \quad \bar{P}_t = P_t P_t^* \quad \text{for all } t \geq 0,$$

where P_t^* denotes the adjoint of P_t in $L^2(\mathbf{R}^d)$. Every \bar{P}_t is a self-adjoint contraction on $L^2(\mathbf{R}^d)$.

Motivated by the work of Kardar [Kar87], we refer to \bar{X} and \bar{P} respectively as the *replica process* and the *replica semigroup*. The corresponding generator is denoted by $\bar{\mathcal{L}}$ and its domain by $\text{Dom}[\bar{\mathcal{L}}]$.

For all $\alpha \geq 0$, we can define the *replica α -potential density* \bar{r}_α as

$$(2.31) \quad \bar{r}_\alpha(x) := \int_0^\infty e^{-\alpha s} \bar{p}_s(x) ds \quad \text{for all } x \in \mathbf{R}^d.$$

Clearly, $\bar{r}_\alpha(x)$ is well defined; but $\bar{r}_\alpha(x)$ can be infinite for some [and even all, in the case that $\alpha = 0$] $x \in \mathbf{R}^d$. The resolvent $\bar{R} := \{\bar{R}_\alpha\}_{\alpha>0}$ of the semigroup \bar{P} can also be defined as follows

$$(2.32) \quad \begin{aligned} (\bar{R}_\alpha \phi)(x) &:= \int_0^\infty e^{-\alpha s} (\bar{P}_s \phi)(x) ds \\ &= \int_{\mathbf{R}^d} \phi(z) \bar{r}_\alpha(z-x) dz, \end{aligned}$$

for all $\alpha > 0$ and $x \in \mathbf{R}^d$. Since \bar{r}_α is a symmetric function on \mathbf{R}^d , it follows that $\bar{R}_\alpha \phi = \phi * \bar{r}_\alpha$.

The preceding quantity which is called the *α -potential* of ϕ makes sense, for example, if $\phi : \mathbf{R}^d \rightarrow \mathbf{R}_+$ is Borel measurable, or when $\phi \in L^p(\mathbf{R}^d)$ for some $p \in [1, \infty]$ because every \bar{P}_s is a contraction on $L^p(\mathbf{R}^d)$.

2.4. On the Heat Equation and Transition Functions

We begin by recalling some generally-known facts about the fundamental [weak] solution to the *heat* [or Kolmogorov] *equation* for \mathcal{L} : We seek to find a function H such that for all $t > 0$ and $x \in \mathbf{R}^d$,

$$(2.33) \quad \begin{cases} \frac{\partial}{\partial t} H_t(x) = (\mathcal{L}H_t)(x), \\ H_0 = \delta_z, \end{cases}$$

where $z \in \mathbf{R}^d$ is fixed. By rewriting the above in terms of the [spatial] Fourier transform \hat{H}_t and using the fact that the [weak] Fourier transform of $\mathcal{L}H_t$ is $-\Psi \cdot \hat{H}_t$ [see (2.23)], we obtain the ordinary differential equation,

$$(2.34) \quad \begin{cases} \frac{\partial}{\partial t} \hat{H}_t(\xi) = -\Psi(\xi) \hat{H}_t(\xi), \\ \hat{H}_0(\xi) = e^{i\xi \cdot z}. \end{cases}$$

The unique solution to this ODE is

$$(2.35) \quad \hat{H}_t(\xi) = e^{i\xi \cdot z - t\Psi(\xi)}.$$

Direct inspection of the Fourier transform reveals that $H_t(x) = p_t(z - x)$. Thus, we find that the fundamental solution to (2.33) is the measurable function $(0, \infty) \times \mathbf{R}^d \times \mathbf{R}^d \ni (t; x, y) \mapsto p_t(y - x)$. In particular, we might observe that in order to have a function solution to (2.33), it is necessary as well as sufficient that the underlying Lévy process X has transition densities.

We are thus led to the natural question: “What are the necessary and sufficient conditions on the characteristic exponent Ψ that ensure the existence of transition densities of the corresponding Lévy processes”? Unfortunately, there is no satisfactory known answer to this question at this time, though several attempts have been made in the first half of the twentieth century; see, for example Blum and Rosenblatt [BR59], Fisz and Varadarajan [FV63], Hartman and Wintner [HW42], and Tucker [Tuc65, Tuc64, Tuc62].

More recently, Bass and Cranston [BC86] applied Malliavin calculus to a family of stochastic differential equations driven by jump noises. Their result can be used to supply good sufficient conditions that ensure the existence of smooth transition functions. And in [NS06], Nourdin and Simon have proved, among other things, that if a Lévy process has transition functions, then so does the same process plus a drift.

We will use the following unpublished result of Hawkes. It typically provides a good-enough sufficient condition for the existence of transition functions. We include a proof in order to document this interesting fact.

PROPOSITION 2.3 (Hawkes [Haw84]). *The following conditions are equivalent:*

- (1) *Condition (2.2) holds and $p_t \in L^2(\mathbf{R}^d)$ for all $t > 0$;*
- (2) *Condition (2.2) holds and $p_t \in L^\infty(\mathbf{R}^d)$ for all $t > 0$;*
- (3) *Condition (2.2) holds and $p_t \in L^2(\mathbf{R}^d)$ for almost every $t > 0$;*
- (4) *Condition (2.2) holds and $p_t \in L^\infty(\mathbf{R}^d)$ for almost every $t > 0$;*
- (5) *$\exp(-\operatorname{Re}\Psi) \in L^t(\mathbf{R}^d)$ for all $t > 0$.*
- (6) *$\exp(-\operatorname{Re}\Psi) \in L^t(\mathbf{R}^d)$ for almost every $t > 0$.*

Moreover, any one of these conditions implies that: (i) $(t, x) \mapsto p_t(x)$ has a continuous version which is uniformly continuous for all $(t, x) \in [\eta, \infty) \times \mathbf{R}^d$ for every $\eta > 0$; and (ii) p_t vanishes at infinity for all $t > 0$.

PROOF (HAWKES [Haw84]). Recall that $\int_{\mathbf{R}^d} p_t(x) dx = 1$ and $p_t = p_{t/2} * p_{t/2}$. Therefore, two applications of Young's inequality yield

$$(2.36) \quad \|p_t\|_{L^\infty(\mathbf{R}^d)} \leq \|p_{t/2}\|_{L^2(\mathbf{R}^d)}^2 \leq \|p_{t/2}\|_{L^\infty(\mathbf{R}^d)} \quad \text{for all } t > 0.$$

Consequently, (1) \Leftrightarrow (2) and (3) \Leftrightarrow (4).

Next let us suppose that (6) holds. Because

$$(2.37) \quad \begin{aligned} |\hat{p}_t(\xi)| &= \left| e^{-t\Psi(\xi)} \right| \\ &\leq e^{-t\operatorname{Re}\Psi(\xi)}, \end{aligned}$$

Plancherel's theorem ensures that

$$(2.38) \quad \begin{aligned} \|p_t\|_{L^2(\mathbf{R}^d)}^2 &= \frac{1}{(2\pi)^d} \|\hat{p}_t\|_{L^2(\mathbf{R}^d)}^2 \\ &\leq \|e^{-2t\operatorname{Re}\Psi}\|_{L^1(\mathbf{R}^d)}. \end{aligned}$$

Since $\operatorname{Re}\Psi \geq 0$, it follows from (6) that $p_t \in L^2(\mathbf{R}^d)$ for every $t > 0$; i.e., (6) \Rightarrow (1). Moreover, we have—in this case—the following inversion formula: For almost all $x \in \mathbf{R}^d$ and every $t > 0$,

$$(2.39) \quad p_t(x) = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} e^{-i\xi \cdot x - t\Psi(\xi)} d\xi.$$

It remains to prove that (1) and equivalently (2) together imply (5). Recall that $\tilde{p}_t(x) := p_t(-x)$ and observe that

$$(2.40) \quad \widehat{p_{t/4} * \tilde{p}_{t/4}} = e^{-(t/2)\operatorname{Re}\Psi}.$$

Therefore, by Plancherel's theorem,

$$(2.41) \quad \begin{aligned} \|e^{-\operatorname{Re}\Psi}\|_{L^t(\mathbf{R}^d)} &= \left\| e^{-(t/2)\operatorname{Re}\Psi} \right\|_{L^2(\mathbf{R}^d)} \\ &= (2\pi)^d \|p_{t/4} * \tilde{p}_{t/4}\|_{L^2(\mathbf{R}^d)}^2 \\ &\leq (2\pi)^d \|p_{t/4} * \tilde{p}_{t/4}\|_{L^\infty(\mathbf{R}^d)}. \end{aligned}$$

Consequently, Young's inequality, implies that

$$(2.42) \quad \|e^{-\operatorname{Re}\Psi}\|_{L^t(\mathbf{R}^d)} \leq (2\pi)^d \|p_{t/4}\|_{L^2(\mathbf{R}^d)}^2,$$

which has the desired effect.

Finally, if (5) holds then the inversion theorem applies and tells us that we can always choose a version of p that satisfies the properties of the final paragraph in the statement of theorem. \square

There is also the following 1970 theorem of J. Zabczyk, which characterizes (2.2) in special though important cases.

PROPOSITION 2.4 (Zabczyk [Zab70, Example (4.6)]). *If $d \geq 2$ and Ψ is a radial function, then (2.2) holds if and only if*

$$(2.43) \quad \lim_{\|\xi\| \rightarrow \infty} \Psi(\xi) = \infty.$$

We now put this beautiful result in context. By the Riemann-Lebesgue lemma, if the process X has transition functions then equality (2.43) holds. Proposition 2.4 states that the converse also holds, provided that $d \geq 2$ and Ψ is radial. In other words, the Riemann–Lebesgue lemma is a necessary and sufficient condition for the existence of transition functions whenever $d \geq 2$ and Ψ is radial. As was mentioned in Zabczyk [Zab70], the preceding is not in general true when $d = 1$. This can be seen by considering Ψ to correspond to two independent one-dimensional Poisson processes.

2.5. On a Family of Isotropic Lévy Processes

The main result of this section will be needed to construct a counterexample in Chapter 4. It is possible that it is known but we were not able to find an explicit reference. So we provide a complete proof. We begin by recalling a few definitions used to study Lévy processes.

We say that a Lévy process $X := \{X_t\}_{t \geq 0}$ is *isotropic* if its characteristic exponent Ψ is a radial function [and hence also real-valued and nonnegative]. Such processes are also known as *radial processes*; see Millar [Mil73].

A [standard] *subordinator* $\tau := \{\tau_t\}_{t \geq 0}$ is a one-dimensional Lévy process that is nondecreasing and $\tau_0 := 0$. According to the Lévy–Khintchine formula [Ber99, Theorem 1.2, p. 13], every subordinator τ is determined by the formula

$$(2.44) \quad \mathbb{E}e^{-\lambda\tau_t} = e^{-t\Phi(\lambda)},$$

where $t, \lambda \geq 0$, and

$$(2.45) \quad \Phi(\lambda) = \int_0^\infty (1 - e^{-\lambda z}) \Pi(dz),$$

for a Borel measure Π on $(0, \infty)$ that satisfies

$$(2.46) \quad \int_0^\infty (1 \wedge x) \Pi(dx) < \infty.$$

The function Φ is the so-called *Laplace exponent* of the subordinator τ .

We have the following lemma.

LEMMA 2.5. *Choose and fix two numbers $p \in (0, 1)$ and $q \in \mathbf{R}$. Then, there exists a subordinator τ on \mathbf{R}_+ whose Laplace exponent satisfies*

$$(2.47) \quad 0 < \inf_{\lambda > e} \frac{\Phi(\lambda)}{\lambda^p (\log \lambda)^{q/2}} \leq \sup_{\lambda > e} \frac{\Phi(\lambda)}{\lambda^p (\log \lambda)^{q/2}} < \infty.$$

PROOF. Define a measure Π via

$$(2.48) \quad \frac{\Pi(dx)}{dx} := \begin{cases} x^{-1-p} (\log(1/x))^{q/2} & \text{if } 0 < x < \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $p \in (0, 1)$, it follows that (2.46) holds, whence Π is a Lévy measure.

We can also apply the definition (2.45) of the Laplace exponent and write $\Phi(\lambda) = \lambda^p Q(\lambda)$, where

$$(2.49) \quad Q(\lambda) = \int_0^{\lambda/2} \frac{1 - e^{-x}}{x^{1+p}} (\log(\lambda/x))^{q/2} dx \quad \text{for } \lambda > 0.$$

In order to complete the proof, we will verify that $Q(\lambda) \asymp (\log \lambda)^{q/2}$ for $\lambda > e$.¹ We do so in the special case that $q \geq 0$; similar arguments can be used to estimate $Q(\lambda)$ in the case that $q < 0$.

Whenever $\lambda > e$, we can write

$$(2.50) \quad Q(\lambda) := I_1 + I_2,$$

where

$$(2.51) \quad \begin{aligned} I_1 &:= \int_1^{\lambda/2} \frac{1 - e^{-x}}{x^{1+p}} (\log(\lambda/x))^{q/2} dx, \\ I_2 &:= \int_0^1 \frac{1 - e^{-x}}{x^{1+p}} (\log(\lambda/x))^{q/2} dx. \end{aligned}$$

Evidently,

$$(2.52) \quad \begin{aligned} I_1 &\leq (\log \lambda)^{q/2} \cdot \int_1^{\infty} \frac{dx}{x^{1+p}} \\ &= \frac{1}{p} (\log \lambda)^{q/2}. \end{aligned}$$

Since $I_1 \geq 0$, it remains to prove that

$$(2.53) \quad I_2 \asymp (\log \lambda)^{q/2} \quad \text{for } \lambda > 1.$$

¹As usual, $h(x) \asymp g(x)$ over a certain range of x 's is short-hand for the statement that, uniformly over that range of x 's, $h(x)/g(x)$ is bounded above and below by positive and finite constants.

We establish this by deriving first an upper, and then a lower, bound for I_2 . Because $1 - \exp(-y) \leq y$ for $y \geq 0$, and since $\sup_{z \in (0,1)} z^\epsilon \log(1/z) < \infty$ for all $\epsilon \in (0, 1)$, it follows that

$$(2.54) \quad \begin{aligned} I_2 &\leq (\log \lambda)^{q/2} \cdot \int_0^1 \left(1 + \frac{\log(1/x)}{\log \lambda}\right)^{q/2} \frac{dx}{x^p} \\ &\leq \text{const} \cdot (\log \lambda)^{q/2}. \end{aligned}$$

And a similar lower bound is obtained via the bounds: (i) $1 - \exp(-x) \geq x/2$; and (ii) $\log(\lambda/x) \geq \log \lambda$; both valid for all $x \in (0, 1)$. \square

The following is the main result of this section. It gives a special construction of an isotropic Lévy process $X := \{X_t\}_{t \geq 0}$ whose characteristic exponent is regularly varying in a special manner.

THEOREM 2.6. *Choose and fix $r \in (0, 2)$ and $q \in \mathbf{R}$. Then, there exists an isotropic Lévy process $X := \{X_t\}_{t \geq 0}$ such that*

$$(2.55) \quad 0 < \inf_{\substack{\xi \in \mathbf{R}^d: \\ \|\xi\| > e}} \frac{\Psi(\xi)}{\|\xi\|^r (\log \|\xi\|)^q} \leq \sup_{\substack{\xi \in \mathbf{R}^d: \\ \|\xi\| > e}} \frac{\Psi(\xi)}{\|\xi\|^r (\log \|\xi\|)^q} < \infty.$$

PROOF. Let $B := \{B(t)\}_{t \geq 0}$ denote a d -dimensional Brownian motion, independent from the subordinator $\tau := \{\tau_t\}_{t \geq 0}$ of Lemma 2.5. Define

$$(2.56) \quad X_t := B(\tau_t) \quad \text{for all } t \geq 0.$$

It is well known—as well as easy to check—that the process $X := \{X_t\}_{t \geq 0}$ is a Lévy process with characteristic exponent

$$(2.57) \quad \mathbb{E} \left[e^{i\xi \cdot X_t} \right] = \exp \left(-t \Phi \left(\frac{\|\xi\|^2}{2} \right) \right) \quad \text{for all } t \geq 0 \text{ and } \xi \in \mathbf{R}^d.$$

That is, $\Psi(\xi) = \Phi(\|\xi\|^2/2)$; Lemma 2.5 [applied with $p := r/2$] completes the remainder of the proof. \square

CHAPTER 3

Positive-Definite Functions, Fourier Analysis, and Probabilistic Potential Theory

A large part of this paper relies heavily on our ensuing analysis of positive-definite functions and their many connections to harmonic analysis. In this chapter we develop the requisite theory and prove Theorem 1.5. We also give intrinsically-probabilistic interpretations to the two central potential-theoretic hypotheses of this paper; namely Conditions 1.1 and 1.10. Even though, most of the results derived in this chapter will be used later in this paper, they might be of independent interest.

3.1. Fourier Analysis

Let us begin by recalling some basic facts from harmonic analysis.

First, let us recall our definition of Fourier transform.

$$(3.1) \quad \hat{g}(\xi) := \int_{\mathbf{R}^d} e^{ix \cdot \xi} g(x) dx \quad \text{for all } \xi \in \mathbf{R}^d \text{ and } g \in L^1(\mathbf{R}^d).$$

This particular normalization is standard in probability theory and leads to the following form of the Parseval identity, which plays a big role in the ensuing theory:

$$(3.2) \quad \int_{\mathbf{R}^d} g(x) h(x) dx = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \hat{g}(\xi) \overline{\hat{h}(\xi)} d\xi \quad \text{for all } g, h \in L^2(\mathbf{R}^d).$$

The preceding is another way to say that the Fourier transform is an “isometry” from $L^2(\mathbf{R}^d)$ onto itself.¹

Recall that a function $g : \mathbf{R}^d \rightarrow \mathbf{R}$ is *tempered* if it is Borel-measurable and there exists $k \geq 0$ such that

$$(3.3) \quad \sup_{x \in \mathbf{R}^d} \frac{|g(x)|}{(1 + |x|)^k} < \infty.$$

¹Strictly speaking, this is not true, as is evidenced by the multiplicative factor of $(2\pi)^{-d/2}$ in the right-hand side of equation (3.2). And that is why the word isometry appears in quotations: In order for the Fourier transform to be a proper isometry from $L^2(\mathbf{R}^d)$ onto itself, we need to use a different normalization than the one used here. We have not done that as it would be nonstandard for probability theory.

Also, recall that g has *at most polynomial growth* [*at least polynomial growth*, resp.] if the preceding holds for some $k \geq 0$ [$k < 0$, resp.]. Finally, we say that $g \in \mathcal{S}$ if g and all of its derivatives have at least polynomial growth. The collection \mathcal{S} is the usual space of *rapidly-decreasing test functions* on \mathbf{R}^d .

The following is an elementary, but important, variant of the Parseval identity.

LEMMA 3.1. *Parseval's identity (3.2) is valid when $g \in \mathcal{S}$ and $h : \mathbf{R}^d \rightarrow \mathbf{R}$ is continuous and tempered.*

We leave the proof to the interested reader.

3.2. Positive-Definite Functions

Recall that a function $g : \mathbf{R}^d \rightarrow \mathbf{R}_+$ is *positive definite* if g is tempered and $(\phi, g * \phi)_{L^2(\mathbf{R}^d)} \geq 0$ for all rapidly-decreasing test functions ϕ . That is,

$$(3.4) \quad \int_{\mathbf{R}^d \times \mathbf{R}^d} \phi(x)\phi(y)g(x-y) dx dy \geq 0 \quad \text{for all } \phi \in \mathcal{S}.$$

The following central result of L. Schwartz characterizes positive-definite functions.

PROPOSITION 3.2 (Schwartz [GV77, Theorem 3, p. 157]). *If $g : \mathbf{R}^d \rightarrow \bar{\mathbf{R}}$ is positive definite, then there exists a tempered measure Γ on \mathbf{R}^d such that $g = \hat{\Gamma}$.*

The preceding has an elementary converse as well: Recall that a Borel measure Γ on \mathbf{R}^d is of *positive type* if $\hat{\Gamma} \geq 0$ in the sense of distributions; that is, $\hat{\Gamma}(\phi) \geq 0$ for all nonnegative $\phi \in \mathcal{S}$. Then it is straightforward to check that if Γ is a tempered measure of positive type on \mathbf{R}^d , then $\hat{\Gamma}$ is positive definite.

Schwartz's theorem is a generalization of the following theorem of Herglotz [$d = 1$] and Bochner [$d \geq 2$]:

PROPOSITION 3.3 (Herglotz [Her11], Bochner [Boc33]). *If $g : \mathbf{R}^d \rightarrow \mathbf{R}$ is continuous and positive definite, then there exists a finite Borel measure Γ on \mathbf{R}^d such that $g = \hat{\Gamma}$.*

Recall also that a continuous function $g : \mathbf{R}^d \rightarrow \mathbf{R}$ is positive definite if and only if it is positive definite in the sense of Herglotz, Bochner, Pólya,

etc.; that is if

$$(3.5) \quad \sum_{i,j=1}^N a_i \bar{a}_j g(x_i - x_j) \geq 0 \text{ for all } a_1, \dots, a_N \in \mathbf{C} \text{ and } x_1, \dots, x_N \in \mathbf{R}^d.$$

The proof uses elementary integration theory, and we merely recall the [easy] steps: Define the finite complex measure

$$(3.6) \quad \mu := \sum_{i=1}^N a_i \delta_{x_i},$$

and choose a sequence $\phi_1, \phi_2, \dots : \mathbf{R}^d \rightarrow \mathbf{R}$ of probability density functions, each in \mathcal{S} , such that $\hat{\phi}_n \rightarrow 1$ pointwise as $n \rightarrow \infty$. Then, one can make precise the following approximation: As $n \rightarrow \infty$,

$$(3.7) \quad \begin{aligned} \sum_{i,j=1}^N a_i \bar{a}_j g(x_i - x_j) &= \int (g * \mu) d\mu \\ &\approx \int (g * \phi_n * \mu) d(\phi_n * \mu). \end{aligned}$$

And the final quantity is nonnegative because g is positive definite and $\text{Re}(\phi_n * \mu), \text{Im}(\phi_n * \mu) \in \mathcal{S}$ for all $n \geq 1$.

3.3. A Preliminary Maximum Principle

Now that we have recalled the basic definitions and properties of positive-definite functions, we can begin our proof of our maximum principle [Theorem 1.5]. But first let us prove the following technical result.

LEMMA 3.4. *If $\phi \in \mathcal{S}$, then there exists a version of $f * \phi$ that is in $C_0(\mathbf{R}^d)$. Consequently, $\bar{R}_\beta(f * \phi) \in C_0(\mathbf{R}^d)$ for every $\beta > 0$.*

PROOF. Because \hat{f} is tempered, the following defines a uniformly continuous function on \mathbf{R}^d :

$$(3.8) \quad h(x) = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} e^{-ix \cdot \xi} \hat{f}(\xi) \hat{\phi}(\xi) d\xi \quad \text{for all } x \in \mathbf{R}^d.$$

In fact, $h \in C_0(\mathbf{R}^d)$ because of the Riemann–Lebesgue lemma. Furthermore, if $\psi \in \mathcal{S}$, then

$$(3.9) \quad \begin{aligned} \int_{\mathbf{R}^d} \psi(x) h(x) dx &= \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \overline{\hat{\psi}(\xi)} \hat{f}(\xi) \hat{\phi}(\xi) d\xi \\ &= \int_{\mathbf{R}^d} \psi(x) (f * \phi)(x) dx. \end{aligned}$$

The first line is justified by the Fubini theorem, and the second by the Parseval identity. It follows from density and the Lebesgue differentiation theorem that $h = f * \phi$ almost everywhere. This proves the first assertion of the lemma. In addition,

$$\begin{aligned}
 (\bar{R}_\beta h)(x) &= \int_{\mathbf{R}^d} \bar{r}_\beta(y-x)h(y) \, dy \\
 (3.10) \qquad &= \int_{\mathbf{R}^d} \bar{r}_\beta(y-x)(f * \phi)(y) \, dy \\
 &= (\bar{R}_\beta(f * \phi))(x).
 \end{aligned}$$

Since $h \in C_0(\mathbf{R}^d)$, it follows from (2.10) that $\bar{R}_\beta(f * \phi) = \bar{R}_\beta h \in C_0(\mathbf{R}^d)$. \square

The following contains a portion of the said maximum principle of Theorem 1.5. It also provides some of the requisite technical estimates that are needed for the remainder of the proof of Theorem 1.5.

PROPOSITION 3.5. *For all $\beta \geq 0$,*

$$(3.11) \quad \Upsilon(\beta) = \sup_{x \in \mathbf{R}^d} (\bar{R}_\beta f)(x) = \operatorname{ess\,sup}_{x \in \mathbf{R}^d} (\bar{R}_\beta f)(x) = \limsup_{x \rightarrow 0} (\bar{R}_\beta f)(x),$$

where $\Upsilon(0) := \lim_{\beta \downarrow 0} \Upsilon(\beta)$.

PROOF. First, we prove the proposition in the case that $\beta > 0$.

In accord with Lemma 3.4, if $\phi \in \mathcal{S}$, then $\bar{R}_\beta(f * \phi)$ is continuous. Therefore, the Plancherel theorem applies pointwise: For all $x \in \mathbf{R}^d$,

$$(3.12) \quad (\bar{R}_\beta(f * \phi))(x) = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \frac{\hat{f}(\xi)\hat{\phi}(\xi)e^{-ix \cdot \xi}}{\beta + 2\operatorname{Re}\Psi(\xi)} \, d\xi.$$

[Without the asserted continuity, we could only deduce this for almost every $x \in \mathbf{R}^d$.] In particular, for all probability densities $\phi \in \mathcal{S}$,

$$(3.13) \quad \sup_{x \in \mathbf{R}^d} (\bar{R}_\beta(f * \phi))(x) \leq \Upsilon(\beta).$$

[Υ was defined in (1.15).] If $\{\phi_n\}_{n=1}^\infty$ is an approximate identity consisting solely of probability densities in \mathcal{S} , then

$$(3.14) \quad \bar{R}_\beta f \leq \liminf_{n \rightarrow \infty} \bar{R}_\beta(f * \phi_n) \quad [\text{pointwise}],$$

by Fatou's lemma. Consequently, (3.13) implies that

$$(3.15) \quad \sup_{x \in \mathbf{R}^d} (\bar{R}_\beta f)(x) \leq \Upsilon(\beta).$$

In order to prove the reverse bound, define the Gaussian mollifiers $\{\phi_n\}_{n=1}^\infty$:

$$(3.16) \quad \phi_n(z) := \left(\frac{n}{2\pi}\right)^{d/2} \exp\left(-\frac{\|z\|^2 n}{2}\right).$$

And observe that

$$(3.17) \quad \begin{aligned} (\bar{R}_\beta(f * \phi_n))(0) &= \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \frac{\hat{f}(\xi) e^{-\|\xi\|^2/(2n)}}{\beta + 2\operatorname{Re}\Psi(\xi)} d\xi \\ &= (1 + o(1))\Upsilon(\beta) \quad \text{as } n \rightarrow \infty, \end{aligned}$$

thanks to the monotone convergence theorem. On the other hand,

$$(3.18) \quad \begin{aligned} (\bar{R}_\beta(f * \phi_n))(0) &= \int_{\mathbf{R}^d} \bar{r}_\beta(y) (f * \phi_n)(y) dy \\ &= \int_{\mathbf{R}^d} (\bar{R}_\beta f)(y) \phi_n(y) dy \\ &\leq \operatorname{ess\,sup}_{x \in \mathbf{R}^d} (\bar{R}_\beta f)(x), \end{aligned}$$

since $\int_{\mathbf{R}^d} \phi_n(y) dy = 1$. This and (3.13) together prove that $\Upsilon(\beta)$ is the maximum of the β -potential of f ; and the maximum β -potential is finite if and only if $\Upsilon(\beta)$ is. We can choose the ϕ_n 's so that in addition to the preceding regularity criteria, every ϕ_n is supported in the ball of radius $1/n$ about the origin. In that way we obtain

$$(3.19) \quad \begin{aligned} (\bar{R}_\beta(f * \phi_n))(0) &= ((\bar{R}_\beta f) * \phi_n)(0) \\ &\leq \sup_{\|x\| < 1/n} (\bar{R}_\beta f)(x). \end{aligned}$$

And this proves (3.11) for every $\beta \in (0, \infty)$.

The case that $\beta := 0$ has to be handled separately because \bar{R}_0 is not a finite measure. Therefore, we next derive (3.11) in the case that $\beta = 0$.

First of all, $\beta \mapsto \Upsilon(\beta)$ is nonincreasing. Therefore, $\Upsilon(0) := \lim_{\beta \downarrow 0} \Upsilon(\beta)$ exists as a nondecreasing limit. Because $\bar{R}_\beta f \leq \bar{R}_0 f$ [pointwise] for all $\beta > 0$, we can deduce that

$$(3.20) \quad \Upsilon(0) \leq \operatorname{ess\,sup}_{x \in \mathbf{R}^d} (\bar{R}_0 f)(x),$$

and

$$(3.21) \quad \Upsilon(0) \leq \limsup_{x \rightarrow 0} (\bar{R}_0 f)(x).$$

For the reverse bound, recall (3.13), and let $\beta \downarrow 0$. Since $\Upsilon(\beta) \rightarrow \Upsilon(0)$, we find that for every probability density $\phi \in \mathcal{S}$ and $x \in \mathbf{R}^d$,

$$(3.22) \quad \lim_{\beta \downarrow 0} (\bar{R}_\beta(f * \phi_n))(x) \leq \Upsilon(0).$$

But the left-hand side is

$$(3.23) \quad \lim_{\beta \downarrow 0} \mathbb{E} \left[\int_0^\infty (f * \phi_n)(\bar{X}_s + x) e^{-\beta s} ds \right] = (\bar{R}_0(f * \phi_n))(x),$$

thanks to the monotone convergence theorem. Another application of Fatou's lemma shows that $(\bar{R}_0 f)(x) \leq \Upsilon(0)$ for all $x \in \mathbf{R}^d$. This establishes (3.11), and hence the proposition, in the case that $\beta = 0$. \square

3.4. Proof of the Maximum Principle

The main goal of this subsection is to establish Theorem 1.5. This subsection also contains a harmonic-analytic estimate that might be of independent interest. We will use that harmonic-analytic to demonstrate Theorem 1.5, as well as the subsequent Theorems 1.8 and 1.11.

In order to motivate our estimate, let us first consider the important special case that the correlation function f is of Riesz type. That is,

$$(3.24) \quad f(z) := \|z\|^{-(d-b)} \quad \text{for } z \in \mathbf{R}^d,$$

where $0^{-1} := \infty$. Clearly, f is locally integrable when $b \in (0, d)$, a condition which we assume, and in fact its Fourier transform is

$$(3.25) \quad \hat{f}(\xi) = \frac{\pi^{d/2} 2^b \Gamma(b/2)}{\Gamma((d-b)/2)} \cdot \|\xi\|^{-b}.$$

See Mattila [Mat95, eq. (12.10), p. 161], for example. Then, it is well known that

$$(3.26) \quad \begin{aligned} \iint f(x-y) \mu(dx) \mu(dy) &= \iint \frac{\mu(dx) \mu(dy)}{\|x-y\|^{d-b}} \\ &= \frac{\pi^{d/2} 2^b \Gamma(b/2)}{\Gamma((d-b)/2)} \cdot \int_{\mathbf{R}^d} \frac{|\hat{\mu}(\xi)|^2}{\|\xi\|^b} d\xi \\ &= \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} |\hat{\mu}(\xi)|^2 \hat{f}(\xi) d\xi. \end{aligned}$$

It is easy to guess this famous identity from an informal application of the Fubini theorem. However, a rigorous derivation of (3.26) requires a good deal of effort; see Mattila [Mat95, Lemma 12.12, p. 162], for instance.

In the language of potential theory, the preceding asserts that the Riesz-type “energy” of the form $\iint \|x - y\|^{-d+b} \mu(dx) \mu(dy)$ is equal to a constant multiple of the Pólya–Szegő-type energy $\int_{\mathbf{R}^d} |\hat{\mu}(\xi)|^2 \|\xi\|^{-b} d\xi$. The following proposition shows that there is a very general lower bound that is valid for every correlation function f .

PROPOSITION 3.6. *For all Borel probability measures μ on \mathbf{R}^d ,*

$$(3.27) \quad \iint f(x - y) \mu(dx) \mu(dy) \geq \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} |\hat{\mu}(\xi)|^2 \hat{f}(\xi) d\xi.$$

PROOF. We will assume, without incurring any loss in generality, that

$$(3.28) \quad \iint f(x - y) \mu(dx) \mu(dy) < \infty;$$

for there is nothing to prove otherwise. We also observe that

$$(3.29) \quad \iint f(x - y) \mu(dx) \mu(dy) = \int (f * \mu) d\mu.$$

This is essentially the famous “reciprocity theorem” of classical potential theory.

Because $f * \mu \in L^1(\mu)$, Lusin’s theorem implies that for all $\epsilon > 0$ there exists a compact set A_ϵ in \mathbf{R}^d such that:

- (i) $\mu(A_\epsilon^c) < \epsilon$; and
- (ii) $f * \mu$ is continuous on A_ϵ .

Let

$$(3.30) \quad \mu_\epsilon(\bullet) := \mu(\bullet \cap A_\epsilon)$$

denote the restriction of μ to A_ϵ , and recall the Gaussian densities $\{\phi\}_{n=1}^\infty$ from (3.16). Because

$$(3.31) \quad \lim_{n \rightarrow \infty} (f * \phi_{n/2} * \mu) = f * \mu, \quad \text{uniformly on } A_\epsilon,$$

it follows from (3.29) that

$$(3.32) \quad \begin{aligned} \iint f(x - y) \mu(dx) \mu(dy) &\geq \int_{A_\epsilon} (f * \mu) d\mu \\ &= \int (f * \mu) d\mu_\epsilon \\ &= \lim_{n \rightarrow \infty} \int (f * \phi_{n/2} * \mu) d\mu_\epsilon \\ &\geq \limsup_{n \rightarrow \infty} \int (f * \phi_{n/2} * \mu_\epsilon) d\mu_\epsilon. \end{aligned}$$

Since $\phi_{n/2} = \phi_n * \phi_n$,

$$(3.33) \quad \iint f(x-y) \mu(dx) \mu(dy) \geq \limsup_{n \rightarrow \infty} \int_{\mathbf{R}^d} (f * \mu_{n,\epsilon})(x) \mu_{n,\epsilon}(x) dx.$$

where $\mu_{n,\epsilon}(x) := (\phi_n * \mu_\epsilon)(x)$. Since $\mu_{n,\epsilon} \in \mathcal{S}$ and $f * \mu_{n,\epsilon}$ is C^∞ and tempered, Lemma 3.1 implies that

$$(3.34) \quad \begin{aligned} \int_{\mathbf{R}^d} (f * \mu_{n,\epsilon})(x) \mu_{n,\epsilon}(x) dx &= \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \hat{f}(\xi) |\hat{\mu}_{n,\epsilon}(\xi)|^2 d\xi \\ &= \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} e^{-\|\xi\|^2/n} |\hat{\mu}_\epsilon(\xi)|^2 \hat{f}(\xi) d\xi. \end{aligned}$$

This, (3.32), and the monotone convergence theorem together imply that

$$(3.35) \quad \iint f(x-y) \mu(dx) \mu(dy) \geq \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} |\hat{\mu}_\epsilon(\xi)|^2 \hat{f}(\xi) d\xi.$$

Since μ_ϵ converges weakly to μ as $\epsilon \rightarrow 0$, we know that $\hat{\mu}_\epsilon$ converges to $\hat{\mu}$ pointwise. The proposition follows from this and Fatou's lemma. \square

An elementary computation [Mat95, Lemma 12.11, p. 161] and Proposition 3.6 above together yield the following corollary.

COROLLARY 3.7. *If f is lower semicontinuous, then for all Borel probability measures μ on \mathbf{R}^d ,*

$$(3.36) \quad \iint f(x-y) \mu(dx) \mu(dy) = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} |\hat{\mu}(\xi)|^2 \hat{f}(\xi) d\xi.$$

Even though we do not use the above result in the sequel, we have chosen to include it as it is an interesting fact about energy forms that are based on lower semicontinuous positive-definite functions. A weaker version of this result has been proved by Khoshnevisan and Xiao [KX09, Theorem 5.2].

We are now ready to prove Theorem 1.5.

PROOF OF THEOREM 1.5. Note that for all $t \geq 0$,

$$(3.37) \quad \begin{aligned} (\bar{P}_t f)(0) &= (P_t^* P_t f)(0) \\ &= (\tilde{p}_t * p_t * f)(0) \\ &= \iint f(x-y) p_t(x) p_t(y) dx dy. \end{aligned}$$

This requires only the Tonelli theorem. Now Proposition 3.6 can be applied to show us that

$$(3.38) \quad (\bar{P}_t f)(0) \geq \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} e^{-2t \operatorname{Re} \Psi(\xi)} \hat{f}(\xi) d\xi.$$

Multiply both sides by $\exp(-\beta t)$ and integrate $[dt]$ to find that

$$(3.39) \quad \begin{aligned} (\bar{R}_\beta f)(0) &\geq \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \frac{\hat{f}(\xi)}{\beta + 2\operatorname{Re}\Psi(\xi)} d\xi \\ &= \Upsilon(\beta). \end{aligned}$$

This and Proposition 3.5 together imply (1.17). Consequently, (1.1) holds if and only if $\Upsilon(\beta) < \infty$ for all $\beta > 0$.

Next we prove that $\bar{R}_\beta f \in C_0(\mathbf{R}^d)$ whenever $\Upsilon(\beta) < \infty$ and f is lower semicontinuous.

When f is lower semicontinuous we can find compactly-supported continuous functions f_n that converge upward to f as $n \rightarrow \infty$. Recall from (2.10) on page 15 that \bar{R}_β maps $C_0(\mathbf{R}^d)$ to $C_0(\mathbf{R}^d)$. Consequently, $\bar{R}_\beta f_n \in C_0(\mathbf{R}^d)$, and from this we may conclude that $\bar{R}_\beta f$ is lower semicontinuous.

Next, let us define

$$(3.40) \quad \pi_\beta(x) := \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \frac{e^{-i\xi \cdot x} \hat{f}(\xi)}{\beta + 2\operatorname{Re}\Psi(\xi)} d\xi.$$

If $\Upsilon(\beta) < \infty$, then $\pi_\beta \in C_0(\mathbf{R}^d)$. Moreover, a few successive applications of Fubini's theorem tell us that for all $\phi \in \mathcal{S}$,

$$(3.41) \quad \int_{\mathbf{R}^d} \pi_\beta(x) \phi(x) dx = \int_{\mathbf{R}^d} (\bar{R}_\beta f)(x) \phi(x) dx.$$

Thus, $\pi_\beta = \bar{R}_\beta f$ a.e.

It remains to prove that if $\Upsilon(1)$ is finite, then so is $\Upsilon(\beta)$ for every $\beta > 0$.

We have shown that if $\Upsilon(1)$ is finite, then $\bar{R}_1 f = \pi_1$ almost everywhere, $\pi_1 \in C_0(\mathbf{R}^d)$; also, $\bar{R}_1 f$ is bounded [Proposition 3.5]. If $h_1 = h_2$ almost everywhere and $h_1, h_2 : \mathbf{R}^d \rightarrow \mathbf{R}_+$ are measurable, then

$$(3.42) \quad \begin{aligned} (\bar{R}_\beta h_1)(x) &= \int_{\mathbf{R}^d} \bar{r}_\beta(y) h_1(x-y) dy \\ &= \int_{\mathbf{R}^d} \bar{r}_\beta(y) h_2(x-y) dy \\ &= (\bar{R}_\beta h_2)(x), \end{aligned}$$

for all $x \in \mathbf{R}^d$. Therefore, (2.10) on page 15 and the fact that $\pi_1 \in C_0(\mathbf{R}^d)$ together imply that $\bar{R}_\beta \pi_1 \in C_0(\mathbf{R}^d)$ for all $\beta > 0$. In particular, $\bar{R}_\beta \bar{R}_1 f \in C_0(\mathbf{R}^d)$ —whence $\bar{R}_\beta \bar{R}_1 f$ is bounded—for every $\beta > 0$. And by the resolvent equation [see Blumenthal and Gettoor [BG68, (8.10), p. 41]],

$$(3.43) \quad (\bar{R}_\beta f)(x) = (\bar{R}_1 f)(x) + (1 - \beta)(\bar{R}_\beta \bar{R}_1 f)(x) \quad \text{for all } x \in \mathbf{R}^d.$$

Thus, it follows from the boundedness of $\bar{R}_1 f$ and $\bar{R}_\beta \bar{R}_1 f$ [for all $\beta > 0$] that $\bar{R}_\beta f$ is bounded for every $\beta > 0$. Consequently, Proposition 3.5 implies the finiteness of $\Upsilon(\beta)$ for every $\beta > 0$. \square

3.5. Probabilistic Potential Theory

Define, for all measurable functions $\phi : \mathbf{R}^d \rightarrow \mathbf{R}_+$, a process $t \mapsto L_t(\phi)$ as follows:

$$(3.44) \quad L_t(\phi) := \int_0^t \phi(\bar{X}_s) ds \quad \text{for all } t \in [0, \infty].$$

The random field $(t, \phi) \mapsto L_t(\phi)$ defined above is often called the *occupation field* of the Lévy process X ; see, for example, Dynkin [Dyn84]. It is well defined, but might well be infinite even in simple cases. The following example highlights this, and also paves the way to the ensuing discussion which yields a probabilistic interpretation of Conditions 1.1 and 1.10.

EXAMPLE 3.8 (After Girsanov, 1962 [McK05, §3.10, pp. 78–81]). Consider $d = 1$, and let X denote one-dimensional Brownian motion, normalized so that

$$(3.45) \quad \mathbb{E} e^{i\xi \cdot X_t} = e^{-t\xi^2/4} \quad \text{for all } t > 0 \text{ and } \xi \in \mathbf{R}.$$

In this way, \bar{X} is normalized to be standard linear Brownian motion; that is,

$$(3.46) \quad \mathbb{E} e^{i\xi \cdot \bar{X}_t} = e^{-t\xi^2/2} \quad \text{for all } t > 0 \text{ and } \xi \in \mathbf{R}.$$

Let $\phi(x) := |x|^{-\alpha}$ for all $x \in \mathbf{R}$, where $\alpha \in (0, 1)$ is fixed. By the occupation density formula [see Corollary (1.6) of Revuz and Yor [RY91, p. 209]],

$$(3.47) \quad L_t(\phi) = \int_{-\infty}^{\infty} \frac{\ell_t^x}{|x|^\alpha} dx,$$

where ℓ denotes the process of local times associated to \bar{X} . According to Trotter's theorem [RY91, Theorem (1.7), p. 209] and the occupation density formula,

$$(3.48) \quad \mathbb{P} \{ \ell_t^\bullet \in C_0(\mathbf{R}) \text{ for all } t > 0 \} = 1.$$

It is a well-known consequence of the Blumenthal zero-one law and Brownian scaling that

$$(3.49) \quad \mathbb{P} \{ \ell_t^0 > 0 \text{ for all } t > 0 \} = 1.$$

Consequently [if we ignore the null sets in the usual way], $L_t(\phi) < \infty$ for some—hence all— $t > 0$ if and only if $\alpha < 1$. At the same time, we note that

$$(3.50) \quad \begin{aligned} \mathbb{E}L_t(\phi) &= \mathbb{E} \int_0^t \frac{ds}{|\bar{X}_s|^\alpha} \\ &= \int_0^t \frac{ds}{s^{\alpha/2}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |z|^{-\alpha} e^{-z^2/2} dz. \end{aligned}$$

This requires only Brownian scaling and the Tonelli theorem. Thus, $\mathbb{E}L_t(\phi)$ is finite for all $t > 0$, if and only if $\alpha < 1$. In rough terms, we have shown that $L_t(\phi) < \infty$ if and only if $\mathbb{E}L_t(\phi) < \infty$. As we shall see, a suitable interpretation of this property can be generalized; see Theorem 3.13 below. \square

It is convenient to use some notation from Markov-process theory: Recall from Markov-process theory that \mathbb{P}_z denotes the law of the underlying Lévy process started at $z \in \mathbf{R}^d$ [so that $\mathbb{P} = \mathbb{P}_0$], and \mathbb{E}_z denotes the corresponding expectation operator. Since the underlying Lévy process X is Lévy, \mathbb{P}_z can be interpreted as the law of $X_\bullet + z$. Thus,

$$(3.51) \quad (\bar{R}_\alpha \phi)(z) = \mathbb{E}_z \int_0^\infty e^{-\alpha s} \phi(\bar{X}_s) ds.$$

Before we state and prove the main result of this section, let us first establish some technical facts which will be needed in the proof of the main result.

LEMMA 3.9. *For all $t, \alpha > 0$ and measurable functions $\phi : \mathbf{R}^d \rightarrow \mathbf{R}_+$,*

$$(3.52) \quad \sup_{x \in \mathbf{R}^d} \mathbb{E}_x \left(\sup_{s > 0} [e^{-\alpha s} L_s(\phi)] \right) \leq \sup_{z \in \mathbf{R}^d} (\bar{R}_\alpha \phi)(z) \leq \chi_\alpha(t) \sup_{x \in \mathbf{R}^d} \mathbb{E}_x L_t(\phi),$$

where

$$(3.53) \quad \chi_\alpha(t) := \frac{e^{\alpha t}}{e^{\alpha t} - 1}.$$

REMARK 3.10. Let us point out the following elementary bound for the right-most term in the preceding display: For all $t, \alpha > 0$,

$$(3.54) \quad e^{-\alpha t} \sup_{x \in \mathbf{R}^d} \mathbb{E}_x L_t(\phi) \leq \sup_{x \in \mathbf{R}^d} \mathbb{E}_x \left(\sup_{s > 0} [e^{-\alpha s} L_s(\phi)] \right).$$

This shows that the quantities on the two extreme ends of (3.52) are one and the same, up to a multiplicative constant that depends on α and t in an explicit way. \square

PROOF OF LEMMA 3.9. Because ϕ is nonnegative, we have the sure inequality,

$$(3.55) \quad e^{-\alpha s} L_s(\phi) \leq \int_0^\infty e^{-\alpha r} \phi(\bar{X}_r) dr, \quad \text{valid for all } s, \alpha > 0.$$

We take suprema over $s > 0$ and then apply expectations $[\mathbb{dP}_x]$ to deduce the first inequality in (3.52). For the second bound, let us note that for all $\alpha, t > 0$ and $z \in \mathbf{R}^d$,

$$(3.56) \quad \begin{aligned} (\bar{R}_\alpha \phi)(z) &= \sum_{n=0}^{\infty} \mathbb{E}_z \int_{nt}^{(n+1)t} e^{-\alpha s} \phi(\bar{X}_s) ds \\ &\leq \sum_{n=0}^{\infty} e^{-\alpha nt} \mathbb{E}_z \int_0^t \phi(\bar{X}_{s+nt}) ds. \end{aligned}$$

This implies the second inequality, because

$$(3.57) \quad \begin{aligned} \mathbb{E}_z \int_0^t \phi(\bar{X}_{s+nt}) ds &= \mathbb{E}_z \mathbb{E}_{X_{nt}} \int_0^t \phi(\bar{X}_s) ds \\ &\leq \sup_{x \in \mathbf{R}^d} \mathbb{E}_x [L_t(\phi)], \end{aligned}$$

in accord with the Markov property. \square

LEMMA 3.11. For all $t > 0$ and measurable $\phi : \mathbf{R}^d \rightarrow \mathbf{R}_+$,

$$(3.58) \quad \sup_{x \in \mathbf{R}^d} \mathbb{E}_x (|L_t(\phi)|^2) \leq 2 \left(\sup_{z \in \mathbf{R}^d} \mathbb{E}_z [L_t(\phi)] \right)^2.$$

PROOF. We can write

$$(3.59) \quad \mathbb{E}_x (|L_t(\phi)|^2) = 2 \int_0^t du \int_u^t dv \mathbb{E}_x (\phi(\bar{X}_u) \phi(\bar{X}_v)).$$

Since

$$(3.60) \quad \begin{aligned} \mathbb{E}_x \left(\int_u^t \phi(\bar{X}_v) dv \mid \bar{X}_s; s \leq u \right) &= \mathbb{E}_{X_u} \int_0^{t-u} \phi(\bar{X}_s) ds \\ &= \mathbb{E}_{X_u} [L_{t-u}(\phi)] \quad \mathbb{P}_x\text{-a.s.}, \end{aligned}$$

it follows that

$$(3.61) \quad \begin{aligned} \mathbb{E}_x (|L_t(\phi)|^2) &= 2 \mathbb{E}_x \int_0^t \phi(\bar{X}_u) \mathbb{E}_{X_u} [L_{t-u}(\phi)] du \\ &\leq 2 \mathbb{E}_x \int_0^t \phi(\bar{X}_u) \cdot \sup_{z \in \mathbf{R}^d} \mathbb{E}_z [L_{t-u}(\phi)] du. \end{aligned}$$

The lemma follows since $L_{t-u}(\phi) \leq L_t(\phi)$. \square

The following constitutes the third, and final, technical lemma of this section.

LEMMA 3.12. *For all $\alpha, t > 0$,*

$$(3.62) \quad \sup_{x \in \mathbf{R}^d} \mathbb{P}_x \left\{ L_t(f) \geq \frac{(\bar{R}_\alpha f)(0)}{2\chi_\alpha(t)} \right\} \geq \frac{1}{8},$$

where $\chi_\alpha(t)$ is defined in Lemma 3.9.

PROOF. Recall the *Paley–Zygmund inequality* [PZ32]: If Z is a nonnegative random variable in $L^2(\mathbb{P})$ with $\mathbb{E}Z > 0$, then

$$(3.63) \quad \mathbb{P} \left\{ Z \geq \frac{1}{2}\mathbb{E}Z \right\} \geq \frac{(\mathbb{E}Z)^2}{4\mathbb{E}(Z^2)}.$$

We can apply this with $Z := L_t(\phi)$ —where $\phi : \mathbf{R}^d \rightarrow \mathbf{R}_+$ is bounded away from zero and infinity—to see that for all $z \in \mathbf{R}^d$,

$$(3.64) \quad \sup_{x \in \mathbf{R}^d} \mathbb{P}_x \left\{ L_t(\phi) \geq \frac{1}{2}\mathbb{E}_z L_t(\phi) \right\} \geq \frac{(\mathbb{E}_z L_t(\phi))^2}{4\mathbb{E}_z(|L_t(\phi)|^2)}.$$

By selecting z appropriately, we can ensure that

$$(3.65) \quad \mathbb{E}_z L_t(\phi) \geq (1 - \epsilon) \cdot \sup_{x \in \mathbf{R}^d} \mathbb{E}_x L_t(\phi),$$

where $\epsilon \in (0, 1)$ is arbitrary but fixed. Let $\epsilon \downarrow 0$ and appeal to the continuity properties of probability measures to deduce that

$$(3.66) \quad \sup_{x \in \mathbf{R}^d} \mathbb{P}_x \left\{ L_t(\phi) \geq \frac{1}{2} \sup_{z \in \mathbf{R}^d} \mathbb{E}_z L_t(\phi) \right\} \geq \frac{(\sup_{z \in \mathbf{R}^d} \mathbb{E}_z L_t(\phi))^2}{4 \sup_{x \in \mathbf{R}^d} \mathbb{E}_x (|L_t(\phi)|^2)} \geq \frac{1}{8};$$

see Lemma 3.11 for the last inequality. A monotone-class argument shows that the preceding holds true for all bounded and measurable functions $\phi \not\equiv 0$. We apply it with $\phi_N := \min(f, N)$ in place of ϕ , where $N \geq 1$ is fixed. In this way we obtain

$$(3.67) \quad \sup_{x \in \mathbf{R}^d} \mathbb{P}_x \left\{ L_t(f) \geq \frac{1}{2} \sup_{z \in \mathbf{R}^d} \mathbb{E}_z L_t(\phi_N) \right\} \geq \frac{1}{8}.$$

This and Lemma 3.9 together tell us that

$$\begin{aligned}
(3.68) \quad & \sup_{x \in \mathbf{R}^d} \mathbb{P}_x \left\{ L_t(f) \geq \frac{(\bar{R}_\alpha \phi_N)(0)}{2\chi_\alpha(t)} \right\} \\
& \geq \sup_{x \in \mathbf{R}^d} \mathbb{P}_x \left\{ L_t(f) \geq \sup_{z \in \mathbf{R}^d} \frac{(\bar{R}_\alpha \phi_N)(z)}{2\chi_\alpha(t)} \right\} \\
& \geq \frac{1}{8}.
\end{aligned}$$

As $N \uparrow \infty$, $(\bar{R}_\alpha \phi_N)(0) \uparrow (\bar{R}_\alpha f)(0)$, and the lemma follows. \square

The next result yields a probabilistic characterization of Condition 1.1 which we recall for the reader's convenience.

$$(\bar{R}_\alpha f)(0) < \infty \quad \text{for all } \alpha > 0.$$

THEOREM 3.13. *Under Condition 1.1,*

$$(3.69) \quad \mathbb{P}_z \{L_t(f) < \infty \text{ for all } t > 0\} = 1 \quad \text{for all } z \in \mathbf{R}^d.$$

Moreover, in this case, $t \mapsto L_t(f)$ grows subexponentially. That is,

$$(3.70) \quad \mathbb{P}_z \left\{ \limsup_{t \rightarrow \infty} \frac{\log L_t(f)}{t} \leq 0 \right\} = 1 \quad \text{for all } z \in \mathbf{R}^d.$$

On the other hand, if Condition 1.1 fails to hold, then

$$(3.71) \quad \mathbb{P}_z \{L_t(f) < \infty \text{ for some } t > 0\} = 0 \quad \text{for some } z \in \mathbf{R}^d.$$

REMARK 3.14. Consider the stochastic heat equation where \dot{F} is space-time white noise. Formally speaking, this means that $f := \delta_0$ is our correlation “function.” In this case, one can [again formally] interpret

$$(3.72) \quad L_t(f) = L_t(\delta_0) = \int_0^t \delta_0(\bar{X}_s) \, ds$$

as the local time of the replica process \bar{X} at zero. And if we interpret Theorem 3.13 loosely as well, then Theorem 1.2 suggests that (1.1) has a mild solution if and only if \bar{X} has local times. This interpretation is correct, as well as easy to check, and leads to deeper connections between SPDEs driven by space-time white noise on one hand and local-time theory on the other hand [EFK09, FKN09]. In the case of the parabolic Anderson model [that is, (1.1) with $\sigma(u) = \text{const} \cdot u$ and $b \equiv 0$], Bertini and Cancrini [BC95] and Hu and Nualart [HN09] discuss other closely-related connections to

local times. In the case that x is a discrete variable [for example, because \mathcal{L} is the generator of a Lévy process on \mathbf{Z}^d ; i.e., the generator of a continuous-time random walk], similar connections were found earlier; see Carmona and Molchanov [CM94], for instance. \square

PROOF OF THEOREM 3.13. If $(\bar{R}_\alpha f)(0) < \infty$ for some $\alpha > 0$, then $(\bar{R}_\beta f)(0) < \infty$ for all $\beta > 0$ by Theorem 1.5. It follows from the first inequality of Lemma 3.9, and Theorem 1.5, that for all $\beta > 0$ and $z \in \mathbf{R}^d$,

$$(3.73) \quad \mathbb{E}_z \left[\sup_{t>0} \left(e^{-\beta t} L_t(f) \right) \right] \leq (\bar{R}_\beta f)(0) < \infty.$$

This implies (3.69); it also implies that

$$(3.74) \quad \limsup_{t \rightarrow \infty} \left[e^{-t\beta} L_t(f) \right] < \infty \quad \text{almost surely } [\mathbb{P}_z].$$

This implies (3.70) because $\beta > 0$ and $z \in \mathbf{R}^d$ are arbitrary.

In order to finish the proof, let us consider the remaining case that $(\bar{R}_\alpha f)(0) = \infty$ for all $\alpha > 0$.

According to Lemma 3.12,

$$(3.75) \quad \sup_{x \in \mathbf{R}^d} \mathbb{P}_x \{ L_t(f) = \infty \} \geq \frac{1}{8} \quad \text{for all } t > 0.$$

In particular, there exists $z \in \mathbf{R}^d$ such that

$$(3.76) \quad \mathbb{P}_z \{ L_t(f) = \infty \text{ for some } t > 0 \} \geq \frac{1}{9}.$$

Because $t \mapsto L_t(f)$ is nondecreasing, the Blumenthal zero-one law applies and implies that

$$(3.77) \quad \mathbb{P}_z \{ L_t(f) = \infty \text{ for some } t > 0 \} = 1;$$

this implies the remaining portion of the theorem. \square

We now have the following consequence of Theorem 3.13. It is particularly useful because its hypothesis are verified by all the examples that we have mentioned in the Introduction.

COROLLARY 3.15. *Suppose f is bounded uniformly on the complement of every open neighborhood of the origin. Then, Condition 1.1 is equivalent to the following: $\mathbb{P}\{L_t(f) < \infty \text{ for some } t > 0\} = 1$.*

PROOF. According to Theorem 3.13, if Condition 1.1 holds then

$$(3.78) \quad \mathbb{P}_z \{ L_t(f) < \infty \text{ for some } t > 0 \} = 1 \quad \text{for all } z \in \mathbf{R}^d.$$

Set $z := 0$ to obtain half of the corollary.

Conversely, suppose Condition 1.1 fails. According to Theorem 3.13, there exists a point $z \in \mathbf{R}^d$ such that

$$(3.79) \quad \mathbb{P}_z \{L_t(f) < \infty \text{ for some } t > 0\} = 0.$$

We need to prove that $z = 0$. This holds because if z were not equal to the origin, then

$$(3.80) \quad \mathbb{P}_z \{L_t(f) < \infty \text{ for all } t \in [0, \tau)\} = 1,$$

where τ denotes the first hitting time of the open ball of radius $\|z\|/2$ around 0. Indeed,

$$(3.81) \quad \sup_{0 \leq t < \tau} L_t(f) \leq \tau \cdot \sup_{\|u\| \geq \|z\|/2} f(u) < \infty,$$

\mathbb{P}_z -almost surely. Since the paths of X are right-continuous, $\mathbb{P}_z\{\tau > 0\} = 1$, and hence (3.79) is contradicted. \square

Condition 1.10 [p. 8] also has a probabilistic interpretation that is given by following proposition.

PROPOSITION 3.16. *If $(\bar{R}_\alpha f)(0) < \infty$ for some, hence all, $\alpha > 0$, then:*

$$(3.82) \quad (\bar{R}_0 f)(0) < \infty \quad \implies \quad L_\infty(f) < \infty \quad a.s.;$$

and

$$(3.83) \quad (\bar{R}_0 f)(0) = \infty \quad \implies \quad L_\infty(f) = \infty \quad a.s.$$

PROOF. If $(\bar{R}_0 f)(0) < \infty$, then because $(\bar{R}_0 f)(0) = \mathbb{E}L_\infty(f) < \infty$, it follows that $L_\infty(f)$ is finite a.s.

If, on the other hand, $(\bar{R}_0 f)(0) = \infty$, then because

$$(3.84) \quad \mathbb{E} \int_0^\infty e^{-\alpha s} f(\bar{X}_s) ds = (\bar{R}_\alpha f)(0),$$

the Paley–Zygmund inequality (3.63) implies that

$$(3.85) \quad \mathbb{P} \left\{ \int_0^\infty e^{-\alpha s} f(\bar{X}_s) ds \geq \frac{1}{2} (\bar{R}_\alpha f)(0) \right\} \geq \frac{|(\bar{R}_\alpha f)(0)|^2}{4\mathbb{E} \left(\left| \int_0^\infty e^{-\alpha s} f(\bar{X}_s) ds \right|^2 \right)}.$$

It follows from this and Lemma 3.11—see also the proof of Lemma 3.12—that

$$(3.86) \quad \mathbb{P} \left\{ \int_0^\infty f(\bar{X}_s) \, ds \geq \frac{1}{2} (\bar{R}_\alpha f)(0) \right\} \geq \frac{(\bar{R}_\alpha f)(0)}{8 \sup_{x \in \mathbf{R}^d} (\bar{R}_\alpha f)(x)} = \frac{1}{8},$$

owing to Theorem 1.5 [p. 5]. Let $\alpha \downarrow 0$ to find that

$$(3.87) \quad \mathcal{P} := \mathbb{P} \left\{ \int_0^\infty f(\bar{X}_s) \, ds = \infty \right\} \geq \frac{1}{8}.$$

But according to Theorem 3.13, $\int_0^T f(\bar{X}_s) \, ds < \infty$, for all $T > 0$ a.s., because $(\bar{R}_\alpha f)(0) < \infty$ for some [hence all] $\alpha > 0$. This implies that

$$(3.88) \quad \mathcal{P} = \mathbb{P} \left\{ \lim_{T \rightarrow \infty} \int_0^T f(\bar{X}_s) \, ds = \infty \right\}.$$

That is: (i) \mathcal{P} is the probability of a tail event; and (ii) \mathcal{P} is strictly positive, in fact $\mathcal{P} \geq 1/8$. By the Hewitt–Savage zero-one law [HS55], $\mathcal{P} = 1$. \square

3.6. A Final Observation

Let us conclude this chapter with an observation that will be used later on in Theorem 4.9 [p. 51] in order to produce a stochastic PDE whose random-field solution exists but is discontinuous densely.

Let $\mathbf{X} := \{X_t\}_{t \geq 0}$ denote a Lévy process on \mathbf{R}^d with characteristic exponent Ψ . Recall that \mathbf{X} has a *one-potential density* v if v is a probability density on \mathbf{R}^d that satisfies the following for all Borel-measurable functions $\phi : \mathbf{R}^d \rightarrow \mathbf{R}_+$:

$$(3.89) \quad \mathbb{E} \left[\int_0^\infty e^{-s} \phi(X_s) \, ds \right] = \int_{\mathbf{R}^d} \phi(x) v(x) \, dx.$$

Because $\phi \geq 0$, the preceding expectation commutes with the ds -integral. Recall that m_s denotes the law of X_s , and restrict attention to only non-negative $\phi \in \mathcal{S}$. In that case,

$$\begin{aligned}
 \int_0^\infty e^{-s} \mathbf{E} \phi(X_s) ds &= \int_0^\infty e^{-s} \left(\int \phi dm_s \right) ds \\
 (3.90) \qquad \qquad \qquad &= \frac{1}{(2\pi)^d} \int_0^\infty e^{-s} ds \int_{\mathbf{R}^d} d\xi \overline{\hat{\phi}(\xi)} e^{-s\Psi(\xi)} \\
 &= \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \frac{\overline{\hat{\phi}(\xi)}}{1 + \Psi(\xi)} d\xi.
 \end{aligned}$$

We compare this to the right-hand side of (3.89), and then apply Plancherel's theorem to the latter, to deduce the following well-known formula:

$$(3.91) \qquad \hat{v}(\xi) = \frac{1}{1 + \Psi(\xi)} \quad \text{for all } \xi \in \mathbf{R}^d.$$

If we consider only the case that \mathbf{X} is symmetric, then \hat{v} is rendered nonnegative, since Ψ is nonnegative in this case. This observation and the Bochner–Schwartz theorem [Theorem 3.2] together imply that v is a correlation function. Because products—and hence integer powers—of correlation functions are themselves correlation functions, Theorem 2.6 yields the following byproduct.

THEOREM 3.17. *Choose and fix $a > 0$ and $b \in \mathbf{R}$. Then, there exists a correlation function v on \mathbf{R}^d such that*

$$(3.92) \qquad \hat{v}(\xi) \asymp \frac{1}{\|\xi\|^a (\log \|\xi\|)^b} \quad \text{for } \xi \in \mathbf{R}^d \text{ with } \|\xi\| > e.$$

CHAPTER 4

The Linear Equation

Before we study the fully nonlinear equation (1.1), we analyse the far simpler linearized form of the same equation [$\sigma \equiv 1$, $b \equiv 0$], and show that it has many interesting features of its own. Because the solutions, if any, to the said linear equations can only be Gaussian random fields, we are able to use the theory of Gaussian processes in order to produce some definitive existence and regularity results. Our results should be compared with our earlier joint effort with Eulalia Nualart [FKN09], in which \dot{F} was space-time white noise. Our earlier effort was, in turn, motivated strongly by the earlier works of Dalang and Frangos [DF98], Dalang [Dal99], and Peszat and Zabczyk [PZ00].

4.1. Existence and uniqueness

The linearized form of (1.1) is the stochastic PDE

$$(4.1) \quad \frac{\partial}{\partial t} u_t(x) = (\mathcal{L}u_t)(x) + \dot{F}_t(x),$$

subject to u_0 being the initial function, which as mentioned in the Introduction is assumed to be a nonrandom bounded and measurable function $u_0 : \mathbf{R}^d \rightarrow \mathbf{R}$. One can follow through the theory of Walsh, and define the *weak solution* to (4.1) as the Gaussian random field $\mathbf{u} := \{u_t(\phi)\}_{t>0, \phi \in \mathcal{S}}$, where [we recall] \mathcal{S} denotes the collection of all rapidly-decreasing test functions on \mathbf{R}^d , and

$$(4.2) \quad u_t(\phi) = \int_{\mathbf{R}^d} u_0(x)(P_t^* \phi)(x) \, dx + \int_0^t \int_{\mathbf{R}^d} (p_{t-s} * \phi)(y) F(ds \, dy).$$

The double integral is a Wiener integral and $\{P_t^*\}_{t \geq 0}$ is the semigroup associated to the dual process $X_t^* := -X_t$ [$t \geq 0$].

There are two main questions that one needs to answer before one proceeds further:

- (a) Is \mathbf{u} well defined?
- (b) What is the largest family of ϕ 's for which $u_t(\phi)$ is well defined?

An affirmative answer to the first question would imply existence of solutions in the general sense of Walsh [Wal186]. Since the analysis of the nonrandom quantity $\int_{\mathbf{R}^d} u_0(x)(P_t^* \phi)(x) dx$ is standard, we can reduce our problem to the special case that $u_0 \equiv 0$. In that case, these question are addressed by the following estimate. Here and throughout, we define

$$(4.3) \quad \mathcal{E}_\lambda(v) := \frac{1}{2(2\pi)^d} \int_{\mathbf{R}^d} \frac{|\hat{v}(\xi)|^2 \hat{f}(\xi)}{\lambda^{-1} + \operatorname{Re}\Psi(\xi)} d\xi$$

for all Schwartz distributions v whose Fourier transform is a function.

LEMMA 4.1. *The weak solution \mathbf{u} to (4.1) with $u_0 \equiv 0$ exists as a well-defined Gaussian random field parametrized by $t > 0$ and $\phi \in \mathcal{S}$. Moreover, for all $t, \lambda > 0$ and $\phi \in \mathcal{S}$,*

$$(4.4) \quad a(t)\mathcal{E}_\lambda(\phi) \leq \mathbb{E}(|u_t(\phi)|^2) \leq b(t)\mathcal{E}_\lambda(\phi),$$

where $a(t) := (1 - e^{-2t/\lambda})$ and $b(t) := e^{2t/\lambda}$.

In fact, Lemma 4.1 holds under far greater generality than the one presented here. For instance, it holds even when transition functions do not necessarily exist, and when the correlation function is a general correlation measure. The less general formulation above suffices for our needs.

PROOF. If $\phi \in \mathcal{S}$ then $\hat{p}_t \hat{\phi} \in \mathcal{S}$ for all $t \geq 0$. Since the Fourier transform is an isometry on \mathcal{S} , this proves that

$$(4.5) \quad P_t^* \phi = p_t * \phi \in \mathcal{S} \quad \text{for every } t \geq 0.$$

Therefore, in accord with (1.3), the second moment $\mathbb{E}(|u_t(\phi)|^2)$ is equal to

$$(4.6) \quad \begin{aligned} & \mathbb{E} \left(\left| \int_0^t \int_{\mathbf{R}^d} (P_{t-s}^* \phi)(y) F(ds dy) \right|^2 \right) \\ &= \frac{1}{(2\pi)^d} \int_0^t ds \int_{\mathbf{R}^d} d\xi \left| e^{-(t-s)\Psi(\xi)} \right|^2 \cdot |\hat{\phi}(\xi)|^2 \hat{f}(\xi) \\ &= \frac{1}{(2\pi)^d} \int_0^t ds \int_{\mathbf{R}^d} d\xi e^{-2s\operatorname{Re}\Psi(\xi)} \cdot |\hat{\phi}(\xi)|^2 \hat{f}(\xi). \end{aligned}$$

And the lemma follows from the preceding and Lemma 3.5 of [FKN09]. \square

There are standard ways to extend the domain of Gaussian random fields. In our case, we proceed as follows: Consider the pseudo-distances $\{\rho_t\}_{t>0}$ defined by

$$(4.7) \quad \rho_t(\phi, \psi) := \left\{ \mathbb{E}(|u_t(\phi) - u_t(\psi)|^2) \right\}^{1/2} \quad \text{for } \phi, \psi \in \mathcal{S}.$$

Because $L^2(\mathbf{P})$ -limits of Gaussian random fields are themselves Gaussian random fields, we deduce the following: Suppose v is a Schwartz distribution such that $\lim_{n \rightarrow \infty} \rho_t(v, v * \phi_n) = 0$ for all $t > 0$, where $\{\phi_n\}_{n=1}^\infty$ is the sequence of Gaussian densities, as defined in (3.16) [p. 29]. Then $u_t(v)$ is well defined in $L^2(\mathbf{P})$, and the totality $\{u_t(v)\}$ of all such random variables forms a Gaussian random field.

We follow [FKN09] and say that (4.1) has a *random-field solution* if we can obtain $u_t(\delta_x)$ in this way for all $x \in \mathbf{R}^d$.

Consider the space \mathcal{H}_0 of all $\phi \in \mathcal{S}$ such that $\mathcal{E}_1(\phi) < \infty$ for all $t > 0$. Evidently, \mathcal{H}_0 can be metrized, using the distance

$$(4.8) \quad \delta(\phi, \psi) := \sqrt{\mathcal{E}_1(\phi - \psi)} \quad \text{for } \psi, \phi \in \mathcal{S}.$$

Define \mathcal{H}_1 to be the completion of \mathcal{H}_0 in the distance δ .

LEMMA 4.2. *Condition 1.1 holds iff $\delta_x \in \mathcal{H}_1$ for some, hence all, $x \in \mathbf{R}^d$.*

PROOF. First of all, we recall (1.15) and check that

$$(4.9) \quad \begin{aligned} \mathcal{E}_\lambda(\delta_x) &= \frac{1}{2(2\pi)^d} \int_{\mathbf{R}^d} \frac{d\xi}{\lambda^{-1} + 2\operatorname{Re}\Psi(\xi)} \\ &= \frac{1}{2} \Upsilon(1/\lambda). \end{aligned}$$

In particular, the value of $\mathcal{E}_\lambda(\delta_x)$ does not depend on $x \in \mathbf{R}^d$. And Theorem 1.5 implies that $\mathcal{E}_\lambda(\delta_x)$ is finite for some $\lambda > 0$ if and only if it is finite for all $\lambda > 0$.

Let us first suppose that $\mathcal{E}_\lambda(\delta_x)$ is finite. We can note that $\delta_x * \phi_n = \phi_n(\bullet - x) \in \mathcal{S}$, where $\{\phi_n\}_{n=1}^\infty$ was defined in (3.16). Therefore, for all $n, m \geq 1$,

$$(4.10) \quad \begin{aligned} &\mathcal{E}_\lambda(\delta_x * \phi_n - \delta_x * \phi_m) \\ &= \frac{1}{2(2\pi)^d} \int_{\mathbf{R}^d} \frac{1}{\lambda^{-1} + 2\operatorname{Re}\Psi(\xi)} \left| 1 - e^{-\|\xi\|^2 \left| \frac{1}{2n} - \frac{1}{2m} \right|} \right|^2 d\xi. \end{aligned}$$

Since $\mathcal{E}_\lambda(\delta_x)$ is finite, the dominated convergence theorem tells us that the sequence $\{\delta_x * \phi_n\}_{n=1}^\infty$ is Cauchy in \mathcal{H}_0 . A calculation similar to the preceding shows that the quantity $\mathcal{E}_\lambda(\delta_x * \phi_n - \delta_x)$ converges to zero as $n \rightarrow \infty$. And therefore, $\delta_x \in \mathcal{H}_1$.

Conversely, if $\delta_x \in \mathcal{H}_1$, then $\mathcal{E}_\lambda(\delta_x * \phi_n - \delta_x * \phi_m) \rightarrow 0$ as $n, m \rightarrow \infty$. We can extract an unbounded subsequence $n_1 \leq n_2 \leq \dots$ of positive integers

such that

$$(4.11) \quad \mathcal{E}_\lambda(\delta_x * \phi_{n_j} - \delta_x * \phi_{n_{j+1}}) \leq 2^{-j} \quad \text{for all } j \geq 1.$$

It follows from (4.10) that if $k^{-1} \leq |n_j^{-1} - n_{j+1}^{-1}|$, then

$$(4.12) \quad \begin{aligned} \mathcal{E}_\lambda(\delta_x - \delta_x * \phi_k) &= \frac{1}{2(2\pi)^d} \int_{\mathbf{R}^d} \frac{1}{\lambda^{-1} + 2\operatorname{Re}\Psi(\xi)} \left| 1 - e^{-\|\xi\|^2/(2k)} \right|^2 \\ &\leq \mathcal{E}_\lambda(\delta_x * \phi_{n_j} - \delta_x * \phi_{n_{j+1}}) \\ &\leq 2^{-j}. \end{aligned}$$

Let $k \rightarrow \infty$ and then $j \rightarrow \infty$, in this order, to deduce from the preceding discussion that

$$(4.13) \quad \lim_{k \rightarrow \infty} \mathcal{E}_\lambda(\delta_x - \delta_x * \phi_k) = 0.$$

Because $v \mapsto \sqrt{\mathcal{E}_\lambda(v)}$ satisfies the triangle inequality, it follows from (4.11) that for all $k \geq 2$,

$$(4.14) \quad \begin{aligned} \sqrt{\mathcal{E}_\lambda(\delta_x)} &\leq \sqrt{\mathcal{E}_\lambda(\delta_x - \delta_x * \phi_k)} + \sqrt{\mathcal{E}_\lambda(\delta_x * \phi_k)} \\ &\leq \sqrt{\mathcal{E}_\lambda(\delta_x - \delta_x * \phi_k)} + \sqrt{\mathcal{E}_\lambda(\delta_x * \phi_k - \delta_x * \phi_{k-1})} \\ &\quad + \sqrt{\mathcal{E}_\lambda(\delta_x * \phi_{k-1})} \\ &\leq \sqrt{\mathcal{E}_\lambda(\delta_x - \delta_x * \phi_k)} + 2^{-(k-1)/2} + \sqrt{\mathcal{E}_\lambda(\delta_x * \phi_{k-1})} \\ &\quad \vdots \\ &\leq \sqrt{\mathcal{E}_\lambda(\delta_x - \delta_x * \phi_k)} + \sum_{j=1}^{k-1} 2^{-j/2} + \sqrt{\mathcal{E}_\lambda(\delta_x * \phi_1)}. \end{aligned}$$

We have shown that the first quantity on the right-hand side converges to zero as $k \rightarrow \infty$; and the second term remains bounded. Finally, the third quantity on the right-hand side of the preceding is finite since $\phi_1 \in \mathcal{S}$. Therefore, it follows that if $\delta_x \in \mathcal{H}_1$ then $\mathcal{E}_\lambda(\delta_x) < \infty$. This concludes our proof. \square

For more general initial functions $u_0 \geq 0$, (4.1) has a random-field solution if and only if Condition 1.1 holds and $(P_t u_0)(x) < \infty$ for all $t > 0$ and $x \in \mathbf{R}^d$. Let us conclude this section with a lemma that provides simple conditions that ensure that $(P_t u_0)(x)$ is finite for all $t > 0$ and $x \in \mathbf{R}^d$.

LEMMA 4.3. *Suppose $\exp(-\operatorname{Re}\Psi) \in L^t(\mathbf{R}^d)$ for all $t > 0$, and $u_0 \in L^\beta(\mathbf{R}^d)$ for some $\beta \in [1, \infty]$. Then, $(P_t u_0)(x) < \infty$ for all $t > 0$ and*

$x \in \mathbf{R}^d$. Moreover, $P_t u_0$ is uniformly bounded and continuous for every fixed $t > 0$.

PROOF. Since P_t is a contraction on $L^\infty(\mathbf{R}^d)$, it suffices to consider only the case that $1 \leq \beta < \infty$.

Choose and fix some $t > 0$. According to Young's inequality,

$$(4.15) \quad \begin{aligned} \|P_t u_0\|_{L^\infty(\mathbf{R}^d)} &= \|\tilde{p}_t * u_0\|_{L^\infty(\mathbf{R}^d)} \\ &\leq \|p_t\|_{L^p(\mathbf{R}^d)} \cdot \|u_0\|_{L^q(\mathbf{R}^d)}, \end{aligned}$$

where $p^{-1} + q^{-1} = 1$. On the other hand, for all $p \in (1, \infty)$,

$$(4.16) \quad \|p_t\|_{L^p(\mathbf{R}^d)} \leq \|p_t\|_{L^\infty(\mathbf{R}^d)},$$

and this is finite, thanks to Proposition 2.3 on page 19. Therefore, it remains to prove continuity.

First consider the case that $\beta < \infty$. In that case, we can bound the quantity

$$(4.17) \quad |(P_t u_0)(x) - (P_t u_0)(x')| = |(\tilde{p}_t * u_0)(x) - (\tilde{p}_t * u_0)(x')|,$$

from above, by

$$(4.18) \quad \begin{aligned} &\int_{\mathbf{R}^d} p_t(y) |u_0(y-x) - u_0(y-x')| \, dy \\ &\leq \left(\int_{\mathbf{R}^d} p_t(y) |u_0(y-x) - u_0(y-x')|^\beta \, dy \right)^{1/\beta} \\ &\leq \|p_t\|_{L^\infty(\mathbf{R}^d)}^{1/\beta} \cdot \|u_0(\bullet-x) - u_0(\bullet-x')\|_{L^\beta(\mathbf{R}^d)}. \end{aligned}$$

It is a classical fact that $u_0 \in L^\beta(\mathbf{R}^d)$ implies that u_0 is continuous in $L^\beta(\mathbf{R}^d)$. Therefore, $P_t u_0 = \tilde{p}_t * u_0$ is continuous.

Next let us consider the case that $\beta = \infty$. In that case, we write

$$(4.19) \quad \begin{aligned} &|(\tilde{p}_t * u_0)(x) - (\tilde{p}_t * u_0)(x')| \\ &\leq \|u_0\|_{L^\infty(\mathbf{R}^d)} \cdot \|p_t(\bullet-x) - p_t(\bullet-x')\|_{L^1(\mathbf{R}^d)}, \end{aligned}$$

which goes to zero because, once again, $p_t \in L^1(\mathbf{R}^d)$ implies that p_t is continuous in $L^1(\mathbf{R}^d)$. \square

4.2. Spatial regularity: Examples

Define for all $x, y \in \mathbf{R}^d$,

$$(4.20) \quad d(x, y) := \left(\frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \frac{1 - \cos(\xi \cdot (x - y))}{1 + 2\operatorname{Re}\Psi(\xi)} \hat{f}(\xi) \, d\xi \right)^{1/2}.$$

Then, d defines a pseudo-distance on \mathbf{R}^d . Let N_d denote the *metric entropy* of $[0, 1]^d$. That is, for all $\epsilon > 0$, $N_d(\epsilon)$ denotes the minimum number of radius- ϵ d -balls required to cover $[0, 1]^d$. We can combine Lemma 4.1 with theorems of Dudley (see, for example, Marcus and Rosen [MR06, Theorem 6.1.2, p. 245]) and Fernique [MR06, Theorem 6.2.2, p. 251], together with Belyaev's dichotomy [MR06, Theorem 5.3.10, p. 213], and deduce the following:

PROPOSITION 4.4. *Suppose $\exp(-\operatorname{Re}\Psi) \in L^t(\mathbf{R}^d)$ for all $t > 0$, and $\Upsilon(1) < \infty$ so that (4.1) has a random-field solution \mathbf{u} with $u_0 \equiv 0$. Then the following are equivalent:*

- (1) $x \mapsto u_t(x)$ has a continuous modification for some $t > 0$;
- (2) $x \mapsto u_t(x)$ has a continuous modification for all $t > 0$;
- (3) The following metric-entropy condition holds:

$$(4.21) \quad \int_{0^+} (\log N_d(\epsilon))^{1/2} \, d\epsilon < \infty.$$

Next we describe a large family of examples for (4.1) that have continuous random-field solutions. Throughout, we write " $h \asymp g$ " in place of " (h/g) is bounded, above and below uniformly, by finite positive constants."

THEOREM 4.5. *Suppose $f(x) = \operatorname{const}/\|x\|^{d-\beta}$ and $\operatorname{Re}\Psi(\xi) \asymp \|\xi\|^\alpha$ for some $\alpha \in [0, 2]$ and $\beta \in (0, d)$. Then (4.1) has a random-field solution if and only if $\alpha + \beta > d$. In this case, (4.21) holds. In fact, we have*

$$(4.22) \quad d(x, y) \asymp g(\|x - y\|) \quad \text{uniformly when } \|x - y\| < 1/e,$$

where for all $r \in (0, 1/e)$,

$$(4.23) \quad g(r) := \begin{cases} r^{(\alpha+\beta-d)/2} & \text{if } \alpha + \beta \in (d+1, d+2), \\ r\sqrt{\log(1/r)} & \text{if } \alpha + \beta = d+2, \\ r & \text{if } \alpha + \beta > d+2. \end{cases}$$

REMARK 4.6. The condition that $\operatorname{Re}\Psi(\xi) \asymp \|\xi\|^\alpha$ implies that the upper and lower Blumenthal–Gettoor indices of Ψ match and are both equal to

α ; see Blumenthal and Gettoor [BG61, Theorem 3.2] and Khoshnevisan and Xiao [KX09] for definitions and further details, including the various connections that exist between those indices and the fractal properties of the underlying Lévy process X . \square

PROOF OF THEOREM 4.5. Recall (1.15). In order to prove the existence of random-field solutions, it suffices to show that $\Upsilon(1) < \infty$. We begin by writing

$$(4.24) \quad \begin{aligned} \Upsilon(1) &= \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \frac{\hat{f}(\xi)}{1 + 2\operatorname{Re}\Psi(\xi)} \, d\xi \\ &:= I_1 + I_2, \end{aligned}$$

where

$$(4.25) \quad \begin{aligned} I_1 &:= \frac{1}{(2\pi)^d} \int_{\|\xi\| \leq 1} \frac{\hat{f}(\xi)}{1 + 2\operatorname{Re}\Psi(\xi)} \, d\xi, \quad \text{and} \\ I_2 &:= \frac{1}{(2\pi)^d} \int_{\|\xi\| > 1} \frac{\hat{f}(\xi)}{1 + 2\operatorname{Re}\Psi(\xi)} \, d\xi. \end{aligned}$$

It is clear from the hypothesis of the Lemma that I_1 is always finite, because $\beta < d$. We now turn our attention to I_2 , and note that

$$(4.26) \quad I_2 \asymp \int_{\|\xi\| > 1} \frac{d\xi}{\|\xi\|^{\alpha+\beta}}.$$

Therefore, $I_2 < \infty$ if and only if $\alpha + \beta > d$. This concludes the first part of the result.

For the second part we assume that

$$(4.27) \quad \|x - y\| \leq 1/e.$$

We write

$$(4.28) \quad |d(x, y)|^2 := \frac{1}{(2\pi)^d} (J_1 + J_2 + J_3),$$

where

$$(4.29) \quad \begin{aligned} J_1 &:= \int_{\|\xi\| \leq 1} \frac{1 - \cos(\xi \cdot (x - y))}{1 + 2\operatorname{Re}\Psi(\xi)} \hat{f}(\xi) \, d\xi, \\ J_2 &:= \int_{\|\xi\| > 1/\|x-y\|} \frac{1 - \cos(\xi \cdot (x - y))}{1 + 2\operatorname{Re}\Psi(\xi)} \hat{f}(\xi) \, d\xi, \\ J_3 &:= \int_{1 < \|\xi\| \leq 1/\|x-y\|} \frac{1 - \cos(\xi \cdot (x - y))}{1 + 2\operatorname{Re}\Psi(\xi)} \hat{f}(\xi) \, d\xi. \end{aligned}$$

We can estimate each J_j separately.

Because $1 - \cos \theta \asymp \theta^2$ for $\theta \in (-1, 1)$,

$$\begin{aligned}
 J_1 &\asymp \|x - y\|^2 \cdot \int_{\|\xi\| \leq 1} \|\xi\|^2 \hat{f}(\xi) \, d\xi \\
 (4.30) \quad &\asymp \|x - y\|^2 \cdot \int_{\|\xi\| \leq 1} \|\xi\|^{2-\beta} \, d\xi \\
 &\asymp \|x - y\|^2 \cdot \int_0^1 \frac{dr}{r^{\beta-d-1}}.
 \end{aligned}$$

Because $\beta < d$, the integral term in the above display is finite, and hence

$$(4.31) \quad J_1 \asymp \|x - y\|^2.$$

We estimate J_2 similarly:

$$\begin{aligned}
 (4.32) \quad J_2 &\leq \text{const} \cdot \int_{\|\xi\| > 1/\|x-y\|} \frac{\hat{f}(\xi) \, d\xi}{\|\xi\|^\alpha} \\
 &\asymp \int_{1/\|x-y\|}^\infty \frac{dr}{r^{\alpha+\beta-d+1}}.
 \end{aligned}$$

The final integral is finite if and only if $\alpha + \beta > d$; and in this case, we have the estimate $0 \leq J_2 \leq \text{const} \cdot \|x - y\|^{\alpha+\beta-d}$.

Finally,

$$\begin{aligned}
 (4.33) \quad J_3 &\asymp \int_{1 < \|\xi\| \leq 1/\|x-y\|} \|\xi\|^{2-\alpha-\beta} \cdot \|x - y\|^2 \, d\xi \\
 &\asymp \|x - y\|^2 \cdot \int_1^{1/\|x-y\|} \frac{dr}{r^{\alpha+\beta-d-1}}.
 \end{aligned}$$

We evaluate the integrals (4.32) and (4.33) for the different cases of $\alpha + \beta$ to obtain the result. \square

The following is an immediate consequence of Proposition 4.4 and Theorem 4.5.

COROLLARY 4.7. *Every random-field solution \mathbf{u} given by Theorem 4.5 has a continuous modification for all $t > 0$.*

We devote the remainder of this section to a special case of (4.1) namely

$$(4.34) \quad \frac{\partial}{\partial t} u_t(x) = (\Delta u_t)(x) + \dot{F}_t(x),$$

where $u_0 \equiv 0$, $x \in \mathbf{R}^3$, $t > 0$, and the Laplacian acts on the x variable only. The noise F is a centered Gaussian noise, as before, that is white in time and homogeneous in space with a correlation function f that satisfies the

following for a fixed $q \in \mathbf{R}$:

$$(4.35) \quad \hat{f}(\xi) \asymp \frac{1}{\|\xi\|(\log \|\xi\|)^q} \quad \text{for } \xi \in \mathbf{R}^3 \text{ with } \|\xi\| > e.$$

According to Theorem 3.17 on page 42, such correlation functions exist.

The following lemma will be useful for the proof of the main result of this section.

LEMMA 4.8. *If $g : \mathbf{R}^3 \mapsto \mathbf{R}_+$ is a Borel-measurable radial function, then*

$$(4.36) \quad \int_{\|x\| > 1/\|y\|} (1 - \cos(x \cdot y))g(x) \, dx \geq \text{const} \cdot \int_{\|x\| > 1/\|y\|} g(x) \, dx,$$

uniformly for all $y \in \mathbf{R}^3 \setminus \{0\}$.

PROOF. Clearly,

$$(4.37) \quad \int_{\|x\| > 1/\|y\|} (1 - \cos(x \cdot y))g(x) \, dx = \int_{1/\|y\|}^{\infty} r^2 R(r) \, dr \int_{\mathbf{S}^2} d\theta (1 - \cos(y \cdot r\theta)),$$

where R is the function on \mathbf{R}_+ defined by $R(\|x\|) := g(x)$ for all $x \in \mathbf{R}^3$. But for all $r > 0$, the $d\theta$ -integral can be computed as

$$(4.38) \quad \int_{\mathbf{S}^2} (1 - \cos(y \cdot r\theta)) \, d\theta = \text{const} \cdot \left(1 - \frac{\sin(r\|y\|)}{r\|y\|}\right),$$

and this is bounded below uniformly, as long as $r > 1/\|y\|$. We combine the preceding two displays to obtain the result. \square

The following is the main result concerning (4.34).

THEOREM 4.9. *Consider the stochastic heat equation (4.34) in \mathbf{R}^3 , where the correlation function f of the noise satisfies (4.35) for a given fixed value $q \in \mathbf{R}$. Then:*

- (a) (4.34) has a random-field solution \mathbf{u} iff $q > 1$;
- (b) $x \mapsto u_t(x)$ has a continuous modification for all $t > 0$ iff $q > 2$.

REMARK 4.10. Theorem 4.9, and general facts about stationary Gaussian processes [see Belyaev's dichotomy [MR06, Theorem 5.3.10, p. 213], for instance], together prove that when $q \in (1, 2]$, the stochastic heat equation (4.35) has a random-field solution \mathbf{u} that almost surely has infinite oscillations in every open space-time set. This example was mentioned at the end of Introduction. \square

PROOF. In order to show existence of a random-field solution, it suffices to show that $\Upsilon(1) < \infty$ if and only if $q > 1$. Because $\Psi(\xi) = \|\xi\|^2$, we may write $\Upsilon(1)$ as follows:

$$(4.39) \quad \begin{aligned} \Upsilon(1) &= \frac{1}{8\pi^3} \int_{\mathbf{R}^3} \frac{\hat{f}(\xi)}{1 + 2\|\xi\|^2} d\xi \\ &:= \frac{I_1 + I_2}{8\pi^3}, \end{aligned}$$

where

$$(4.40) \quad I_1 := \int_{\|\xi\| < e} \frac{\hat{f}(\xi)}{1 + 2\|\xi\|^2} d\xi \quad \text{and} \quad I_2 := \int_{\|\xi\| > e} \frac{\hat{f}(\xi)}{1 + 2\|\xi\|^2} d\xi.$$

Direct inspection reveals that

$$(4.41) \quad I_1 \asymp \int_0^e \frac{r}{(\log r)^q} dr \quad \text{and} \quad I_2 \asymp \int_e^\infty \frac{1}{r(\log r)^q} dr.$$

It follows readily from this that $\Upsilon(1) < 1$ if and only if $q > 1$.

We now turn our attention to the second part of the proof. Throughout, we assume that $\|x - y\| \leq 1/e$, and consider the following integral:

$$(4.42) \quad \begin{aligned} |d(x, y)|^2 &= \frac{1}{8\pi^3} \int_{\mathbf{R}^3} \frac{1 - \cos(\xi \cdot (x - y))}{1 + 2\|\xi\|^2} \hat{f}(\xi) d\xi \\ &:= \frac{J_1 + J_2 + J_3}{8\pi^3}, \end{aligned}$$

where

$$(4.43) \quad \begin{aligned} J_1 &:= \int_{\|\xi\| \leq e} \frac{1 - \cos(\xi \cdot (x - y))}{1 + 2\|\xi\|^2} \hat{f}(\xi) d\xi, \\ J_2 &:= \int_{\|\xi\| > 1/\|x - y\|} \frac{1 - \cos(\xi \cdot (x - y))}{1 + 2\|\xi\|^2} \hat{f}(\xi) d\xi, \\ J_3 &:= \int_{e < \|\xi\| \leq 1/\|x - y\|} \frac{1 - \cos(\xi \cdot (x - y))}{1 + 2\|\xi\|^2} \hat{f}(\xi) d\xi. \end{aligned}$$

We estimate each of the integral separately. The first term can be dealt with easily, and we obtain the following, using similar computations to those in the proof of Theorem 4.5:

$$(4.44) \quad J_1 \asymp \|x - y\|^2.$$

The estimation of the second term requires a little bit more work, viz.,

$$\begin{aligned}
J_2 &\leq \frac{1}{2} \int_{\|\xi\| > 1/\|x-y\|} \frac{1 - \cos(\xi \cdot (x-y))}{\|\xi\|^2} \hat{f}(\xi) \, d\xi \\
&\leq \text{const} \cdot \int_{\|\xi\| > 1/\|x-y\|} \frac{\hat{f}(\xi)}{\|\xi\|^2} \, d\xi \\
(4.45) \quad &\leq \text{const} \cdot \int_{1/\|x-y\|}^{\infty} \frac{1}{r(\log r)^q} \, dr \\
&= \text{const} \cdot \left(\log \frac{1}{\|x-y\|} \right)^{-q+1}.
\end{aligned}$$

Finally, we consider the final term J_3 :

$$\begin{aligned}
J_3 &\asymp \|x-y\|^2 \int_{e < \|\xi\| \leq 1/\|x-y\|} \hat{f}(\xi) \, d\xi \\
(4.46) \quad &\asymp \|x-y\|^2 \int_e^{1/\|x-y\|} \frac{r}{(\log r)^q} \, dr \\
&\leq \text{const} \cdot \left(\log \frac{1}{\|x-y\|} \right)^{-q}.
\end{aligned}$$

Upon combining the above estimates we obtain the bound

$$(4.47) \quad |d(x, y)|^2 \leq \text{const} \cdot \left(\log \frac{1}{\|x-y\|} \right)^{-q+1}.$$

Next, we compute a similar lower bound for $|d(x, y)|^2$. Since the integrands are nonnegative throughout, we may consider only J_2 . In that case, Lemma 4.8 yields the following:

$$\begin{aligned}
J_2 &\geq \text{const} \cdot \int_{\|\xi\| \geq 1/\|x-y\|} \frac{\hat{f}(\xi)}{\|\xi\|^2} \, d\xi \\
(4.48) \quad &\geq \text{const} \cdot \left(\log \frac{1}{\|x-y\|} \right)^{-q+1}.
\end{aligned}$$

Thus far, we have proved that

$$(4.49) \quad d(x, y) \asymp |\log(\|x-y\|)|^{(1-q)/2},$$

uniformly, as long as $\|x-y\| < 1/e$. From this, we obtain

$$(4.50) \quad \log N_d(\epsilon) \asymp \epsilon^{2/(1-q)},$$

valid for $0 < \epsilon < 1/e$. In particular, the metric-entropy condition (4.21) applies if and only if $q > 2$. Since the other conditions of Proposition 4.4

hold [for elementary reasons], the second part of the theorem follows from Proposition 4.4. \square

CHAPTER 5

The Nonlinear Equation

The primary goal of this chapter is to study the fully-nonlinear stochastic heat equation (1.1) as described in the introduction.

In the first part, we derive a series of *a priori* estimates that ultimately lead to the proof of Theorem 1.2. The latter theorem shows that the finite-potential Condition 1.1 is sufficient for the existence of a mild solution to the stochastic heat equation. As a byproduct, that theorem also yields a temporal growth rate for the solution. This means that under some natural conditions on the multiplicative nonlinearity σ , the mild solution will not be intermittent.

The second part is devoted to the proofs of Theorems 1.8 and 1.11, and thereby establishing the fact that, in contrast to the preceding discussion, if “there is enough symmetry and nonlinearity,” then the mild solution to the stochastic heat equation is weakly intermittent.

In the third and final part, we give a partial answer to a deep question of David Nualart who asked about the “effect of drift” on the intermittence of the solution. In particular, we show that if the drift is exactly linear—which corresponds to a massive and/or dissipative version of (1.1)—then there is frequently an explicit phase transition which describes the amount of drift needed in order to offset the intermittent multiplicative effect of the underlying noise.

5.1. Existence and Uniqueness

The main goal of this section is to prove Theorem 1.2. With that aim in mind, we can formulate (1.1), in mild form, as follows:

$$(5.1) \quad \begin{aligned} u_t(x) = (P_t u_0)(x) &+ \int_0^t ds \int_{\mathbf{R}^d} dy p_{t-s}(y-x) b(u_s(y)) \\ &+ \int_0^t \int_{\mathbf{R}^d} p_{t-s}(y-x) \sigma(u_s(y)) F(ds dy). \end{aligned}$$

As is customary, we seek to find a mild solution that satisfies the following integrability condition:

$$(5.2) \quad \sup_{t \in [0, T]} \sup_{x \in \mathbf{R}^d} \mathbf{E} (|u_t(x)|^2) < \infty \quad \text{for all } T > 0.$$

In order to prove Theorem 1.2 we apply a familiar fixed-point argument, though the details of this argument are not entirely standard.

Let $\mathcal{F} := \{\mathcal{F}_t\}_{t \geq 0}$ denote the right-continuous complete filtration generated by the noise F . Specifically, for every positive t , we define \mathcal{F}_t^0 to be the σ -algebra generated by random variables of the form of the Wiener integral $\int_{[0, t] \times \mathbf{R}^d} \phi_s(x) F(ds dx)$, as ϕ ranges over $L^2([0, t] \times \mathbf{R}^d)$. Define \mathcal{F}_t^1 to be the P-completion of \mathcal{F}_t^0 , and finally define

$$(5.3) \quad \mathcal{F}_t := \bigcap_{s > t} \mathcal{F}_s^1$$

as the right-continuous extension.

We recall from Walsh [Wal86] that a random field $\{v_t(x)\}_{t \geq 0, x \in \mathbf{R}^d}$ is *predictable* if it can be realized as an $L^2(\mathbf{P})$ -limit of finite linear combination of random fields of the type

$$(5.4) \quad z_t(x)(\omega) := X(\omega) \mathbf{1}_{(a, b] \times A}(t, x) \quad \text{for } t > 0, x \in \mathbf{R}^d, \text{ and } \omega \in \Omega,$$

where: $0 < a < b < \infty$, $A \subseteq \mathbf{R}^d$ is compact, and X is an \mathcal{F}_t -measurable and bounded random variable.¹ Define for all predictable random fields v ,

$$(5.5) \quad (\mathcal{A}v)_t(x) := \int_{[0, t] \times \mathbf{R}^d} p_{t-s}(y-x) \sigma(v_s(y)) F(ds dy),$$

and

$$(5.6) \quad (\mathcal{B}v)_t(x) := \int_0^t ds \int_{\mathbf{R}^d} dy p_{t-s}(y-x) b(v_s(y)),$$

provided that the integrals exist: The first integral must exist in the sense of Walsh [Wal86]; and the second in the sense of Lebesgue.

Define for all $\beta, p > 0$, and all predictable random fields v ,

$$(5.7) \quad \|v\|_{\beta, p} := \sup_{t > 0} \sup_{x \in \mathbf{R}^d} \left[e^{-\beta t} \mathbf{E} (|v_t(x)|^p) \right]^{1/p}.$$

It is easy to see that the preceding defines a [pseudo-] norm on random fields, for every fixed choice of $\beta, p > 0$. In fact, these are one among

¹We are using the standard “ $(\Omega, \mathcal{F}, \mathbf{P})$ ” notation of probability for the underlying probability space, of course.

many possible infinite-dimensional L^p -norms. And the corresponding L^p -type space is denoted by $\mathbf{B}_{\beta,p}$. We make the following definition which will be in force throughout the rest of the paper.

DEFINITION 5.1. Let $\mathbf{B}_{\beta,p}$ denote the collection of all [equivalence classes of modifications of] predictable random fields $X := \{X_t(x)\}_{t \geq 0, x \in \mathbf{R}^d}$ such that $\|X\|_{\beta,p} < \infty$.

One can easily check easily that $\|\cdot\|_{\beta,p}$ defines a pseudo-norm on $\mathbf{B}_{\beta,p}$. Moreover, if we identify $X \in \mathbf{B}_{\beta,p}$ with $Y \in \mathbf{B}_{\beta,p}$ when $\|X - Y\|_{\beta,p} = 0$, then [the resulting collection of equivalence classes in] $\mathbf{B}_{\beta,p}$ becomes a Banach space. Because $\|X - Y\|_{\beta,p} = 0$ if and only if X and Y are modifications of one another, it follows that—after the usual identification of a process with its modifications— $\mathbf{B}_{\beta,p}$ is a Banach space of [equivalence classes of] functions with finite $\|\cdot\|_{\beta,p}$ norm.

Our next two lemmas contain *a priori* estimates on Walsh-type stochastic integrals, as well as certain Lebesgue integrals. Among other things, these lemmas show that \mathcal{B} and \mathcal{A} are bounded linear maps from predictable processes to predictable processes. These lemmas are motivated strongly by the theory of optimal regularity for parabolic equations, as is our entire approach to the proof of Theorem 1.2; see Lunardi [Lun95]. We follow the main idea of optimal regularity, and aim to find a good function space such that if u_0 resides in that function space, then u_t has to live in the same function space for all t . As we shall soon see, the previously-defined Banach spaces $\{\mathbf{B}_{\beta,p}\}_{\beta,p>0}$ form excellent candidates for those function spaces. In a rather different context, this general idea appears also in Dalang and Mueller [DM03]. Those authors show that $L^2(\mathbf{R}^d)$ is also a good candidate for such a function space provided that $\sigma(0) = 0$.

Here and throughout, we will use the following notation on Lipschitz functions.

CONVENTION 5.2. If $g : \mathbf{R}^d \rightarrow \mathbf{R}$ is Lipschitz continuous, then we can find finite constants C_g and D_g such that

$$(5.8) \quad |g(x)| \leq C_g + D_g|x| \quad \text{for all } x \in \mathbf{R}^d.$$

To be concrete, we choose $C_g := |g(0)|$ and $D_g := \text{Lip}_g$, to be concrete.

As mentioned above, the next two results describe *a priori* estimates for the Walsh-integral-processes $\mathcal{B}v$ and $\mathcal{A}v$ when v is a nice predictable random field. Together, they imply that the random linear operators \mathcal{A} and \mathcal{B} map

each and every $\mathcal{B}_{\beta,p}$ into itself boundedly and continuously. The respective operator norms are both described in terms of a replica potential of the correlation function f .

LEMMA 5.3. *For all integers $p \geq 2$, real numbers $\beta > 0$, and predictable random fields v and w ,*

$$(5.9) \quad \|\mathcal{B}v\|_{\beta,p} \leq \frac{p}{\beta} \left(\frac{C_b}{e} + D_b \|v\|_{\beta,p} \right),$$

and

$$(5.10) \quad \|\mathcal{B}v - \mathcal{B}w\|_{\beta,p} \leq \frac{p \text{Lip}_b}{\beta} \|v - w\|_{\beta,p}.$$

PROOF. On one hand, the triangle inequality implies that $\mathbb{E}(|(\mathcal{B}v)_t(x)|^p)$ is bounded above by the following quantity:

$$\int_0^t ds_1 \int_{\mathbf{R}^d} dy_1 \cdots \int_0^t ds_p \int_{\mathbf{R}^d} dy_p \prod_{k=1}^p p_{t-s_k}(t_k - x) \cdot \mathbb{E} \left(\prod_{j=1}^p |b(v_{s_j}(y_j))| \right).$$

On the other hand, the generalized Hölder inequality tells us that

$$(5.11) \quad \mathbb{E} \left(\prod_{j=1}^p |b(v_{s_j}(y_j))| \right) \leq \prod_{j=1}^p \|b(v_{s_j}(y_j))\|_p.$$

Therefore, we can conclude that

$$(5.12) \quad \mathbb{E}(|(\mathcal{B}v)_t(x)|^p) \leq \left(C_b t + D_b \int_0^t ds \int_{\mathbf{R}^d} dy p_{t-s}(y-x) \|v_s(y)\|_p \right)^p.$$

We multiply the preceding by $\exp(-\beta t)$ and take the the $(1/p)$ -th root to find that

$$(5.13) \quad \begin{aligned} & \left[e^{-\beta t} \mathbb{E}(|(\mathcal{B}v)_t(x)|^p) \right]^{1/p} \\ & \leq C_b t e^{-\beta t/p} + D_b \int_0^\infty ds \int_{\mathbf{R}^d} dy e^{-\beta(t-s)/p} p_{t-s}(y-x) e^{-\beta s/p} \|v_s(y)\|_p \\ & \leq C_b t e^{-\beta t/p} + \frac{p D_b}{\beta} \|v\|_{\beta,p}. \end{aligned}$$

The first display of the lemma follows because $t \exp(-\beta t/p) \leq p/(e\beta)$ for all $t > 0$. In order to obtain the second display we note that

$$(5.14) \quad |(\mathcal{B}v)_t(x) - (\mathcal{B}w)_t(x)| \leq \text{Lip}_b \cdot (\mathcal{B}_1(|v-w|))_t(x),$$

where \mathcal{B}_1 is defined exactly as \mathcal{B} was, but with $b(x)$ replaced by $b_1(x) = x$. Because we may choose $C_{b_1} = 0$ and $D_{b_1} = 1$, the second assertion of the lemma follows from the first. \square

LEMMA 5.4. *For all even integers $p \geq 2$, real numbers $\beta > 0$, and predictable random fields v and w ,*

$$(5.15) \quad \|\mathcal{A}v\|_{\beta,p} \leq z_p (C_\sigma + D_\sigma \|v\|_{\beta,p}) \sqrt{(\bar{R}_{2\beta/p} f)(0)},$$

and

$$(5.16) \quad \|\mathcal{A}v - \mathcal{A}w\|_{\beta,p} \leq z_p \text{Lip}_\sigma \|v - w\|_{\beta,p} \sqrt{(\bar{R}_{2\beta/p} f)(0)}.$$

PROOF. According to Davis's formulation [Dav76] of the Burkholder–Davis–Gundy inequality, $E(|(\mathcal{A}v)_t(x)|^p)$ is bounded above by z_p^p times the expectation of

$$(5.17) \quad \left| \int_0^t ds \int_{\mathbf{R}^d} dy \int_{\mathbf{R}^d} dz V_s(y, z) p_{t-s}(y-x) p_{t-s}(z-x) f(z-y) \right|^{p/2},$$

where

$$(5.18) \quad V_s(y, z) := \sigma(v_s(y)) \sigma(v_s(z)).$$

By the generalized Hölder inequality,

$$(5.19) \quad E \left(\prod_{j=1}^{p/2} V_{s_j}(y_j, z_j) \right) \leq \prod_{j=1}^{p/2} \|\sigma(v_{s_j}(y_j))\|_p \|\sigma(v_{s_j}(z_j))\|_p.$$

Consequently, $E(|(\mathcal{A}v)_t(x)|^p)$ is bounded above by z_p^p times

$$(5.20) \quad \left(\int_0^t ds \int_{\mathbf{R}^d} dy \int_{\mathbf{R}^d} dz V'_s(y, z) p_{t-s}(y-x) p_{t-s}(z-x) f(z-y) \right)^{p/2},$$

where

$$(5.21) \quad V'_s(y, z) := \|\sigma(v_s(y))\|_p \|\sigma(v_s(z))\|_p.$$

We can note that for all $s > 0$ and $y \in \mathbf{R}^d$,

$$(5.22) \quad \begin{aligned} \|\sigma(v_s(y))\|_p &\leq C_\sigma + D_\sigma \|v_s(y)\|_p \\ &\leq C_\sigma + D_\sigma e^{\beta s/p} \|v\|_{\beta,p} \\ &\leq e^{\beta s/p} (C_\sigma + D_\sigma \|v\|_{\beta,p}). \end{aligned}$$

Therefore, a line or two of computation yield

$$(5.23) \quad \mathbb{E} (|(\mathcal{A}v)_t(x)|^p) \leq e^{\beta t} z_p^p (C_\sigma + D_\sigma \|v\|_{\beta,p})^p \left(\int_0^\infty e^{-2\beta s/p} \mathcal{H}_s \, ds \right)^{p/2},$$

where

$$(5.24) \quad \begin{aligned} \mathcal{H}_s &:= \int_{\mathbf{R}^d} da \int_{\mathbf{R}^d} db f(a-b) p_s(a) p_s(b) \\ &= (P_s^* P_s f)(0) \\ &= (\bar{P}_s f)(0). \end{aligned}$$

And hence,

$$(5.25) \quad \mathbb{E} (|(\mathcal{A}v)_t(x)|^p) \leq e^{\beta t} z_p^p (C_\sigma + D_\sigma \|v\|_{\beta,p})^p ((\bar{R}_{2\beta/p} f)(0))^{p/2},$$

The first assertion of the lemma follows immediately from this.

In order to deduce the second assertion we note that

$$(5.26) \quad \|\mathcal{A}v - \mathcal{A}w\|_{\beta,p} \leq \text{Lip}_\sigma \cdot \mathcal{A}_1(|v-w|),$$

where \mathcal{A}_1 is the same as \mathcal{A} , but with $\sigma(x)$ replaced by $\sigma_1(x) = x$. Therefore, the first assertion of the lemma implies the second. This completes the proof. \square

We are ready to begin a more-or-less standard iterative construction that is used to prove Theorem 1.2.

Let

$$(5.27) \quad u_t^0(x) := u_0(x),$$

and define iteratively: For all $n \geq 0$,

$$(5.28) \quad u_t^{n+1}(x) = (P_t u_0)(x) + (\mathcal{B}u^n)_t(x) + (\mathcal{A}u^n)_t(x).$$

LEMMA 5.5. *Choose and fix $\beta > 0$ and an even integer $p \geq 2$. If*

$$(5.29) \quad \frac{pD_b}{\beta} + z_p D_\sigma \sqrt{(\bar{R}_{2\beta/p} f)(0)} < 1,$$

then $\sup_{n \geq 0} \|u^n\|_{\beta,p} < \infty$.

PROOF. Because $P_t u_0$ is bounded, uniformly in modulus, by $\sup_{x \in \mathbf{R}^d} |u_0(x)|$, the triangle inequality implies that

$$(5.30) \quad \|u^{n+1}\|_{\beta,p} \leq \sup_{x \in \mathbf{R}^d} |u_0(x)| + \|\mathcal{B}u^n\|_{\beta,p} + \|\mathcal{A}u^n\|_{\beta,p}.$$

Lemmas 5.3 and 5.4, and a few lines of direct computation, together imply that

$$(5.31) \quad \|u^{n+1}\|_{\beta,p} \leq A + B\|u^n\|_{\beta,p},$$

where

$$(5.32) \quad A := \sup_{x \in \mathbf{R}^d} |u_0(x)| + \frac{pC_b}{\beta e} + z_p C_\sigma \sqrt{(\bar{R}_{2\beta/p} f)(0)},$$

and

$$(5.33) \quad B := \frac{pD_b}{\beta} + z_p D_\sigma \sqrt{(\bar{R}_{2\beta/p} f)(0)}.$$

Iteration yields the bound

$$(5.34) \quad \|u^{n+1}\|_{\beta,p} \leq A \left(1 + B + \dots + B^{n-1} + B^n \sup_{x \in \mathbf{R}^d} |u_0(x)| \right).$$

Consequently, if $B < 1$ then

$$(5.35) \quad \sup_{k \geq 1} \|u^k\|_{\beta,p} \leq \frac{A}{1-B}.$$

Since $\|u^0\|_{\beta,p} \leq \sup_{x \in \mathbf{R}^d} |u_0(x)| < \infty$, the lemma follows. \square

We now have all the technical estimates for the proof of Theorem 1.2.

5.1.1. Proof of Theorem 1.2. Without loss of generality, we can find $\beta > 0$ such that $Q(p, \beta) < 1$, otherwise there is nothing to prove. Choose and fix such a β .

Thanks to Lemma 5.5, every u^n is well defined and $\|u^n\|_{\beta,p}$ is finite, uniformly in n . In particular, $u^n \in \mathbf{B}_{\beta,p}$. Next we apply Lemmas 5.3 and 5.4—with $C_g := |g(0)|$ and $D_g := \text{Lip}_g$ —to find that

$$(5.36) \quad \begin{aligned} \|u^{n+1} - u^n\|_{\beta,p} &\leq \|\mathcal{B}u^n - \mathcal{B}u^{n-1}\|_{\beta,p} + \|\mathcal{A}u^n - \mathcal{A}u^{n-1}\|_{\beta,p} \\ &\leq \|u^n - u^{n-1}\|_{\beta,p} \cdot Q(p, \beta). \end{aligned}$$

Because $Q(p, \beta) < 1$, the preceding implies that

$$(5.37) \quad \sum_{n=1}^{\infty} \|u^{n+1} - u^n\|_{\beta,p} \leq \text{const} \cdot \sum_{n=1}^{\infty} \{Q(p, \beta)\}^n < \infty.$$

Therefore, we can find a predictable random field $u^\infty \in \mathbf{B}_{\beta,p}$ such that

$$(5.38) \quad \lim_{n \rightarrow \infty} \|u^n - u^\infty\|_{\beta,p} = 0.$$

Our arguments can be adjusted to show also that

$$(5.39) \quad \lim_{n \rightarrow \infty} \|\mathcal{B}u^n - \mathcal{B}u^\infty\|_{\beta,p} = 0,$$

as well as

$$(5.40) \quad \lim_{n \rightarrow \infty} \|\mathcal{A}u^n - \mathcal{A}u^\infty\|_{\beta,p} = 0.$$

This proves that u^∞ is another solution to (5.1). As we have mentioned, u is the almost-surely unique solution to (5.1). Therefore, u^∞ is equal to u , up to evanescence. It follows that $u \in \mathbf{B}_{\beta,p}$ for all $\beta > 0$ such that $Q(p, \beta) < 1$. This proves the theorem. \square

5.2. Intermittency

In this section we prove Theorems 1.8 and 1.11, which state that the solution to the stochastic heat equation can be weakly intermittent in the presence of enough symmetry and nonlinearity. The basic idea is to follow our earlier work on space-time white noise [FK09] and apply the renewal-theory methods of Choquet and Deny [CD60]. However, this turns out to be a difficult adaptation which appears to require a fair bit of harmonic analysis.

It might help to recall the definition (1.15) [page 4] of the function Υ , and the relation [Theorem 1.5, page 5] between the positive number $\Upsilon(\beta)$ and the maximum value of the replica β -potential of the correlation function f , as well as the positive number $(\bar{R}_\beta f)(0)$ of that replica potential at the origin.

PROOF OF THEOREM 1.8. We can assume, without loss of generality, that there exists $\beta > 0$ such that

$$(5.41) \quad \Upsilon(\beta) \geq \frac{2}{L_\sigma^2},$$

for there is nothing left to prove otherwise. Now consider any such β .

Since $b(x) = 0$, $u_0 \geq \eta$, and $\sigma(u) \geq L_\sigma|u|$, we can apply (5.1) to deduce that for all $x, y \in \mathbf{R}^d$ and $t > 0$,

$$\begin{aligned}
& \mathbb{E}(|u_t(x)u_t(y)|) \\
(5.42) \quad & \geq \mathbb{E}(u_t(x)u_t(y)) \\
& \geq \eta^2 + \int_0^t ds \int_{\mathbf{R}^d} dz \int_{\mathbf{R}^d} dz' W_s(z, z') p_{t-s}(z-x) p_{t-s}(z'-y) f(z-z'),
\end{aligned}$$

where

$$\begin{aligned}
(5.43) \quad W_s(z, z') & := \mathbb{E}(\sigma(u_s(z))\sigma(u_s(z'))) \\
& \geq L_\sigma^2 \mathbb{E}(|u_s(z)u_s(z')|).
\end{aligned}$$

Consider the following \mathbf{R}_+ -valued functions on $(\mathbf{R}^d)^2$:

$$\begin{aligned}
(5.44) \quad H_\beta(a, b) & := \int_0^\infty e^{-\beta t} \mathbb{E}(|u_t(a)u_t(b)|) dt, \\
G_\beta(a, b) & := \int_0^\infty e^{-\beta t} p_t(a)p_t(b) dt, \\
F(a, b) & := f(a-b).
\end{aligned}$$

Also consider the linear operator \mathcal{A}_β defined as follows: For all nonnegative Borel-measurable functions $h : (\mathbf{R}^d)^2 \rightarrow \mathbf{R}_+$,

$$(5.45) \quad (\mathcal{A}_\beta h)(x, y) := (Fh * \tilde{G}_\beta)(x, y).$$

A line or two of computation shows that the preceding is simply a quick way to write the following:

$$\begin{aligned}
(5.46) \quad & (\mathcal{A}_\beta h)(x, y) \\
& = \int_{\mathbf{R}^d} da \int_{\mathbf{R}^d} db F(a, b) h(a, b) G_\beta(x-a, y-b) \\
& = \int_0^\infty e^{-\beta t} dt \int_{\mathbf{R}^d} da \int_{\mathbf{R}^d} db f(a-b) h(a, b) p_t(x-a) p_t(y-a).
\end{aligned}$$

With the preceding definitions under way, we can write equation (5.42) in short hand as follows:

$$(5.47) \quad H_\beta(x, y) \geq \frac{\eta^2}{\beta} + L_\sigma^2 (\mathcal{A}_\beta H_\beta)(x, y).$$

Since $F \geq 0$ on $(\mathbf{R}^d)^2$, we can apply the preceding to find the following pointwise bounds:

$$\begin{aligned}
(5.48) \quad H_\beta &\geq \frac{\eta^2}{\beta} \mathbf{1} + \mathbf{L}_\sigma^2 \left(\mathcal{A}_\beta \left\{ \frac{\eta^2}{\beta} + \mathbf{L}_\sigma^2(\mathcal{A}_\beta H_\beta) \right\} \right) \\
&= \frac{\eta^2}{\beta} \mathbf{1} + \mathbf{L}_\sigma^2 \frac{\eta^2}{\beta} \mathcal{A}_\beta \mathbf{1} + \mathbf{L}_\sigma^4 \mathcal{A}_\beta^2 H_\beta, \\
&\geq \frac{\eta^2}{\beta} \mathbf{1} + \mathbf{L}_\sigma^2 \frac{\eta^2}{\beta} \mathcal{A}_\beta \mathbf{1} + \mathbf{L}_\sigma^4 \mathcal{A}_\beta^2 \left(\frac{\eta^2}{\beta} + \mathbf{L}_\sigma^2 \mathcal{A}_\beta H_\beta \right) \\
&= \frac{\eta^2}{\beta} \mathbf{1} + \mathbf{L}_\sigma^2 \frac{\eta^2}{\beta} \mathcal{A}_\beta \mathbf{1} + \mathbf{L}_\sigma^4 \frac{\eta^2}{\beta} \mathcal{A}_\beta^2 \mathbf{1} + \mathbf{L}_\sigma^6 \mathcal{A}_\beta^3 H_\beta,
\end{aligned}$$

where $\mathbf{1}(x, y) := 1$ for all $x, y \in \mathbf{R}^d$. By applying induction we may arrive at the following simplified bound:

$$\begin{aligned}
(5.49) \quad H_\beta &\geq \frac{\eta^2}{\beta} \cdot \sum_{\ell=0}^k \mathbf{L}_\sigma^{2\ell} \mathcal{A}_\beta^\ell \mathbf{1} + \mathbf{L}_\sigma^{2(k+1)} \mathcal{A}_\beta^{k+1} H_\beta \\
&\geq \frac{\eta^2}{\beta} \cdot \sum_{\ell=0}^k \mathbf{L}_\sigma^{2\ell} \mathcal{A}_\beta^\ell \mathbf{1}.
\end{aligned}$$

Because the parameter $k \geq 0$ is arbitrary, it follows that

$$(5.50) \quad H_\beta \geq \frac{\eta^2}{\beta} \cdot \sum_{\ell=0}^{\infty} \mathbf{L}_\sigma^{2\ell} \mathcal{A}_\beta^\ell \mathbf{1}.$$

In order to better understand the behavior of this infinite sum, we begin by inspecting only the first few terms.

The first term in the sum is identically 1. And the more interesting second term can be written as \mathbf{L}_σ^2 multiplied by

$$\begin{aligned}
(5.51) \quad (\mathcal{A}_\beta \mathbf{1})(x, y) &= (F * \tilde{G}_\beta)(x, y) \\
&= \int_0^\infty e^{-\beta t} dt \int_{\mathbf{R}^d} da \int_{\mathbf{R}^d} db f(a-b) p_t(a-x) p_t(b-y) \\
&\geq \frac{1}{(2\pi)^d} \int_0^\infty e^{-\beta t} dt \int d\xi \hat{f}(\xi) e^{-2t \operatorname{Re} \Psi(\xi)} e^{i\xi \cdot (x-y)},
\end{aligned}$$

thanks to Proposition 3.6 [p. 31]. We change the order of the double integral $[dt d\xi]$ to find that

$$(5.52) \quad (\mathcal{A}_\beta \mathbf{1})(x, y) \geq \frac{1}{(2\pi)^d} \int \frac{e^{i\xi \cdot (x-y)} \hat{f}(\xi)}{\beta + 2 \operatorname{Re} \Psi(\xi)} d\xi.$$

[Theorem 1.5 and condition (1.1) together justify the use of Fubini's theorem.]

In order to bound the third term in the infinite sum in (5.50), we need to estimate the following quantity:

$$(5.53) \quad \begin{aligned} (\mathcal{A}_\beta^2 \mathbf{1})(x, y) &= \left(F(\mathcal{A}_\beta \mathbf{1}) * \tilde{G}_\beta \right)(x, y) \\ &\geq \frac{1}{(2\pi)^d} \int \frac{\hat{f}(\xi) d\xi}{\beta + 2\operatorname{Re}\Psi(\xi)} Z_\xi(x, y), \end{aligned}$$

where

$$(5.54) \quad Z_\xi(x, y) := \int_0^\infty e^{-\beta t} dt \int da \int db f(a-b) e^{i\xi \cdot (a-b)} p_t(a-x) p_t(b-y).$$

[The same ideas that were applied to the second term can be applied here, in exactly the same manner, to produce this bound.] The Fourier transform of the function

$$(5.55) \quad a \mapsto \exp(i\xi \cdot a) p_t(a-x)$$

is

$$(5.56) \quad \zeta \mapsto \exp(i(\xi + \zeta) \cdot x - t\Psi(\xi + \zeta)).$$

Therefore, Proposition 3.6 implies the following:

$$(5.57) \quad \begin{aligned} Z_\xi(x, y) &\geq \frac{1}{(2\pi)^d} \int_0^\infty e^{-\beta t} dt \int d\zeta \hat{f}(\zeta) e^{i(\xi + \zeta) \cdot (x-y) - 2t\operatorname{Re}\Psi(\xi + \zeta)} \\ &= \frac{1}{(2\pi)^d} \int \frac{e^{i(\xi + \zeta) \cdot (x-y)} \hat{f}(\zeta)}{\beta + 2\operatorname{Re}\Psi(\xi + \zeta)} d\zeta. \end{aligned}$$

And therefore,

$$(5.58) \quad \begin{aligned} (\mathcal{A}_\beta^2 \mathbf{1})(x, y) &\geq \frac{1}{(2\pi)^{2d}} \int \frac{\hat{f}(\xi_1) d\xi_1}{\beta + 2\operatorname{Re}\Psi(\xi_1)} \int \frac{\hat{f}(\xi_2) d\xi_2}{\beta + 2\operatorname{Re}\Psi(\xi_1 + \xi_2)} e^{i(\xi_1 + \xi_2) \cdot (x-y)}. \end{aligned}$$

Now, we can apply induction to deduce that we have the following estimate [used to analyse the ℓ th term in the infinite sum in (5.50)] in general: For all integers $\ell \geq 1$ and $x, y \in \mathbf{R}^d$,

$$(5.59) \quad \begin{aligned} (\mathcal{A}_\beta^\ell \mathbf{1})(x, y) &\geq \frac{1}{(2\pi)^{\ell d}} \int \frac{\hat{f}(\xi_1) d\xi_1}{\beta + 2\operatorname{Re}\Psi(\xi_1)} \int \frac{\hat{f}(\xi_2) d\xi_2}{\beta + 2\operatorname{Re}\Psi(\xi_1 + \xi_2)} \cdots \\ &\quad \cdots \int \frac{\hat{f}(\xi_\ell) d\xi_\ell}{\beta + 2\operatorname{Re}\Psi(\xi_1 + \cdots + \xi_\ell)} e^{i \sum_{j=1}^\ell \xi_j \cdot (x-y)}. \end{aligned}$$

Although the preceding multiple integral is manifestly nonnegative, the integrand itself is complex valued. Fortunately, we wish to only understand the behavior of $\mathcal{F}_\beta^\ell \mathbf{1}$ on the diagonal of $(\mathbf{R}^d)^2$. In that case, the integrand is real and nonnegative. As such, we can estimate the integrand directly. In order to do so, let us set $y := x$ in (5.59), and then plug the result in (5.50), to find that

$$(5.60) \quad \inf_{x \in \mathbf{R}^d} \int_0^\infty e^{-\beta t} \mathbf{E} (|u_t(x)|^2) dt \\ \geq \frac{\eta^2}{\beta} \cdot \sum_{\ell=0}^\infty \frac{L_\sigma^{2\ell}}{(2\pi)^{\ell d}} \int_{\mathbf{R}^d} d\xi_1 \cdots \int_{\mathbf{R}^d} d\xi_\ell \prod_{j=1}^\ell \frac{\hat{f}(\xi_j)}{(\beta + 2\operatorname{Re}\Psi(\xi_1 + \cdots + \xi_j))}.$$

A change of variables yields the following:

$$(5.61) \quad \inf_{x \in \mathbf{R}^d} \int_0^\infty e^{-\beta t} \mathbf{E} (|u_t(x)|^2) dt \\ \geq \frac{\eta^2}{\beta} \cdot \sum_{\ell=0}^\infty \frac{L_\sigma^{2\ell}}{(2\pi)^{\ell d}} \int_{\mathbf{R}^d} dz_1 \cdots \int_{\mathbf{R}^d} dz_d \prod_{j=1}^\ell \frac{\hat{f}(z_j - z_{j-1})}{(\beta + 2\operatorname{Re}\Psi(z_j))},$$

where $z_0 := 0$.

The preceding holds even without Condition 1.7 [p. 6]. But now we recall that Condition 1.7 is in place, and use it to produce the announced lower bound on $\inf_{x \in \mathbf{R}^d} \bar{\gamma}_x(2)$.

Define

$$(5.62) \quad \Sigma := \left\{ x := (x_1, \dots, x_d) \in \mathbf{R}^d : \operatorname{sign}(x_1) = \cdots = \operatorname{sign}(x_d) \right\}.$$

Equivalently, $\Sigma := \mathbf{R}_+^d \cup \mathbf{R}_-^d$. Because the terms under the product sign in (5.61) are all individually nonnegative, Condition 1.7 assures us that the following holds:

$$(5.63) \quad \inf_{x \in \mathbf{R}^d} \int_0^\infty e^{-\beta t} \mathbf{E} (|u_t(x)|^2) dt \\ \geq \frac{\eta^2}{\beta} \cdot \sum_{\ell=0}^\infty \frac{L_\sigma^{2\ell}}{(2\pi)^{\ell d}} \int_\Sigma dz_1 \cdots \int_\Sigma dz_d \prod_{j=1}^\ell \frac{\hat{f}(z_j - z_{j-1})}{(\beta + 2\operatorname{Re}\Psi(z_j))}.$$

If $z_1, \dots, z_d \in \Sigma$, then the absolute value of the k th coordinate of $z_j - z_{j-1}$ is less than or equal to the absolute value of the k th coordinate of z_j for all $k = 1, \dots, d$; therefore

$$(5.64) \quad \hat{f}(z_j - z_{j-1}) \geq \hat{f}(z_j),$$

thanks to Condition 1.7. Consequently,

$$\begin{aligned}
(5.65) \quad & \inf_{x \in \mathbf{R}^d} \int_0^\infty e^{-\beta t} \mathbf{E} (|u_t(x)|^2) dt \\
& \geq \frac{\eta^2}{\beta} \cdot \sum_{\ell=0}^\infty \frac{L_\sigma^{2\ell}}{(2\pi)^{\ell d}} \int_{\Sigma^\ell} dz \prod_{j=1}^\ell \frac{\hat{f}(z_j)}{(\beta + 2\operatorname{Re}\Psi(z_j))} \\
& = \frac{\eta^2}{\beta} \sum_{\ell=0}^\infty \left(\frac{L_\sigma^2}{(2\pi)^d} \int_{\Sigma} \frac{\hat{f}(z)}{(\beta + 2\operatorname{Re}\Psi(z))} dz \right)^\ell.
\end{aligned}$$

In particular, if there exists $\beta > 0$ such that

$$(5.66) \quad \frac{1}{(2\pi)^d} \int_{\Sigma} \frac{\hat{f}(\xi) d\xi}{\beta + 2\operatorname{Re}\Psi(\xi)} \geq L_\sigma^{-2},$$

then

$$(5.67) \quad \int_0^\infty e^{-\beta t} \mathbf{E} (|u_t(x)|^2) dt = \infty \quad \text{for all } x \in \mathbf{R}^d.$$

Thanks to symmetry considerations, Condition 1.7 has the following consequence:

$$\begin{aligned}
(5.68) \quad & \int_{\Sigma} \frac{\hat{f}(\xi) d\xi}{\beta + 2\operatorname{Re}\Psi(\xi)} = 2^{-d+1} \int_{\mathbf{R}^d} \frac{\hat{f}(\xi) d\xi}{\beta + 2\operatorname{Re}\Psi(\xi)} \\
& = 2^{-d+1} \Upsilon(\beta).
\end{aligned}$$

This and (5.66) together imply that (5.67) holds whenever

$$(5.69) \quad \Upsilon(\beta) \geq \frac{2^{d-1}}{L_\sigma^2}.$$

Now we can apply a real-variable argument to prove that if (5.67) holds for some $\beta > 0$, then

$$(5.70) \quad \bar{\gamma}_x(2) \geq \beta \quad \text{for all } x \in \mathbf{R}^d.$$

This and (5.41) together imply the theorem.

Indeed, let us suppose to the contrary that (5.67) holds for our β , and yet $\inf_{x \in \mathbf{R}^d} \bar{\gamma}_x(2) < \beta$ for the very same β . It follows immediately that there exists $x \in \mathbf{R}^d$, $\delta \in (0, \beta)$, and $C \in (0, \infty)$ such that

$$(5.71) \quad \mathbf{E} (|u_t(x)|^2) \leq C e^{(\beta-\delta)t} \quad \text{for all } t > 0.$$

And hence, (5.67) cannot hold in this case. This produces a contradiction, and shows that (5.67) implies the theorem. \square

PROOF OF THEOREM 1.11. We begin as we did with (5.42), but can no longer apply the inequality in (5.43). To circumvent that, note that for all $q_0 \in (0, q)$ there exists $A := A(q_0) \in (0, \infty)$ such that $\sigma(y) \geq q_0|y|$ as soon as $|y| > A$. We have assumed that $\mathbf{P}\{u_s(y) > 0\} = 1$ for all $s > 0$ and $y \in \mathbf{R}^d$. Therefore,

$$\begin{aligned}
(5.72) \quad W_s(z, z') &\geq q_0^2 \mathbf{E} \left(u_s(z) u_s(z') ; u_s(z) \wedge u_s(z') > A \right) \\
&\geq q_0^2 \mathbf{E} \left(u_s(z) u_s(z') \right) - q_0^2 A^2 - q_0 A \mathbf{E} \left(u_s(z) ; u_s(z) > A \right) \\
&\quad - q_0 A \mathbf{E} \left(u_s(z') ; u_s(z') > A \right) \\
&\geq q_0^2 \mathbf{E} \left(u_s(z) u_s(z') \right) - q_0^2 A^2 - q_0 A \left\{ \mathbf{E} \left(u_s(z) \right) + \mathbf{E} \left(u_s(z') \right) \right\}.
\end{aligned}$$

On the other hand, (5.1) guarantees that

$$\begin{aligned}
(5.73) \quad 0 &\leq \mathbf{E} \left(u_t(x) \right) \\
&= \left(P_t u_0 \right) (x) \\
&\leq \|u_0\|_{L^\infty(\mathbf{R}^d)}.
\end{aligned}$$

Consequently,

$$(5.74) \quad W_s(z, z') \geq q_0^2 \left\{ \mathbf{E} \left(u_s(z) u_s(z') \right) - A_* \right\},$$

where

$$(5.75) \quad A_* := \max \left(A^2, 2\|u_0\|_{L^\infty(\mathbf{R}^d)} \right).$$

Now we apply the recursion argument in the proof of Theorem 1.8, and find the following pointwise bounds:

$$\begin{aligned}
(5.76) \quad H_\beta &\geq \frac{\eta^2}{\beta} + q_0^2 \left\{ \mathcal{A}_\beta H_\beta - A_* \mathcal{A}_\beta \mathbf{1} \right\} \\
&\geq \frac{\eta^2}{\beta} + q_0^2 \left(\frac{\eta^2}{\beta} - A_* \right) \mathcal{A}_\beta \mathbf{1} + q_0^4 \mathcal{A}_\beta^2 H_\beta - q_0^4 A_* \mathcal{A}_\beta^2 \mathbf{1} \\
&\quad \vdots \\
&\geq \frac{\eta^2}{\beta} + \left(\frac{\eta^2}{\beta} - A_* \right) \sum_{\ell=1}^N q_0^{2\ell} \mathcal{A}_\beta^\ell \mathbf{1} + q_0^{2(N+1)} \left(\mathcal{A}_\beta^{N+1} H_\beta - A_* \mathcal{A}_\beta^{N+1} \mathbf{1} \right);
\end{aligned}$$

valid for every integer $N \geq 1$. We apply the following obvious inequalities: $\eta^2/\beta \geq \eta^2/\beta - A_*$ to the first term on the right; and $H_\beta \geq \eta^2/\beta$ to the last

bracketed term, to find that for all integers $N \geq 1$,

$$(5.77) \quad H_\beta \geq \left(\frac{\eta^2}{\beta} - A_* \right) \sum_{\ell=0}^{N+1} q_0^{2\ell} \mathcal{A}_\beta^\ell \mathbf{1}.$$

We can now let $N \uparrow \infty$ and apply the same estimate that we used to derive (5.65), and deduce the following bound:

$$(5.78) \quad \begin{aligned} \inf_{x \in \mathbf{R}^d} \int_0^\infty e^{-\beta t} \mathbf{E} (|u_t(x)|^2) dt \\ \geq \left(\frac{\eta^2}{\beta} - A_* \right) \cdot \sum_{\ell=0}^\infty \left(\frac{q_0^2}{(2\pi)^d} \int_\Sigma \frac{\hat{f}(z)}{(\beta + 2\operatorname{Re}\Psi(z))} dz \right)^\ell \\ = \left(\frac{\eta^2}{\beta} - A_* \right) \cdot \sum_{\ell=0}^\infty \left(\frac{q_0^2}{2^{d-1}} \Upsilon(\beta) \right)^\ell. \end{aligned}$$

Thanks to Theorem 1.5, the preceding implies the following bound:

$$(5.79) \quad \inf_{x \in \mathbf{R}^d} \int_0^\infty e^{-\beta t} \mathbf{E} (|u_t(x)|^2) dt \geq \left(\frac{\eta^2}{\beta} - A_* \right) \cdot \sum_{\ell=0}^\infty \left(\frac{q_0^2}{2^{d-1}} (\bar{R}_\beta f)(0) \right)^\ell.$$

Because $(\bar{R}_0 f)(0) = \infty$, we can find a $\beta_0 > 0$ such that

$$(5.80) \quad (\bar{R}_{\beta_0} f)(0) \geq \frac{2^{d-1}}{q_0^2}.$$

For that choice of β_0 , the preceding sum diverges. Therefore,

$$(5.81) \quad \int_0^\infty e^{-\beta_0 t} \mathbf{E} (|u_t(x)|^2) dt = \infty \quad \text{provided that} \quad \eta > \sqrt{\beta_0 A_*}.$$

This and an elementary real-variable argument; (5.71) and together prove the theorem. \square

5.3. The Massive and Dissipative Operators

David Nualart asked us about the effect of the drift coefficient b in (1.1) on the intermittent behavior of the solution to the stochastic heat equation (1.1). At present, we have only an answer to this in a special but physically-interesting family of cases.

Indeed, let us consider the stochastic heat equation

$$(5.82) \quad \frac{\partial}{\partial t} u_t(x) = (\mathcal{L}u_t)(x) + \frac{\lambda}{2} u_t(x) + \sigma(u_t(x)) \dot{F}_t(x),$$

where $x \in \mathbf{R}^d$, $t > 0$, $\lambda \in \mathbf{R}$, and \dot{F} is as before. Moreover, $\sigma : \mathbf{R} \rightarrow \mathbf{R}$ is Lipschitz continuous, also as before. That is, (5.82) corresponds to the drift-free stochastic heat equation for the *massive operator* $\mathcal{L}^{(\lambda)} := \mathcal{L} + (\lambda/2)I$ when $\lambda > 0$, the *dissipative operator* $\mathcal{L}^{(\lambda)} = \mathcal{L} - |\lambda/2|I$ when $\lambda < 0$, and the *free operator* $\mathcal{L}^{(0)} = \mathcal{L}$ when $\lambda = 0$.² Of course, Theorem 1.2—applied with $b(u) := \lambda u/2$ —guarantees us of the existence and uniqueness of a mild solution to (5.82).

The operator $\mathcal{L}^{(\lambda)}$ is the generator of the semigroup $\{P_t^{(\lambda)}\}_{t \geq 0}$ defined by

$$(5.83) \quad (P_t^{(\lambda)}\phi)(x) := e^{\lambda t/2}(P_t\phi)(x).$$

This can be seen immediately by a semi-formal differential of $t \mapsto P_t^{(\lambda)}$ at $t = 0$; and it is easy to make the argument rigorous as well. The corresponding “transition functions” are given by

$$(5.84) \quad p_t^{(\lambda)}(y - x) := e^{\lambda t/2}p_t(y - x).$$

Then the domain $\text{Dom}[\mathcal{L}^{(\lambda)}]$ of the definition of $\mathcal{L}^{(\lambda)}$ is the same as $\text{Dom}[\mathcal{L}]$, and

$$(5.85) \quad \mathcal{L}^{(\lambda)}\phi = \mathcal{L}\phi + \frac{\lambda}{2}\phi \quad \text{for all } \phi \in \text{Dom}[\mathcal{L}^{(\lambda)}].$$

Let $\{P_t^{*(\lambda)}\}_{t \geq 0}$ denote the adjoint [or dual, in probabilistic terms] semigroup. That is,

$$(5.86) \quad (P_t^{*(\lambda)}\phi)(x) := e^{\lambda t/2}(P_t^*\phi)(x).$$

with corresponding transition functions,

$$(5.87) \quad p_t^{*(\lambda)}(y - x) := e^{\lambda t/2}p_t(x - y).$$

And finally there is also a corresponding replica semigroup,

$$(5.88) \quad (\bar{P}_t^{(\lambda)}\phi)(x) := e^{\lambda t}(\bar{P}_t\phi)(x),$$

whose resolvent is described by the following:

$$(5.89) \quad (\bar{R}_\alpha^{(\lambda)}\phi)(x) := \int_0^\infty e^{-(\alpha-\lambda)s}(\bar{P}_s\phi)(x) ds \quad \text{for all } \alpha \geq \lambda.$$

We might note that $\bar{R}_\alpha^{(\lambda)}f = \bar{R}_{\alpha-\lambda}f$ is merely a shift of the the free replica resolvent of f . Therefore, the proof of Theorem 1.2 goes through unhindered, and after accounting for the mentioned shift, produces the following:

²The operator $\mathcal{L}^{(\lambda)}$ is also known as the “relativistic” form of \mathcal{L} .

THEOREM 5.6. *Suppose $u_0 : \mathbf{R}^d \rightarrow \mathbf{R}$ is bounded and measurable. Then, under Condition 1.1, the mild solution to (5.82) satisfies the following: For all integers $p \geq 2$ and $\lambda \in \mathbf{R}$,*

$$(5.90) \quad \bar{\gamma}_*(p) \leq \lambda + \frac{p}{2} \inf \left\{ \alpha > 0 : (\bar{R}_\alpha f)(0) < \frac{1}{z_p^2 \text{Lip}_\sigma^2} \right\},$$

where z_p denotes the largest positive zero of the Hermite polynomial He_p .

Recall the function Q of Theorem 1.2. Since

$$(5.91) \quad Q(p, \beta) \geq \max \left(\frac{p\lambda}{2\beta}, z_p \text{Lip}_\sigma \sqrt{(\bar{R}_{2\beta/p} f)(0)} \right)$$

a few lines of arithmetic show that Theorem 5.6 provides us with a better upper bound than Theorem 1.2 for the top Liapounov L^p -exponent of the mild solution to (1.1). Next, we produce instances where the solution is intermittent.

First of all, note that according to (5.90),

$$(5.92) \quad \bar{\gamma}_*(2) \leq \lambda + \inf \left\{ \alpha > 0 : (\bar{R}_\alpha f)(0) < \frac{1}{\text{Lip}_\sigma^2} \right\},$$

because $z_2 = 2$. We apply similar “shifting arguments” together with Theorem 1.8 to deduce that the following offers a converse, under some symmetry and regularity conditions. We note, in advance, that when $d = 1$, the preceding estimate and the following essentially match up.

THEOREM 5.7. *Suppose that both Conditions 1.1 and 1.7 hold, $\eta := \inf_{x \in \mathbf{R}^d} u_0(x) > 0$, and there exists $L_\sigma \in (0, \infty)$ such that $\sigma(z) \geq L_\sigma |z|$ for all $z \in \mathbf{R}$. Then,*

$$(5.93) \quad \inf_{x \in \mathbf{R}^d} \bar{\gamma}_x(2) \geq \lambda + \sup \left\{ \alpha > 0 : (\bar{R}_\alpha f)(0) \geq \frac{2^{d-1}}{L_\sigma^2} \right\},$$

where $\sup \emptyset := 0$.

We end this chapter with the example mentioned on page 10 of the Introduction.

EXAMPLE 5.8. Consider the case that $\mathcal{L} = -(-\Delta)^{q/2}$ is a power of the Laplacian. In that case, \mathcal{L} is the generator of an isotropic stable process of index q , and $q \in (0, 2]$ necessarily. Consider also the case that $f(x) = \|x\|^{-d+b}$ is a Riesz kernel, where $b \in (0, d)$. Then [see (3.25) on page 30],

$$(5.94) \quad \hat{f}(\xi) = \frac{C_{d,b}}{\|\xi\|^b}, \quad \text{where} \quad C_{d,b} := \frac{\pi^{d/2} 2^b \Gamma(b/2)}{\Gamma((d-b)/2)}.$$

And therefore,

$$\begin{aligned}
(\bar{R}_\alpha f)(0) &= \Upsilon(\alpha) && \text{[by Theorem 1.5]} \\
(5.95) \quad &= \frac{C_{d,b}}{(2\pi)^d} \cdot \int_{\mathbf{R}^d} \frac{\|\xi\|^{-b}}{\alpha + 2\|\xi\|^q} d\xi \\
&= \frac{C_{d,b}\alpha^{-1+(d-b)/q}}{(2\pi)^d \cdot 2^{(d-b)/q}} \cdot \int_{\mathbf{R}^d} \frac{dz}{\|z\|^b + \|z\|^{q+b}} \quad [z := (2/\alpha)^{1/q}\xi].
\end{aligned}$$

In other words,

$$(5.96) \quad (\bar{R}_\alpha f)(0) = \frac{\mathfrak{A}_{d,q,b}}{\alpha^{1-\nu}},$$

where

$$(5.97) \quad \nu := \frac{d-b}{q} \quad \text{and} \quad \mathfrak{A}_{d,q,b} := \frac{C_{d,b}}{(2\pi)^d \cdot 2^\nu} \cdot \int_{\mathbf{R}^d} \frac{dz}{\|z\|^b + \|z\|^{q+b}}.$$

Since $b \in (0, d)$, $\mathfrak{A}_{d,q,b}$ —and hence $(\bar{R}_\alpha f)(0)$ —is finite if and only if $q + b > d$; this is the sufficient condition for the existence of a unique mild solution to the resulting stochastic PDE. And not surprisingly, this “ $q + b > d$ ” condition is the necessary and sufficient condition for the existence of a solution to the linear equation [Theorem 4.5, p. 48]. Moreover, when $q + b > d$, we can apply Theorems 5.6 and 5.7 to find that for all $x \in \mathbf{R}^d$,

$$(5.98) \quad \lambda + \left(\frac{\mathfrak{A}_{d,q,b} L_\sigma^2}{2^{d-1}} \right)^{1/(1-\nu)} \leq \bar{\gamma}_x(2) \leq \lambda + (\mathfrak{A}_{d,q,b} \text{Lip}_\sigma^2)^{1/(1-\nu)}.$$

In particular, consider the massive/dissipative “parabolic Anderson model,”

$$(5.99) \quad \frac{\partial}{\partial t} u_t(x) = - \left((\Delta)^{q/2} u_t \right) (x) + \frac{\lambda}{2} u_t(x) + \kappa u_t(x) \dot{F}_t(x),$$

where $u_0 : \mathbf{R} \rightarrow \mathbf{R}$ is measurable and bounded uniformly away from zero and infinity, and $\kappa \neq 0$. The preceding discussion shows that the parabolic Anderson model has a solution if $q + b > d$. And when $q + b > d$, we obtain the following bounds for the upper L^2 -Liapounov exponent of the solution: For all $x \in \mathbf{R}^d$,

$$(5.100) \quad \lambda + \left(\frac{\mathfrak{A}_{d,q,b} \kappa^2}{2^{d-1}} \right)^{1/(1-\nu)} \leq \bar{\gamma}_x(2) \leq \lambda + (\mathfrak{A}_{d,q,b} \kappa^2)^{1/(1-\nu)}.$$

Theorem 1.2 shows also that $\bar{\gamma}_*(p) < \infty$ for all $p \geq 2$. Therefore, we have no weak intermittency if $\lambda \leq -(\mathfrak{A}_{d,q,b} \kappa^2)^{1/(1-\nu)}$, whereas there is weak intermittency if $\lambda > -(\mathfrak{A}_{d,q,b} \kappa^2 / (2^{d-1}))^{1/(1-\nu)}$. Our condition is sharp when, and only when, $d = 1$. In that one-dimensional case, we have a solution if

and only if $q + b > 1$, and if this inequality holds then

$$(5.101) \quad \begin{aligned} \mathfrak{A}_{1,q,b} &= \frac{C_{1,b}}{2^\nu \pi} \cdot \int_0^\infty \frac{dz}{z^b + z^{b+q}} \\ &= \frac{C_{1,b}}{2^\nu \pi q} \cdot \text{B} \left(\frac{1-b}{q}, 1 - \frac{1-b}{q} \right), \end{aligned}$$

where $\text{B}(x, y) := \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the beta function. Thanks to (5.94), this implies that

$$(5.102) \quad \mathfrak{A}_{1,q,b} = \frac{2^{b-\nu}\Gamma(b/2)}{\pi^{1/2}q\Gamma((1-b)/2)} \text{B} \left(\frac{1-b}{q}, 1 - \frac{1-b}{q} \right),$$

and we find that weak intermittency holds if and only if

$$(5.103) \quad \lambda > - \left(\mathfrak{A}_{1,q,b} \kappa^2 \right)^{1/(1-\nu)}.$$

Another simple though tedious computation shows that this example [applied with $q := 2$] includes the material that led to (1.26) of page 10.

CHAPTER 6

Temperate Solutions to Parabolic SPDEs

In this chapter we continue the study of the stochastic heat equation of the following form:

$$(6.1) \quad \frac{\partial}{\partial t} u_t(x) = (\mathcal{L}u_t(x)) + \sigma(u_t(x))\dot{F}_t(x),$$

where $\sigma : \mathbf{R}^d \rightarrow \mathbf{R}$ is Lipschitz continuous, as before. But here u_0 can be a finite Borel measure.

Such a problem arises naturally in various specific cases; see, for example, Bertini and Cancrini [BC95], Carmona and Molchanov [CM94], and Molchanov [Mo191]. It is also motivated by the physical literature on statistical mechanics, where u_0 typically represents the initial [probability] distribution of a particle system in a random environment that interacts with it random environment. In fact, the role of u_0 as a measure is typically taken for granted and little or no mention of u_0 is made explicitly in the physics literature [KS91, Kar87, KPZ86].

Thus, our present goal is to continue and produce solutions to (6.1), even though the initial data is a measure.

The fact that u_0 is no longer a function poses some fundamental problems for the existing theory of SPDEs. In fact, we argue next that in order to make sense of (6.1) in the case that u_0 is a finite Borel measure, we need to impose regularity conditions on both the semigroup $\{P_t\}_{t>0}$ and the initial measure. More importantly, we need to develop a slightly less restrictive concept of a solution than the well known, as well as standard, notion of a mild solution.

Let us begin with a few observations.

It follows from the properties of the Walsh stochastic integral [Wal86, Chapter 2] that if \mathbf{u} is a predictable random field which is the mild solution

of (5.1) [with $b \equiv 0$] and satisfies (5.2), then

$$(6.2) \quad \begin{aligned} & \mathbb{E} \left(|u_t(x)|^2 \right) \\ &= |(P_t u_0)(x)|^2 + \mathbb{E} \left(\left| \int_0^t \int_{\mathbf{R}^d} p_{t-s}(y-x) \sigma(u_s(y)) F(ds dy) \right|^2 \right) \\ &\geq |(P_t u_0)(x)|^2, \end{aligned}$$

where as usual,

$$(6.3) \quad (P_t u_0)(x) := \int_{\mathbf{R}^d} p_t(y-x) u_0(dy).$$

Thus, in particular, (5.2) implies that the following is a necessary condition for the existence of a mild solution:

$$(6.4) \quad \sup_{t \in (0, T)} \sup_{x \in \mathbf{R}^d} |(P_t u_0)(x)| < \infty \quad \text{for all } T > 0.$$

It is clear that the latter condition rules out many interesting choices of u_0 . For instance, consider the case that $\mathcal{L} = \Delta$ and one of the most natural choices $u_0 := \delta_z$ for initial data. In that case,

$$(6.5) \quad (P_t \delta_z)(a) = \frac{e^{-\|a-z\|^2/(4t)}}{(4\pi t)^{d/2}},$$

and hence

$$(6.6) \quad \sup_{t \in (0, T)} \sup_{x \in \mathbf{R}^d} |(P_t \delta_z)(x)| = \infty \quad \text{for all } T > 0 \text{ and } z \in \mathbf{R}^d.$$

Thus, we can never have a mild solution to (6.1) when the initial data is a point mass. But when $\sigma(u) = \text{const} \cdot u$ and $\mathcal{L} = \Delta$, (6.1) is expected to have a solution defined by the Feynman–Kac formula in many cases where f is “nice”; see Bertini and Cancrini [BC95] and Carmona and Molchanov [CM94], for example. It is therefore natural to try to use a different definition of a solution to handle such problems.

It turns out that one can turn the suggested strategy into a fruitful rigorous approach. In fact, in this chapter we consider a different notion of solution which satisfies, among other things, the following condition in place of the more restrictive condition (5.2):

$$(6.7) \quad \sup_{x \in \mathbf{R}^d} \mathbb{E} \left(|u_t(x)|^2 \right) < \infty \quad \text{for almost every } t > 0.$$

Now suppose we want to know when (6.1), with $u_0 = \delta_z$, has a random-field solution which satisfies (6.7). Similar considerations as those of the

previous paragraph tell us that a necessary condition is that p_t is a bounded function for almost all $t > 0$. Thanks to Hawkes's theorem [Proposition 2.3, p. 19], (6.7) implies that

$$(6.8) \quad \exp(-\operatorname{Re}\Psi) \in L^t(\mathbf{R}^d) \quad \text{for all } t > 0.$$

Thus, we will need to assume, at the very least, that (6.8) holds.

Recall that a mild solution \mathbf{u} of (6.1) is a predictable random field that satisfies (5.1) with $b \equiv 0$. To be specific, one requires that for all nonrandom pairs $(t, x) \in (0, \infty) \times \mathbf{R}^d$, the random equation (5.1) holds almost surely. The following is a slightly less stringent notion of a solution.

DEFINITION 6.1. Let $\mathbf{u} := \{u_t(x)\}_{t>0, x \in \mathbf{R}^d}$ be a predictable random field. We say that \mathbf{u} is a *temperate solution* to (6.1) if there exists a null set $N_0 \subset \mathbf{R}_+$ such that for all $t \notin N_0$, the following holds for every $x \in \mathbf{R}^d$:

$$(6.9) \quad u_t(x) = (P_t u_0)(x) + \int_0^t \int_{\mathbf{R}^d} p_{t-s}(y-x) \sigma(u_s(y)) F(ds dy) \quad \text{a.s.}$$

As we shall see in the next section, the stochastic integral in (6.9) can be defined properly though, technically speaking, it is not a Walsh integral. Regardless, we have the following, which is the main result of this chapter.

THEOREM 6.2. *Suppose $\sigma : \mathbf{R}^d \rightarrow \mathbf{R}$ is Lipschitz continuous and u_0 is a finite Borel measure on \mathbf{R}^d such that*

$$(6.10) \quad \int_{\mathbf{R}^d} \frac{|\hat{u}_0(\xi)| + \hat{f}(\xi)}{1 + 2\operatorname{Re}\Psi(\xi)} d\xi < \infty.$$

If, in addition, (6.8) holds, then there exists a temperate solution $\mathbf{u} := \{u_t(x)\}_{t>0, x \in \mathbf{R}^d}$ to (6.1).

There is also a corresponding [limited] uniqueness result for these temperate solutions; see Proposition 6.14.

Let us remark that (6.10) holds if and only if:

- (a) Condition (1.1) holds; and
- (b)

$$(6.11) \quad \int_{\mathbf{R}^d} \frac{|\hat{u}_0(\xi)|}{1 + 2\operatorname{Re}\Psi(\xi)} d\xi < \infty.$$

In addition, in the case that u_0 is a positive-definite function, the preceding display holds if and only if u_0 has bounded potentials; more precisely, that $(\bar{R}_\alpha u_0)(0) = \sup_{x \in \mathbf{R}^d} (\bar{R}_\alpha u_0)(x) < \infty$ for all $\alpha > 0$; see Theorem 1.5 for a more precise statement.

COROLLARY 6.3. *Suppose $d = 1$, $\sigma : \mathbf{R} \rightarrow \mathbf{R}$ is Lipschitz continuous, and Condition 1.1 and the following are both valid:*

$$(6.12) \quad \int_{-\infty}^{\infty} \frac{d\xi}{1 + 2\operatorname{Re}\Psi(\xi)} < \infty.$$

Then, (6.1) has a temperate solution for every finite initial measure u_0 .

REMARK 6.4. Condition (6.12) is equivalent to the property that the replica process \bar{X} has local times; see Hawkes [Haw86]. In that case, it can be shown, as in our earlier work with E. Nualart [FKN09], that (6.1) has a temperate solution even when \dot{F} is space-time white noise. We have stated Corollary 6.3 for $d = 1$ only because (2.14) [see p. 15] implies fairly readily that (6.12) can never hold when $d \geq 2$. \square

REMARK 6.5. Corollary 6.3 has the following remarkable consequence: When (6.12) holds, the stochastic heat equation (6.1) has a temperate solution for every finite measure u_0 ; this includes every $u_0 \in L^1(\mathbf{R})$. This property fails to hold for a mild solution that satisfies the usual integrability condition (5.2). \square

We conclude this section with our final main result concerning temperate solutions to (6.1). This result yields an upper bound for the growth of the solution. Since our solution is not in general a mild one, we will need to also adapt our previously-used notion of *Liapounov exponents*. Thus, we introduce the following, which seems to have a number of desirable mathematical properties.

DEFINITION 6.6. We define the *integrated $L^p(\mathbf{P})$ -Liapounov exponent* $\lambda_*(p)$ of the temperate solution to (6.1)—when it exists—as

$$(6.13) \quad \lambda_*(p) := \inf \left\{ \beta > 0 : \int_0^\infty e^{-\beta t} \left\{ \sup_{x \in \mathbf{R}^d} \mathbf{E} (|u_t(x)|^p) \right\}^{2/p} dt < \infty \right\},$$

where $\inf \emptyset := \infty$.

THEOREM 6.7. *Suppose (6.8) holds, $\sigma : \mathbf{R} \rightarrow \mathbf{R}$ is Lipschitz continuous and nonrandom, and u_0 is a nonrandom finite Borel measure on \mathbf{R}^d that satisfies (6.10). Then the temperate solution \mathbf{u} to (6.1) satisfies the following:*

$$(6.14) \quad \lambda_*(p) \leq \inf \left\{ \beta > 0 : (\bar{R}_\beta f)(0) < \frac{1}{2z_p^2 \operatorname{Lip}_\sigma^2} \right\}.$$

According to Theorem 1.5, the integral condition (6.10) implies that $(\bar{R}_\alpha f)(0) < \infty$ for all $\alpha > 0$, whence $\lim_{\beta \rightarrow \infty} (\bar{R}_\beta f)(0) = 0$, thanks to the dominated convergence theorem. Consequently, Theorem 6.7 implies, among other things, that $\lambda_*(p) < \infty$ for all $p \geq 2$.

It would be interesting to find nontrivial conditions on f and \mathcal{L} that ensure that the preceding temperate solution satisfies $\lambda_*(2) > 0$, and in this way derive a version of weak intermittency for temperate solutions. Such an undertaking would also require nondegeneracy conditions on u_0 . But it seems hard to find nontrivial conditions that can be placed on an initial measure u_0 that guarantee the strict positivity of $\lambda_*(2)$ without assuming that u_0 is a bounded measurable function [and not just a measure] which is also bounded away from zero.

We end this section by making a final observation about weak intermittency.

Suppose $(\bar{R}_0 f)(0) < \infty$. Then, $\lambda_*(p) = 0$ provided that

$$(6.15) \quad \text{Lip}_\sigma < \frac{1}{\sqrt{2z_p^2(\bar{R}_0 f)(0)}}.$$

We can apply (6.15) with $p = 2$ to deduce that if σ is sufficiently smooth in the sense that

$$(6.16) \quad \text{Lip}_\sigma < \frac{1}{\sqrt{2(\bar{R}_0 f)(0)}},$$

then $\lambda_*(2) = 0$, and “weak intermittency” [with respect to the new Liapounov exponents $\{\lambda_*(p)\}_{p>2}$] does not hold.

6.1. Stochastic Convolutions

Define for all predictable random fields $Z := \{Z_t(x)\}_{t>0, x \in \mathbf{R}^d}$,

$$(6.17) \quad (\tilde{p} * Z \dot{F})_t(x) := \int_0^t \int_{\mathbf{R}^d} p_{t-s}(y-x) Z_s(y) F(ds dy),$$

provided that the preceding Walsh-type stochastic integral exists. Note, in particular, that whenever $v := \{v_t(x)\}_{t>0, x \in \mathbf{R}^d}$ is a predictable random field,

$$(6.18) \quad \mathcal{A}v = \tilde{p} * (\sigma \circ v) \dot{F},$$

provided that the Walsh stochastic integrals are well defined.

According to the theory of Walsh [Wal86], the stochastic-convolution process $\tilde{p} * Z\dot{F}$ is well defined provided that the quantity

$$(6.19) \quad \int_0^t ds \int_{\mathbf{R}^d} dy \int_{\mathbf{R}^d} dz \mathbb{E}(|Z_s(y)Z_s(z)|) p_{t-s}(y-x)p_{t-s}(z-x)f(y-z)$$

is finite for all choices of $(t, x) \in (0, \infty) \times \mathbf{R}^d$. In that case,

$$(6.20) \quad \begin{aligned} & \mathbb{E} \left(\left| (\tilde{p} * Z\dot{F})_t(x) \right|^2 \right) \\ &= \int_0^t ds \int_{\mathbf{R}^d} dy \int_{\mathbf{R}^d} dz \mathbb{E} (Z_s(y)Z_s(z)) p_{t-s}(y-x)p_{t-s}(z-x)f(y-z). \end{aligned}$$

Define for all $\beta > 0$ and all predictable random fields $\mathbf{v} := \{v_t(x)\}_{t>0, x \in \mathbf{R}^d}$,

$$(6.21) \quad \mathcal{N}_\beta(\mathbf{v}) := \left(\int_0^\infty e^{-\beta t} \sup_{x \in \mathbf{R}^d} \mathbb{E}(|v_t(x)|^2) dt \right)^{1/2}.$$

Each \mathcal{N}_β is a norm on equivalence classes of square-integrable predictable processes that are modifications of one another.

DEFINITION 6.8. Let $Z := \{Z_t(x)\}_{t>0, x \in \mathbf{R}^d}$ be a random field. We say that Z is (p, F) -Walsh integrable if Z is predictable and the quantity in (6.19) is finite. We frequently abuse notation and write “Walsh integrable” in place of “ (p, F) -Walsh integrable.”

In other words, Walsh-integrable random fields are random fields for which the stochastic-convolution process $\tilde{p} * Z\dot{F}$ is defined by the stochastic integration theory of Walsh [Wal86].

The following is a key “embedding” theorem for our extension of stochastic convolutions.

LEMMA 6.9. *If $Z := \{Z_t(x)\}_{t>0, x \in \mathbf{R}^d}$ is Walsh integrable, then*

$$(6.22) \quad \mathcal{N}_\beta(\tilde{p} * Z\dot{F}) \leq \mathcal{N}_\beta(Z) \cdot \sqrt{(\bar{R}_\beta f)(0)} \quad \text{for all } \beta > 0.$$

PROOF. Owing to (6.20), the following is valid for all $t > 0$ and $x \in \mathbf{R}^d$:

$$\begin{aligned}
& \mathbb{E} \left(\left| (\tilde{p} * Z\dot{F})_t(x) \right|^2 \right) \\
& \leq \int_0^t \sup_{a \in \mathbf{R}^d} \mathbb{E} \left(|Z_s(a)|^2 \right) ds \int_{\mathbf{R}^d} dy \int_{\mathbf{R}^d} dz p_{t-s}(y-x) p_{t-s}(z-x) f(y-z) \\
(6.23) \quad & = \int_0^t \sup_{a \in \mathbf{R}^d} \mathbb{E} \left(|Z_s(a)|^2 \right) (\bar{P}_{t-s} f)(0) ds.
\end{aligned}$$

Since the right-most quantity is independent of $x \in \mathbf{R}^d$, we can take the supremum over all $x \in \mathbf{R}^d$, multiply the preceding by $\exp(-\beta t)$ and then integrate $[dt]$ to deduce the lemma. \square

We now use Lemma 6.9 to extend the Walsh definition of a stochastic convolution as follows. Suppose $(\bar{R}_\alpha f)(0) < \infty$ for some $\alpha > 0$; because of Theorem 1.5 we know that $(\bar{R}_\beta f)(0) < \infty$ for all $\beta > 0$. Define \mathbf{L}_β^2 to be the collection of all [equivalence classes of modifications of] predictable random fields Z such that $\mathcal{N}_\beta(Z) < \infty$.¹ Clearly, \mathbf{L}_β^2 is a Banach space for every $\beta > 0$.

LEMMA 6.10. *If $Z \in \mathbf{L}_\beta^2$ for some $\beta > 0$, then there exist $Z^1, Z^2, \dots \in \mathbf{L}_\beta^2$ such that $\lim_{n \rightarrow \infty} Z^n = Z$ in \mathbf{L}_β^2 , and Z^n is Walsh integrable for all $n \geq 1$.*

PROOF. We may assume without loss of generality that $Z_t(x) \geq 0$ for all $t > 0$ and $x \in \mathbf{R}^d$; for otherwise we can consider the predictable random fields $(Z_t(x))^+$ and $(Z_t(x))^-$ separately [they too are in \mathbf{L}_β^2].

For all $t > 0$, $x \in \mathbf{R}^d$, and $n \geq 1$, let $Z_t^n(x)$ denote the minimum of $Z_t(x)$ and n . Then Z^n converges in \mathbf{L}_β^2 to Z as $n \rightarrow \infty$, thanks to the dominated

¹As is customary in the study of L^p spaces, we treat the elements of \mathbf{L}_β^2 slightly carelessly as if they are random fields, rather than equivalence classes of random fields. This is done, as in the case of L^p spaces, merely to simplify the otherwise-cumbersome notation. There should be no loss in precision, as ought to be clear from the context.

convergence theorem. And

$$\begin{aligned}
& \int_0^t ds \int_{\mathbf{R}^d} dy \int_{\mathbf{R}^d} dz \mathbb{E} (|Z_s^n(y) Z_s^n(z)|) p_{t-s}(y-x) p_{t-s}(z-x) f(y-z) \\
& \leq n^2 \int_0^t ds \int_{\mathbf{R}^d} dy \int_{\mathbf{R}^d} dz p_s(y-x) p_s(z-x) f(y-z) \\
(6.24) \quad & = n^2 \int_0^t (\bar{P}_s f)(0) ds \\
& \leq n^2 e^{\beta t} (\bar{R}_\beta f)(0),
\end{aligned}$$

which is finite. \square

Now let us choose and fix some $Z \in \mathbf{L}_\beta^2$. According to Lemma 6.10 we can find $Z^n \rightarrow Z$ [in \mathbf{L}_β^2] such that every Z^n is (p, F) -Walsh integrable. Lemma 6.9 tells us that $\lim_{n \rightarrow \infty} (\tilde{p} * Z^n \dot{F})$ exists in \mathbf{L}_β^2 . Thus, we can define the stochastic-convolution process $\tilde{p} * Z \dot{F}$ for every $Z \in \mathbf{L}_\beta^2$ as follows:

$$(6.25) \quad (\tilde{p} * Z \dot{F})_t(x) := \lim_{n \rightarrow \infty} (\tilde{p} * Z^n \dot{F})_t(x),$$

where the limit takes place in \mathbf{L}_β^2 . Of course, the preceding limit does not depend on the choice of $\{Z^n\}_{n=1}^\infty$. But it also does not depend on β because

$$(6.26) \quad \mathbf{L}_\beta^2 \subseteq \mathbf{L}_\alpha^2 \quad \text{if } \alpha \leq \beta.$$

We abuse notation slightly and write for all $t > 0$ and $x \in \mathbf{R}^d$,

$$(6.27) \quad \int_0^t \int_{\mathbf{R}^d} p_{t-s}(y-x) Z_s(y) F(ds dy) := (\tilde{p} * Z \dot{F})_t(x) \quad \text{for } Z \in \bigcup_{\beta > 0} \mathbf{L}_\beta^2.$$

It is easy to see that many of the standard properties of the Walsh stochastic convolution transfer, by limiting arguments, to those of the present extension. Chief among them is the fact that $Z \mapsto (\tilde{p} * Z \dot{F})$ is a random linear map from \mathbf{L}_β^2 to itself. In other words, if $Z, W \in \mathbf{L}_\beta^2$ and $a, b \in \mathbf{R}$, then

$$(6.28) \quad \tilde{p} * (aZ + bW) \dot{F} = a(\tilde{p} * Z \dot{F}) + b(\tilde{p} * W \dot{F}).$$

We emphasize that the preceding means that, as elements of \mathbf{L}_β^2 , the two sides agree. In particular, because of the Fubini theorem, we obtain the following equivalent formulation of (6.28): There exists a null set $N_0 \subset \mathbf{R}_+$ such that for all $t \notin N_0$ and all $x \in \mathbf{R}^d$,

$$(6.29) \quad \mathbb{P} \left\{ \left(\tilde{p} * (aZ + bW) \dot{F} \right)_t(x) = a \left(\tilde{p} * Z \dot{F} \right)_t(x) + b \left(\tilde{p} * W \dot{F} \right)_t(x) \right\}$$

is equal to one.

We conclude this section by emphasizing that the stochastic convolution process $t \mapsto (\tilde{p} * Z\dot{F})_t$ is hereby not defined for all $t > 0$. However, it is defined for almost all $t > 0$; and if $(\tilde{p} * Z\dot{F})_t$ is defined for a given $t > 0$, then $(\tilde{p} * Z\dot{F})_t(x)$ is well-defined for every $x \in \mathbf{R}^d$, and $x \mapsto (\tilde{p} * Z\dot{F})_t(x)$ is a random field in the usual sense.

Our stochastic convolution is thus quite different from the existing ones in the literature. For example, for the most commonly-used infinite-dimensional stochastic convolution [see, for example, Chapter 5 of Da Prato and Zabczyk [DPZ92, §5.1.2]], each random process $x \mapsto (\tilde{p} * Z\dot{F})_t(x)$ is defined for every $t > 0$, typically as an element of a certain Hilbert space. But, in that case, the random variables $(\tilde{p} * Z\dot{F})_t(x)$ cannot in general be defined for every x .

6.2. A Priori Estimates

We now proceed to establish *a priori* estimates for Walsh-type stochastic integrals of the form $\mathcal{A}v$, thereby showing that the random linear operator \mathcal{A} maps every \mathbf{L}_β^2 to itself continuously and boundedly. As such, the following result should be compared to Lemma 5.4 on page 59.

LEMMA 6.11. *For all $\beta > 0$ and predictable random fields v and w ,*

$$(6.30) \quad \mathcal{N}_\beta(\mathcal{A}v) \leq \left(\frac{C_\sigma}{\beta^{1/2}} + D_\sigma \mathcal{N}_\beta(v) \right) \sqrt{2(\bar{R}_\beta f)(0)},$$

and

$$(6.31) \quad \mathcal{N}_\beta(\mathcal{A}v - \mathcal{A}w) \leq \text{Lip}_\sigma \mathcal{N}_\beta(v - w) \sqrt{2(\bar{R}_\beta f)(0)}.$$

PROOF. According to (5.20) and (5.22), $E(|(\mathcal{A}v)_t(x)|^2)$ is bounded above by

$$(6.32) \quad \int_0^t Q_s^2 ds \int_{\mathbf{R}^d} dy \int_{\mathbf{R}^d} dz p_{t-s}(y-x) p_{t-s}(z-x) f(z-y)$$

$$= \int_0^t Q_s^2 ds \int_{\mathbf{R}^d} dy \int_{\mathbf{R}^d} dz p_{t-s}(y) p_{t-s}(z) f(z-y)$$

$$(6.33) \quad = \int_0^t Q_s^2 \cdot (\bar{P}_{t-s} f)(0) ds,$$

where

$$(6.34) \quad Q_s := C_\sigma + D_\sigma \sup_{a \in \mathbf{R}^d} \|v_s(a)\|_2.$$

Therefore,

$$(6.35) \quad \int_0^\infty e^{-\beta t} \mathbf{E} (|(\mathcal{A}v)_t(x)|^2) dt \leq \int_0^\infty e^{-\beta s} Q_s^2 ds \cdot \int_0^\infty e^{-\beta t} (\bar{P}_t f)(0) dt \\ = \int_0^\infty e^{-\beta s} Q_s^2 ds \cdot (\bar{R}_\beta f)(0).$$

Since

$$(6.36) \quad Q_s^2 \leq 2C_\sigma^2 + 2D_\sigma^2 \sup_{a \in \mathbf{R}^d} \mathbf{E} (|v_s(a)|^2),$$

the first part of the lemma follows from the Minkowski-type bound

$$(6.37) \quad (|a| + |b|)^{1/2} \leq |a|^{1/2} + |b|^{1/2},$$

valid for all $a, b \in \mathbf{R}$.

The first follows from the second as in the proof of Lemma 5.4. \square

According to Theorem 1.2, if u_0 and v_0 are bounded and measurable functions, then there exist a.s.-unique mild solutions u and v to the stochastic heat equation (6.1). Our next lemma shows that if u_0 and v_0 are suitably close, then u and v are as well.

LEMMA 6.12. *Assume Condition 1.1 holds. Let u and v be two mild solutions to (6.1) whose initial functions $u_0, v_0 : \mathbf{R}^d \rightarrow \mathbf{R}$ are both bounded and in $L^1(\mathbf{R}^d)$. Then there exists β_0 which depends only on f and Lip_σ , such that for all $\beta > \beta_0$,*

$$(6.38) \quad \mathcal{N}_\beta(u - v) \leq \frac{4}{(2\pi)^d} \int_{\mathbf{R}^d} \frac{|\hat{u}_0(\xi) - \hat{v}_0(\xi)|}{\beta + 2\text{Re}\Psi(\xi)} d\xi.$$

PROOF. Throughout we suppose that $c > 1$. Now let us consider a fixed $x \in \mathbf{R}^d$ and $t > 0$, and define u^n and v^n to be the respective Picard approximation to u and v at the n th stage. Then,

$$(6.39) \quad \|u_t^n(x) - v_t^n(x)\|_2^2 \leq 2 |(P_t u_0)(x) - (P_t v_0)(x)|^2 + 2\mathcal{Q}_t^{n-1},$$

where \mathcal{Q}_t^{n-1} is defined as

$$(6.40) \quad \left\| \int_0^t \int_{\mathbf{R}^d} p_{t-s}(y-x) (\sigma(u_s^{n-1}(y)) - \sigma(v_s^{n-1}(y))) F(ds dy) \right\|_2^2.$$

Define for all $t > 0$ and $n \geq 1$,

$$(6.41) \quad H_t^n := \sup_{a \in \mathbf{R}^d} \mathbf{E} (|u_t^n(a) - v_t^n(a)|^2).$$

Then it follows from our construction of stochastic convolutions [and (6.19)] that \mathcal{Q}_t^{n-1} is bounded above by

$$(6.42) \quad \begin{aligned} \text{Lip}_\sigma^2 \int_0^t H_s^{n-1} ds \int_{\mathbf{R}^d} dy \int_{\mathbf{R}^d} dz p_{t-s}(y-x)p_{t-s}(z-x)f(z-y) \\ = \text{Lip}_\sigma^2 \cdot \int_0^t H_s^{n-1}(\bar{P}_{t-s}f)(0) ds. \end{aligned}$$

And hence,

$$(6.43) \quad \int_0^\infty e^{-\beta t} \mathcal{Q}_t^{n-1} dt \leq \text{Lip}_\sigma^2 \cdot \int_0^\infty e^{-\beta t} H_t^{n-1} dt \cdot (\bar{R}_\beta f)(0).$$

This and (6.39) together prove that

$$(6.44) \quad \begin{aligned} \int_0^\infty e^{-\beta t} H_t^n dt \leq 2 \int_0^\infty e^{-\beta t} \sup_{x \in \mathbf{R}^d} |(P_t u_0)(x) - (P_t v_0)(x)|^2 dt \\ + 2\text{Lip}_\sigma^2 \cdot \int_0^\infty e^{-\beta t} H_t^{n-1} dt \cdot (\bar{R}_\beta f)(0). \end{aligned}$$

Condition 1.1 and the monotone convergence theorem together imply that $\lim_{\beta \rightarrow \infty} (\bar{R}_\beta f)(0) = 0$. We can choose β_0 so large that

$$(6.45) \quad 4\text{Lip}_\sigma^2 (\bar{R}_{\beta_0} f)(0) < 1.$$

Since

$$(6.46) \quad \int_0^\infty e^{-\beta t} H_t^n dt = \{\mathcal{N}_\beta(u^n - v^n)\}^2,$$

it follows that for all $\beta > \beta_0$,

$$(6.47) \quad \begin{aligned} \mathcal{N}_\beta(u^n - v^n) &\leq 2 \left(\int_0^\infty e^{-\beta t} \sup_{x \in \mathbf{R}^d} |(P_t u_0)(x) - (P_t v_0)(x)|^2 dt \right)^{1/2} \\ &\leq 2 \int_0^\infty e^{-\beta t/2} \sup_{x \in \mathbf{R}^d} |(P_t u_0)(x) - (P_t v_0)(x)| dt, \end{aligned}$$

thanks to Minkowski's inequality. Recall from the proof of Theorem of 1.2 that, among other things,

$$(6.48) \quad \|u^n - u\|_2 + \|v^n - v\|_2 \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Therefore, Fatou's lemma implies that for all $\beta > \beta_0$,

$$(6.49) \quad \mathcal{N}_\beta(u - v) \leq 2 \int_0^\infty e^{-\beta t/2} \sup_{x \in \mathbf{R}^d} |(P_t u_0)(x) - (P_t v_0)(x)| dt.$$

In order to complete the lemma, we choose β as large as needed in order to ensure that the preceding inequality holds, and define

$$(6.50) \quad q := u_0 - v_0.$$

Hawkes's theorem [Proposition 2.3, p. 19] assures us that p_t and \hat{p}_t are both in $L^1(\mathbf{R}^d)$; in Fourier-analytic language, p_t is in the Fourier [or Wiener] algebra

$$(6.51) \quad A(\mathbf{R}^d) := \left\{ h \in L^1(\mathbf{R}^d) : \hat{h} \in L^1(\mathbf{R}^d) \right\},$$

for all $t > 0$. It is easy to see that the Fourier transform is one-to-one and onto on $A(\mathbf{R}^d)$. And continuous elements of $A(\mathbf{R}^d)$ are in fact in $C_0(\mathbf{R}^d)$, by the Riemann–Lebesgue lemma. Since u_0 and v_0 are integrable, it follows also that $P_t u_0, P_t v_0 \in A(\mathbf{R}^d)$ for all $t > 0$, and that $P_t u_0$ and $P_t v_0$ are both continuous. The inversion theorem of Fourier analysis can be applied pointwise to the elements of $A(\mathbf{R}^d)$. Therefore,

$$(6.52) \quad \begin{aligned} \sup_{x \in \mathbf{R}^d} |(P_t q)(x)| &\leq \|\widehat{P_t q}\|_{L^1(\mathbf{R}^d)} \\ &= \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \left| e^{-t\Psi(-\xi)} \hat{q}(\xi) \right| d\xi \\ &\leq \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} e^{-t\operatorname{Re}\Psi(\xi)} |\hat{u}_0(\xi) - \hat{v}_0(\xi)| d\xi. \end{aligned}$$

And hence,

$$(6.53) \quad \frac{1}{2} \int_0^\infty e^{-\beta t/2} \sup_{x \in \mathbf{R}^d} |(P_t q)(x)| dt \leq \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \frac{|\hat{u}_0(\xi) - \hat{v}_0(\xi)|}{\beta + 2\operatorname{Re}\Psi(\xi)} d\xi.$$

This proves the lemma. \square

6.3. Proof of Existence

Having prepared the background material, we are ready to demonstrate Theorem 6.2. Before we begin the proof, let us recall the following:

DEFINITION 6.13. Let $\{\psi_n\}_{n=1}^\infty$ denote a collection of measurable functions from \mathbf{R}^d to \mathbf{R}_+ such that: (i) $n \mapsto \|\psi_n\|_{L^1(\mathbf{R}^d)}$ is uniformly bounded; and (ii) $\lim_{n \rightarrow \infty} \hat{\psi}_n = 1$ pointwise. Then we say that $\{\psi_n\}_{n=1}^\infty$ is a *weak mollifier*.

Clearly, weak mollifiers are mollifiers in the usual sense. But the converse is not in general true, because weak mollifiers need not have very good smoothness properties.

We need this definition because our strategy of the proof is the following: Let $\{\psi_n\}_{n=1}^\infty$ denote a weak mollifier. Then, according to Theorem 1.2, we can solve (6.1) subject to the initial condition $u_0 * \psi_n$, since $u_0 * \psi_n$ is a bounded and measurable function. If $\mathbf{u}^{(n)}$ denotes the solution, we then proceed to show that $\mathbf{u} := \lim_{n \rightarrow \infty} \mathbf{u}^{(n)}$ exists in a suitable sense and is a temperate solution to (6.1). Now we write down the details of this argument.

PROOF OF THEOREM 6.2. Recall that u_0 is a finite Borel measure and define $u_0^{(n)} := \psi_n * u_0$, where $\{\psi_n\}_{n=1}^\infty$ is a weak mollifier. According to Theorem 1.2, there is a solution $\mathbf{u}^{(n)}$ to the following SPDE:

$$(6.54) \quad \frac{\partial}{\partial t} u_t^{(n)}(x) = (\mathcal{L}u_t^{(n)})(x) + \sigma(u_t^{(n)}(x))\dot{F}_t(x),$$

and the solution is unique up to evanescence. Theorem 1.2 also assures us that

$$(6.55) \quad C_\alpha := \sup_{t>0} \sup_{x \in \mathbf{R}^d} e^{-\alpha t} \mathbf{E} \left(\left| u_t^{(n)}(x) \right|^2 \right) < \infty,$$

provided that $\alpha > 0$ is sufficiently large. Note that if $\theta > \alpha$, then

$$(6.56) \quad \begin{aligned} \mathcal{N}_\theta(u^{(n)}) &\leq C_\alpha \int_0^\infty e^{-(\theta-\alpha)t} dt \\ &= \frac{C_\alpha}{\theta - \alpha} < \infty. \end{aligned}$$

That is, $\mathbf{u}^{(n)} \in \mathbf{L}_\theta^2$ for all θ sufficiently large. According to Lemma 6.12, for all $n, m \geq 1$,

$$(6.57) \quad \mathcal{N}_\beta \left(\mathbf{u}^{(n)} - \mathbf{u}^{(m)} \right) \leq \frac{4}{(2\pi)^d} \int_{\mathbf{R}^d} \frac{|\hat{u}_0(\xi)|}{\beta + 2\operatorname{Re}\Psi(\xi)} \cdot \left| \hat{\psi}_n(\xi) - \hat{\psi}_m(\xi) \right| d\xi,$$

provided that $\beta > \beta_0$ for a β_0 that depends only on f and $\operatorname{Lip}_\sigma$. Therefore, it follows that if $\beta > \beta_0$, then $\{\mathbf{u}^{(n)}\}_{n=1}^\infty$ is a Cauchy sequence in \mathbf{L}_β^2 . Let $\mathbf{u} := \{u_t(x)\}_{t>0, x \in \mathbf{R}^d}$ denote the limit. By definition,

$$(6.58) \quad \lim_{n \rightarrow \infty} \mathbf{u}^{(n)} = \mathbf{u} \quad \text{in } \bigcap_{\beta > \beta_0} \mathbf{L}_\beta^2.$$

And therefore, Lemma 6.11 and our extension of stochastic convolutions together imply that

$$(6.59) \quad \lim_{n \rightarrow \infty} \left(\mathbf{u}^{(n)} - \mathcal{A}\mathbf{u}^{(n)} \right) = \mathbf{u} - \mathcal{A}\mathbf{u} \quad \text{in } \bigcap_{\beta > \beta_0} \mathbf{L}_\beta^2,$$

where $\mathcal{A}u^{(n)}$ is defined in (5.5) on page 56; and $\mathcal{A}u$ is defined in the same way, but now interpreted as a stochastic convolution in the sense of the present chapter.

In particular, for all $T > 0$,

$$(6.60) \quad \int_0^T \sup_{x \in \mathbf{R}^d} \mathbb{E} \left(\left| [u_t^{(n)}(x) - (\mathcal{A}u^{(n)})_t(x)] - [u_t(x) - (\mathcal{A}u)_t(x)] \right|^2 \right) dt \rightarrow 0,$$

as $n \rightarrow \infty$. This follows simply because

$$(6.61) \quad \int_0^T \kappa(s) ds \leq e^{\beta T} \int_0^\infty e^{-\beta s} \kappa(s) ds,$$

for all nonnegative measurable $\kappa : \mathbf{R}_+ \rightarrow \mathbf{R}_+$.

Since $u^{(n)}$ is a mild solution to (6.1) with initial data $u_0^{(n)} = \psi_n * u_0$, Tonelli's theorem implies that

$$(6.62) \quad u_t^{(n)}(x) - (\mathcal{A}u^{(n)})_t(x) = ((P_t u_0) * \psi_n)(x).$$

Hawkes's theorem [Proposition 2.3, p. 19] implies that $P_t u_0$ is uniformly continuous for every $t > 0$. Therefore, for every $t > 0$ fixed,

$$(6.63) \quad \lim_{n \rightarrow \infty} (P_t u_0) * \psi_n = P_t u_0 \quad \text{uniformly.}$$

It follows from the preceding and Fatou's lemma that for all $T > 0$,

$$(6.64) \quad \int_0^T \sup_{x \in \mathbf{R}^d} \mathbb{E} \left(|u_t(x) - (P_t u_0)(x) - (\mathcal{A}u)_t(x)|^2 \right) dt = 0.$$

This proves the theorem. \square

The following is the final result of this section. It shows that the temperate solution to (6.1) is unique among a natural family of possible solutions. It is entirely possible that one can establish the uniqueness of the temperate solution among a larger family than that offered below. But we are not aware of such further improvements.

PROPOSITION 6.14. *The temperate solution u , provided by the preceding proof, does not depend on the choice of the weak mollifier, up to evanescence.*

PROOF. Let $\{\kappa_n\}_{n=1}^\infty$ be another weak mollifier, and define v to be the temperate solution that the preceding proof yields based on $\{\kappa_n\}_{n=1}^\infty$; that is, v^n is obtained from $\{\kappa_l\}_{l=1}^\infty$ in the same way that u^n was obtained from $\{\psi_l\}_{l=1}^\infty$.

Let β_0 be as in the proof of Theorem 6.2 [see also Lemma 6.12], and recall that β_0 does not depend on $\{\psi_n\}_{n=1}^\infty$, $\{\kappa_n\}_{n=1}^\infty$, etc. In accord with

Fatou's lemma,

$$(6.65) \quad \begin{aligned} \mathcal{N}_\beta(\mathbf{u} - \mathbf{v}) &\leq \liminf_{n \rightarrow \infty} \mathcal{N}_\beta(u^n - v^n) \\ &\leq \frac{4}{(2\pi)^d} \liminf_{n \rightarrow \infty} \int_{\mathbf{R}^d} \frac{|\hat{u}_0(\xi)|}{\beta + 2\operatorname{Re}\Psi(\xi)} \cdot \left| \hat{\psi}_n(\xi) - \hat{\kappa}_n(\xi) \right| d\xi, \end{aligned}$$

for every $\beta > \beta_0$, owing to Lemma 6.12. And the preceding is zero by the dominated convergence theorem. This proves the proposition. \square

6.4. More A Priori Estimates

Let u_0 be a finite Borel measure on \mathbf{R}^d that satisfies (6.10). In order to investigate the large-time behavior of the temperate solution to (6.1), we need to introduce a suitable family of Banach spaces. The task of the present section is precisely to do that. Our analysis of the temperate solution to (6.1) will resume after this section.

Define for all $\beta > 0$, finite real numbers $p \geq 1$, and predictable random fields $\mathbf{v} := \{v_t(x)\}_{t>0, x \in \mathbf{R}^d}$,

$$(6.66) \quad \mathcal{N}_{\beta,p}(\mathbf{v}) := \left(\int_0^\infty e^{-\beta t} \sup_{x \in \mathbf{R}^d} \{ \mathbf{E}(|v_t(x)|^p) \}^{2/p} dt \right)^{1/2},$$

so that $\mathcal{N}_{\beta,2}(v)$ is the same quantity as $\mathcal{N}_\beta(v)$; the latter was defined earlier in (6.21). And, just as was the case with \mathcal{N}_β when $p = 2$, every $\mathcal{N}_{\beta,p}$ is a norm on equivalence classes of p -times integrable predictable processes that are modifications of one another.

DEFINITION 6.15. Let \mathbf{L}_β^p denote the collection of all predictable random fields $v := \{v_t(x)\}_{t>0, x \in \mathbf{R}^d}$ such that $\mathcal{N}_{\beta,p}(v) < \infty$.

Thus, our present definition of \mathbf{L}_β^2 agrees with our older one.

The following generalizes Lemma 6.11 to the case that p is an integer ≥ 2 .

LEMMA 6.16. *For all $\beta > 0$, even integers $p \geq 2$, and Walsh-integrable random fields v and w ,*

$$(6.67) \quad \mathcal{N}_{\beta,p}(\mathcal{A}v) \leq z_p \left(\frac{C_\sigma}{\beta^{1/2}} + D_\sigma \mathcal{N}_{\beta,p}(v) \right) \sqrt{2(\bar{R}_\beta f)(0)},$$

and

$$(6.68) \quad \mathcal{N}_{\beta,p}(\mathcal{A}v - \mathcal{A}w) \leq z_p \operatorname{Lip}_\sigma \mathcal{N}_{\beta,p}(v - w) \sqrt{2(\bar{R}_\beta f)(0)}.$$

PROOF. The ensuing argument is a generalization of the proof of Lemma 5.4. Indeed, by (5.20), $E(|(\mathcal{A}v)_t(x)|^p)$ is at most z_p^p times

$$(6.69) \quad \left(\int_0^t ds \int_{\mathbf{R}^d} dy \int_{\mathbf{R}^d} dz V'_s(y, z) p_{t-s}(y-x) p_{t-s}(z-x) f(y-z) \right)^{p/2},$$

where

$$(6.70) \quad \begin{aligned} V'_s(y, z) &:= \|\sigma(v_s(y))\|_p \cdot \|\sigma(v_s(z))\|_p \\ &\leq \left(C_\sigma + D_\sigma \sup_{q \in \mathbf{R}^d} \|v_s(q)\|_p \right)^2 \\ &\leq 2\Theta_s, \end{aligned}$$

where

$$(6.71) \quad \Theta_s := C_\sigma^2 + D_\sigma^2 \sup_{q \in \mathbf{R}^d} \|v_s(q)\|_p^2.$$

It follows that $E(|(\mathcal{A}v)_t(x)|^p)$ is bounded above by $2^{p/2} z_p^p$ times

$$(6.72) \quad \begin{aligned} &E(|(\mathcal{A}v)_t(x)|^p) \\ &\leq 2^{p/2} z_p^p \left(\int_0^t \Theta_s ds \int_{\mathbf{R}^d} dy \int_{\mathbf{R}^d} dz p_{t-s}(y) p_{t-s}(z) f(y-z) \right)^{p/2} \\ &= 2^{p/2} z_p^p \left(\int_0^t \Theta_s (\bar{P}_{t-s} f)(0) ds \right)^{p/2}, \end{aligned}$$

thanks a direct computation that was made already during the course of the proof of Lemma 5.4.

Let us examine the preceding display next: The right-most quantity is independent of the variable x . Therefore, we can maximize the extreme terms in that display over all $x \in \mathbf{R}^d$, raise the resulting inequality to the power $2/p$ on all sides, and then finally multiply by $\exp(-\beta t)$ and integrate $[dt]$ to obtain the following:

$$(6.73) \quad \mathcal{N}_{\beta,p}(v) \leq z_p \sqrt{2 \int_0^\infty e^{-\beta s} \Theta_s ds \cdot (\bar{R}_\beta f)(0)}$$

The first bound of the lemma follows after a few direct computations. The second bound follows from the first in a manner that has been pointed out several times already; see, for example, the end of the proof of Lemma 5.4 for an outline. \square

Note that if v is a Walsh-integrable random field, then so is $\sigma \circ v$, and then $\mathcal{A}v$ is none other than the stochastic convolution $\tilde{p} * (\sigma \circ v) \dot{F}$. Therefore,

the following is a refinement of Lemma 6.16 that works in the setting of stochastic convolutions.

LEMMA 6.17. *Recall that $\sigma : \mathbf{R} \rightarrow \mathbf{R}$ is a predetermined nonrandom and Lipschitz-continuous function. Choose and fix a real $\beta > 0$ and an even integer $p \geq 2$. Then, for all predictable random fields v and w ,*

$$(6.74) \quad \mathcal{N}_{\beta,p}(\tilde{p} * (\sigma \circ v)\dot{F}) \leq z_p \left(\frac{C_\sigma}{\beta^{1/2}} + D_\sigma \mathcal{N}_{\beta,p}(v) \right) \sqrt{2(\bar{R}_\beta f)(0)},$$

and

$$(6.75) \quad \begin{aligned} \mathcal{N}_{\beta,p}(\tilde{p} * (\sigma \circ v)\dot{F} - \tilde{p} * (\sigma \circ w)\dot{F}) \\ \leq z_p \text{Lip}_\sigma \mathcal{N}_{\beta,p}(v - w) \sqrt{2(\bar{R}_\beta f)(0)}. \end{aligned}$$

PROOF. We can, and will, assume without loss of generality that the three quantities $(\bar{R}_\beta f)(0)$, $\mathcal{N}_{\beta,p}(v)$, and $\mathcal{N}_{\beta,p}(v - w)$ are all finite.

Since $p \geq 2$, it follows that the stochastic convolution $\tilde{p} * v\dot{F}$ is well defined. Moreover, we can find a sequence v^1, v^2, \dots of Walsh-integrable random fields such that $\tilde{p} * v^n\dot{F}$ converges to $\tilde{p} * v\dot{F}$ in \mathbf{L}_β^2 . In particular, there exists a Lebesgue-null set $N_0 \subset \mathbf{R}_+$ such that for all $t \notin N_0$,

$$(6.76) \quad \lim_{n \rightarrow \infty} \sup_{x \in \mathbf{R}^d} \mathbf{E}(|v_t^n(x) - v_t(x)|^2) = 0.$$

Therefore, we may apply Fatou's lemma to the [pseudo-] norm $\mathcal{N}_{\beta,p}$ and deduce the following:

$$(6.77) \quad \begin{aligned} \mathcal{N}_{\beta,p}(\tilde{p} * v\dot{F}) &\leq \liminf_{n \rightarrow \infty} \mathcal{N}_{\beta,p}(\tilde{p} * v^n\dot{F}) \\ &\leq z_p \liminf_{n \rightarrow \infty} \mathcal{N}_{\beta,p}(v^n) \sqrt{(\bar{R}_\beta f)(0)}, \end{aligned}$$

because we can apply Lemma 6.16 to each Walsh-integrable process v^n [with $\sigma(u) := u$]. Among other things, this proves that $\tilde{p} * v\dot{F} \in \mathbf{L}_\beta^p$, and

$$(6.78) \quad \mathcal{N}_{\beta,p}(\tilde{p} * v\dot{F}) \leq z_p \mathcal{N}_{\beta,p}(v) \sqrt{2(\bar{R}_\beta f)(0)}.$$

The first inequality of the lemma follows from the above and Minkowski's inequality, since $\sigma(v_t(x)) \leq C_\sigma + D_\sigma |v_t(x)|$ pointwise.

Because we also have that $\mathcal{N}_{\beta,p}(w) < \infty$, we can deduce the second inequality from Lemma 6.16 as well. \square

We will not need the entire strength of the preceding lemma itself. We have stated it for two reasons: First of all, it suggests that our extension

of the stochastic convolution has good continuity properties, viewed as elements of the Banach spaces \mathbf{L}_β^p ; and also, this is a simple setting in which an approximation method introduced via the proof of Lemma 6.17. We will use that method later on.

6.5. An Upper Bound for Growth

The goal of this section is to establish Theorem 6.7.

PROOF OF THEOREM 6.7. Let $\{\psi^n\}_{n=1}^\infty$ be a weak mollifier, and let $\mathbf{u}^{(n)} := \{u_t^{(n)}(x)\}_{t>0, x \in \mathbf{R}^d}$ be the solution to (6.1) with initial data $\psi^n * u_0$. By the Minkowski inequality, the following is valid for every integer $n \geq 1$:

$$(6.79) \quad \mathcal{N}_{\beta,p}(\mathbf{u}^{(n)}) \leq \mathcal{N}_{\beta,p}(P_\bullet u_0) + \mathcal{N}_{\beta,p}(\tilde{p} * (\sigma \circ u^{(n)})\dot{F}).$$

The first term on the right is estimated easily, thanks to Minkowski's inequality, as follows:

$$(6.80) \quad \begin{aligned} \mathcal{N}_{\beta,p}(P_\bullet u_0) &= \left(\int_0^\infty e^{-\beta t} \sup_{x \in \mathbf{R}^d} |(P_t u_0)(x)|^2 dt \right)^{1/2} \\ &\leq \int_0^\infty e^{-\beta t/2} \sup_{x \in \mathbf{R}^d} |(P_t u_0)(x)| dt \\ &\leq \frac{2}{(2\pi)^d} \int_{\mathbf{R}^d} \frac{|\hat{u}_0(\xi)|}{\beta + 2\operatorname{Re}\Psi(\xi)} d\xi; \end{aligned}$$

see (6.53). Therefore, Lemma 6.17 implies that

$$(6.81) \quad \begin{aligned} \mathcal{N}_{\beta,p}(\mathbf{u}^{(n)}) &\leq \frac{2}{(2\pi)^d} \int_{\mathbf{R}^d} \frac{|\hat{u}_0(\xi)|}{\beta + 2\operatorname{Re}\Psi(\xi)} d\xi \\ &\quad + z_p \left(\frac{C_\sigma}{\beta^{1/2}} + D_\sigma \mathcal{N}_{\beta,p}(\mathbf{u}^{(n)}) \right) \sqrt{2(\bar{R}_\beta f)(0)}. \end{aligned}$$

Let us assume, for the time being, that we could prove that $\mathcal{N}_{\beta,p}(\mathbf{u}^{(n)})$ is finite—that is, $\mathbf{u}^{(n)}$ is in the Banach space \mathbf{L}_β^p —provided that²

$$(6.82) \quad z_p D_\sigma \sqrt{2(\bar{R}_\beta f)(0)} < 1.$$

Then we could rearrange (6.81) and find that

$$(6.83) \quad \mathcal{N}_{\beta,p}(\mathbf{u}^{(n)}) \leq c \left(\frac{2}{(2\pi)^d} \int_{\mathbf{R}^d} \frac{|\hat{u}_0(\xi)|}{\beta + 2\operatorname{Re}\Psi(\xi)} d\xi + \frac{z_p C_\sigma}{\beta^{1/2}} \right),$$

²(6.82) holds for all β sufficiently large because thanks to (1.1), $\lim_{\beta \rightarrow \infty} (\bar{R}_\beta f)(0) = 0$.

where

$$(6.84) \quad c := \frac{1}{1 - z_p D_\sigma \sqrt{2(\bar{R}_\beta f)(0)}}.$$

And therefore, the proof of Lemma 6.17 implies that

$$(6.85) \quad \begin{aligned} \mathcal{N}_{\beta,p}(\mathbf{u}) &\leq \liminf_{n \rightarrow \infty} \mathcal{N}_{\beta,p}(\mathbf{u}^{(n)}) \\ &\leq c \left(\frac{2}{(2\pi)^d} \int_{\mathbf{R}^d} \frac{|\hat{u}_0(\xi)|}{\beta + 2\operatorname{Re}\Psi(\xi)} d\xi + \frac{z_p C_\sigma}{\beta^{1/2}} \right). \end{aligned}$$

In other words, we have shown the following: If we could prove that $\mathbf{u}^{(n)} \in \mathbf{L}_\beta^p$ for every n , then in fact $\{\mathbf{u}^{(n)}\}_{n=1}^\infty$ is bounded in \mathbf{L}_β^p , whence $\mathbf{u} \in \mathbf{L}_\beta^p$ and Theorem 6.7 follows.

Now we proceed as follows: Let

$$(6.86) \quad u_t^{(n,0)}(x) := (\psi^n * u_0)(x) \quad \text{for all } t \geq 0 \text{ and } x \in \mathbf{R}^d,$$

and iteratively define

$$(6.87) \quad u_t^{(n,k)}(x) := (P_t u_0)(x) + \int_0^t \int_{\mathbf{R}^d} p_{t-s}(y-x) \sigma \left(u_s^{(n,k-1)}(y) \right) F(dy ds).$$

By the Minkowski inequality and (6.53),

$$(6.88) \quad \begin{aligned} \mathcal{N}_{\beta,p}(\mathbf{u}^{(n,0)}) &= \left(\int_0^\infty e^{-\beta t} \sup_{x \in \mathbf{R}^d} |(P_t(\psi^n * u_0))(x)|^2 dt \right)^{1/2} \\ &\leq \frac{2}{(2\pi)^d} \int_{\mathbf{R}^d} \frac{|\hat{\psi}_n(\xi)| \cdot |\hat{u}_0(\xi)|}{\beta + 2\operatorname{Re}\Psi(\xi)} d\xi, \end{aligned}$$

which is finite; it is in fact bounded uniformly in n since $\sup_{n \geq 1} \|\psi_n\|_{L^1(\mathbf{R}^d)}$ is finite by the very definition of weak mollifiers. Now the argument that led to (6.81) leads also to the following:

$$(6.89) \quad \begin{aligned} \mathcal{N}_{\beta,p}(\mathbf{u}^{(n,k+1)}) &\leq \frac{2}{(2\pi)^d} \int_{\mathbf{R}^d} \frac{|\hat{u}_0(\xi)|}{\beta + 2\operatorname{Re}\Psi(\xi)} d\xi \\ &\quad + z_p \left(\frac{C_\sigma}{\beta^{1/2}} + D_\sigma \mathcal{N}_{\beta,p}(\mathbf{u}^{(n,k)}) \right) \sqrt{2(\bar{R}_\beta f)(0)}. \end{aligned}$$

This, (6.88), and induction together prove that if (6.82) is in effect, then

$$(6.90) \quad \sup_{k \geq 1} \mathcal{N}_{\beta,p}(\mathbf{u}^{(n,k)}) < \infty.$$

By the Borel–Cantelli lemma, and owing to Theorem 1.2 and its proof,

$$(6.91) \quad \lim_{k \rightarrow \infty} \sup_{x \in \mathbf{R}^d} \mathbb{E} \left(\left| u_t^{(n,k)}(x) - u_t^{(n)}(x) \right|^2 \right) = 0 \quad \text{for all } t > 0.$$

[In fact, this is essentially (6.76), but we have noted further that “almost all t ” can be replaced by “all t ” in the present setting, since $\mathbf{u}^{(n)}$ is a mild solution to (6.1).] Therefore, Fatou’s lemma proves that

$$(6.92) \quad \mathcal{N}_{\beta,p}(\mathbf{u}^{(n)}) \leq \liminf_{k \rightarrow \infty} \mathcal{N}_{\beta,p}(\mathbf{u}^{(n,k)}) < \infty.$$

This establishes Theorem 6.7. □

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Index

- $\mathcal{A}v$, a stochastic integral, 56
- $\mathbf{B}_{\beta,p}$, a family of stochastic Banach spaces, 57
- $C_0(\mathbf{R}^d)$, the space of all real-valued continuous functions on \mathbf{R}^d that vanish at infinity, 14
- $\text{Dom}[\mathcal{L}]$, the domain of the generator, 15
- \mathbf{L}_{β}^2 , an L^2 space of predictable random fields, 81
- \mathbf{L}_{β}^p , an L^p space of predictable random fields, 89
- \mathcal{L} , the generator, 16
- \mathcal{N}_{β} , a family of Hilbertian norms, 80
- $\mathcal{N}_{\beta,p}$, a family of norms, 89
- P_t , $\{P_t\}_{t \geq 0}$, etc., the underlying semigroup, 14
- \bar{P}_t , $\{\bar{P}_t\}_{t \geq 0}$, etc., the replica semigroup, 17
- R_{α} , $\{R_{\alpha}\}_{\alpha > 0}$, etc., the resolvent, 14
- \bar{R}_{α} , $\{\bar{R}_{\alpha}\}_{\alpha > 0}$, etc., the replica resolvent, 18
- \mathcal{S} , the space of all rapidly-decreasing test functions., 26
- $X^* := -X$, the dual process, 17
- X , the underlying Lévy process, 13
- \bar{X} , the replica process, 17
- $p_t(x)$, $\{p_t(x)\}_{t > 0, x \in \mathbf{R}^d}$, etc., the underlying transition functions, 13
- $\bar{p}_t(x)$, $\{\bar{p}_t(x)\}_{t > 0, x \in \mathbf{R}^d}$, etc., the replica transition functions, 17
- r_{α} , α -potential density, 14
- \bar{r}_{α} , the replica α -potential density, 18
- $\bar{\gamma}_*(p)$, the maximum upper $L^p(\mathbf{P})$ -Liapounov exponent of the solution., 3
- $\bar{\gamma}_x(p)$, the upper $L^p(\mathbf{P})$ -Liapounov exponent of the solution at x , 3
- Generator, 15
- Positive-definite function, 26
- Replica
 - process, 18
 - semigroup, 18
- Temperate solution, 77