A LOCAL-TIME CORRESPONDENCE FOR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. It is frequently the case that a white-noise-driven parabolic and/or hyperbolic stochastic partial differential equation (SPDE) can have random-field solutions only in spatial dimension one. Here we show that in many cases, where the "spatial operator" is the L^2-generator of a Lévy process X, a linear SPDE has a random-field solution if and only if the symmetrization of X possesses local times. This result gives a probabilistic reason for the lack of existence of random-field solutions in dimensions strictly bigger than one.

In addition, we prove that the solution to the SPDE is [Hölder] continuous in its spatial variable if and only if the said local time is [Hölder] continuous in its spatial variable. We also produce examples where the random-field solution exists, but is almost surely unbounded in every open subset of space-time. Our results are based on first establishing a quasi-isometry between the linear L^2-space of the weak solutions of a family of linear SPDEs, on one hand, and the Dirichlet space generated by the symmetrization of X, on the other hand.

We study mainly linear equations in order to present the local-time correspondence at a modest technical level. However, some of our work has consequences for nonlinear SPDEs as well. We demonstrate this assertion by studying a family of parabolic SPDEs that have additive nonlinearities. For those equations we prove that if the linearized problem has a random-field solution, then so does the nonlinear SPDE. Moreover, the solution to the linearized equation is [Hölder] continuous if and only if the solution to the nonlinear equation is. And the solutions are bounded and unbounded together as well. Finally, we prove that in the cases that the solutions are unbounded, they almost surely blow up at exactly the same points.

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Contents

1. Introduction

We consider the *stochastic heat equation* inspired by the fundamental works of Pardoux (1975a, 1975b, 1972), Krylov and Rozovskii (1979a, 1979b, 1977), and Funaki (1983). Let $\dot{w}$ denote space-time white noise, t is nonnegative, x is in $\mathbf{R}^d$, and the Laplacian acts on the x variable. Then, we have:

$$\left|\begin{aligned} &\partial_t H(t\,,x) = (\Delta H)(t\,,x) + \dot{w}(t\,,x),\\ &H(0\,,x) = 0.\end{aligned}\right. \tag{1.1}$$

Let us consider also the *stochastic wave equation* of Cabaña (1970):

$$\left|\begin{aligned} &\partial_{tt} W(t\,,x) = (\Delta W)(t\,,x) + \dot{w}(t\,,x),\\ &W(0\,,x) = \partial_t W(0\,,x) = 0.\end{aligned}\right. \tag{1.2}$$

One of the common features of (1.1) and (1.2) is that they suffer from a *curse of dimensionality*. Namely, these equations can have random-field solutions only in dimension one. Moreover, this curse of dimensionality appears to extend beyond the linear parabolic setting of (1.1), or the linear hyperbolic setting of (1.2). For instance, see Perkins (2002, Corollary III.4.3) for an example from superprocesses, and Walsh (1986, Chapter 9) for an example from statistical mechanics.

One can informally ascribe this curse of dimensionality to the "fact" that while the Laplacian smooths, white noise roughens. In one dimension, the roughening effect of white noise turns out to be small relative to the smoothing properties of the Laplacian, and we thus have a random-field solution. However, in dimensions greater than one white noise is much too rough, and the Laplacian cannot smooth the solution enough to yield a random field.

Dalang and Frangos (1998) were able to construct a first fully-rigorous explanation of the curse of dimensionality. They do so by first replacing white noise by a Gaussian noise that is white in time and colored in space. And then they describe precisely the roughening effect of the noise on the solution, viewed as a random generalized function. See also Brzeźniak and van Neerven (2003), Dalang and Mueller (2003), Millet and Sanz-Solé (1999), Peszat (2002), and Peszat and Zabczyk (2000). More recently, Dalang and Sanz-Solé (2005) study fully nonlinear stochastic wave equations driven by noises that are white in time and colored in space, and operators that are arbitrary powers of the Laplacian.

In this article we present a different explanation of this phenomenon. Our approach is to describe accurately the smoothing effect of the Laplacian in the presence of white noise. Whereas the answer of Dalang and Frangos (1998) is analytic, ours is probabilistic. For instance, we will see soon that (1.1) and (1.2) have solutions only in dimension one because d-dimensional Brownian motion has local times only in dimension one [Theorem 2.1]. Similarly, when $d = 1$, the solution to (1.1) [and/or (1.2)] is continuous in x because the local time of one-dimensional Brownian motion is continuous in its spatial variable.

The methods that we employ also give us a local-time paradigm that makes precise the claim that the stochastic PDEs (1.1) and (1.2) "have random-field solutions in dimension $d = 2 - \epsilon$ for all $\epsilon \in (0\,,2)$." See Example 7.4 below, where we introduce a family of SPDEs on fractals.

An outline of this paper follows: In §2 we describe a suitable generalization of the stochastic PDEs (1.1) and (1.2) that have sharp local-time correspondences. This section contains the main results of the paper. The existence theorem of §2 is proved in §3. Section 4 contains the proof of the necessary and sufficient local-time condition for continuity of our SPDEs in their space variables. Although we are not aware of any interesting connections between local times and temporal regularity of the solutions of SPDEs, we have included §5 that contains a sharp analytic condition for temporal continuity of the SPDEs of §2. We also produce examples of SPDEs that have random-field solutions which are almost surely unbounded in every open space-time set [Example 5.5].

Section 6 discusses issues of Hölder continuity in either space, or time, variable. In §7 we establish a very general (but somewhat weak) connection between Markov processes, their local times, and solutions to various linear SPDEs. The material of that section is strongly motivated by the recent article of Da Prato (2007) who studies Kolmogorov SPDEs that are not unlike those studied here, but also have multiplicative nonlinearities. We go on to produce examples where one can make sense of a random-field solution to (1.1) in dimension $2-\epsilon$ for all $\epsilon \in (0\,,2)$. These solutions in fact turn out to be jointly Hölder continuous, but we will not dwell on that here. Finally, in §8 we discuss a parabolic version of the SPDEs of §2 that have additive nonlinearities; we prove the existence of solutions and describe exactly when and where these solutions blow up.

The chief aim of this paper is to point out various interesting and deep connections between the local-time theory of Markov processes and families of stochastic partial differential equations. In all cases, we have strived to study the simplest SPDEs that best highlight these connections. But it would also be interesting to study much more general equations.

There are other connections between local times of Markov processes and Gaussian processes that appear to be different from those presented here. For a sampler of those isomorphism theorems see Brydges, Fröhlich, and Spencer (1982), Dynkin (1984), and Eisenbaum (1995). Marcus and Rosen (2006) contains an excellent and complete account of this theory. Diaconis and Evans (2002) have introduced yet a different isomorphism theorem.

Finally, we conclude by mentioning what we mean by "local times," as there are many [slightly] different versions in the literature. Given a stochastic process $Y := \{Y_t\}_{t\geq 0}$ on $\mathbf{R}^d$, consider the occupation measure[s],

$$(1.3)\qquad Z(t\,,\varphi) := \int_0^t \varphi(Y_s)\,ds \qquad \text{for all } t \geq 0 \text{ and measurable } \varphi : \mathbf{R}^d \to \mathbf{R}_+.$$

We can identify each $Z(t\,,\bullet)$ with a measure in the usual way. Then, we say that Y has local times when $Z(t\,,dx) \ll dx$ for all t. The local times of Y are themselves defined by $Z(t\,,x) := Z(t\,,dx)/dx$. It follows that if Y has local times, then $Z(t\,,\varphi) = \int_{\mathbf{R}^d} Z(t\,,x)\varphi(x)\,dx$ a.s. for every $t \geq 0$ and all measurable functions $\varphi : \mathbf{R}^d \to \mathbf{R}_+$. And the converse holds also.

2. More general equations

In order to describe when (1.1) and (1.2) have random-field solutions, and why, we study more general equations.

Let $\mathscr{L}$ denote the generator of a d-dimensional Lévy process $X := \{X_t\}_{t\geq 0}$ with characteristic exponent Ψ. We can normalize things so that $\mathrm{E}\exp(i\xi \cdot X_t) = \exp(-t\Psi(\xi))$, and consider $\mathscr{L}$ as an L^2-generator with domain

$$\text{Dom}\,\mathscr{L} := \left\{ f \in L^2(\mathbf{R}^d) : \int_{\mathbf{R}^d} |\hat{f}(\xi)|^2\, |\Psi(\xi)|\, d\xi < \infty \right\}. \tag{2.1}$$

As usual, $\hat{f}$ denotes the Fourier transform of f; we opt for the normalization

$$\hat{f}(\xi) := \int_{\mathbf{R}^d} e^{i\xi\cdot x} f(x)\, dx \qquad \text{for all } f \in L^1(\mathbf{R}^d). \tag{2.2}$$

[The general L^2-theory of Markov processes is described in great depth in Fukushima, Ōshima, and Takeda (1994) in the symmetric case, and Ma and Röckner (1992) for the general case.]

In this way, we can—and will—view $\mathscr{L}$ as a generalized convolution operator with Fourier multiplier $\hat{\mathscr{L}}(\xi) := -\overline{\Psi(\xi)}$.

We consider two families of stochastic partial differential equations. The first is the stochastic heat equation for $\mathscr{L}$:

$$\left| \begin{aligned} &\partial_t H(t\,,x) = (\mathscr{L} H)(t\,,x) + \dot{w}(t\,,x), \\ &H(0\,,x) = 0, \end{aligned} \right. \tag{2.3}$$

where x ranges over $\mathbf{R}^d$ and t over $\mathbf{R}_+ := [0\,,\infty)$, and the operator $\mathscr{L}$ acts on the x variable.

We also consider hyperbolic SPDEs of the following wave type:

$$\left| \begin{aligned} &\partial_{tt} W(t\,,x) = (\mathscr{L} W)(t\,,x) + \dot{w}(t\,,x), \\ &W(0\,,x) = \partial_t W(0\,,x) = 0. \end{aligned} \right. \tag{2.4}$$

Recall that $X := \{X_t\}_{t\geq 0}$ is a Lévy process whose generator is $\mathscr{L}$. Let X' denote an independent copy of X and define its symmetrization, a la Paul Lévy, by

$$\bar{X}_t := X_t - X'_t \qquad \text{for all } t \geq 0. \tag{2.5}$$

It is a standard fact that $\bar{X}$ is a symmetric Lévy process with characteristic exponent $2\text{Re}\,\Psi$.

The following is our first main result, though the precise meaning of its terminology is yet to be explained.

Theorem 2.1. *The stochastic heat equation* (2.3) *has random-field solutions if and only if the symmetric Lévy process $\bar{X}$ has local times. The same is true for the stochastic wave equation* (2.4)*, provided that the process X is itself symmetric.*

Remark 2.2. Brzeźniak and van Neerven (2003) consider parabolic equations of the type (2.3), where $\mathscr{L}$ is a pseudo-differential operator with a symbol that is bounded below in our language [bounded above in theirs]. If we apply their theory with our constant-symbol operator $\mathscr{L}$, then their bounded-below condition is equivalent to the symmetry of the Lévy process X, and their main result is equivalent to the parabolic portion of Theorem 2.1, under the added condition that X is symmetric. □

It is well-known that when $d \geq 2$, Lévy processes in $\mathbf{R}^d$ *do not* have local times (Hawkes, 1986). It follows from this that neither (2.3) nor (2.4) [under a symmetry assumption on X] can ever have random-field solutions in dimension greater than one. But it is possible that there are no random-field solutions even in dimension one. Here is one such example; many others exist.

Example 2.3. Suppose X is a strictly stable process in $\mathbf{R}$ with stability index $\alpha \in (0\,,2]$. It is very well known that the symmetric Lévy process $\bar{X}$ has local times if and only if $\alpha > 1$; see, for example, Hawkes (1986). If X is itself symmetric, then $\mathscr{L} = -(-\Delta)^{\alpha/2}$ is the α-dimensional fractional Laplacian (Stein, 1970, Chapter V, §1.1), and also the stochastic wave equation (2.4) has a random-field solution if and only if $\alpha > 1$. □

The local-time correspondence of Theorem 2.1 is not a mere accident. In fact, the next two theorems suggest far deeper connections between the solutions to the linear SPDEs of this paper and the theory of local times of Markov processes. We emphasize that the next two theorems assume the existence of a random-field solution to one of the stochastic PDEs (2.3) and/or (2.4). Therefore, they are inherently one dimensional statements.

Theorem 2.4. *Assume that $d = 1$ and the stochastic heat equation* (2.3) *has a random-field solution $\{H(t\,,x)\}_{t\geq 0, x\in\mathbf{R}}$. Then, the following are equivalent:*

1. *There exists $t > 0$ such that $x \mapsto H(t\,,x)$ is a.s. continuous.*
2. *For all $t > 0$, $x \mapsto H(t\,,x)$ is a.s. continuous.*
3. *The local times of $\bar{X}$ are a.s. continuous in their spatial variable.*

The same equivalence is true for the solution W to the stochastic wave equation (2.4)*, provided that the process X is itself symmetric.*

Theorem 2.5. *Assume that $d = 1$ and the stochastic heat equation* (2.3) *has a random-field solution $\{H(t\,,x)\}_{t\geq 0, x\in\mathbf{R}}$. Then, the following are equivalent:*

1. *There exists $t > 0$ such that $x \mapsto H(t\,,x)$ is a.s. Hölder continuous.*
2. *For all $t > 0$, $x \mapsto H(t\,,x)$ is a.s. Hölder continuous.*
3. *The local times of $\bar{X}$ are a.s. Hölder continuous in their spatial variable.*

If the process X is itself symmetric, then the preceding conditions are also equivalent to the Hölder continuity of the solution W to the stochastic wave equation (2.4) *in the spatial variable. Finally, the critical Hölder indices of $x \mapsto H(t\,,x)$, $x \mapsto W(t\,,x)$ and that of the local times of $\bar{X}$ are the same.*

Blumenthal and Getoor (1961) have introduced several "indices" that describe various properties of a Lévy process. We recall below their *lower index* β'':

$$\beta'' := \liminf_{|\xi|\to\infty} \frac{\log \mathrm{Re}\,\Psi(\xi)}{\log|\xi|} = \sup\left\{\alpha \geq 0 : \lim_{|\xi|\to\infty} \frac{\mathrm{Re}\,\Psi(\xi)}{|\xi|^\alpha} = \infty\right\}. \tag{2.6}$$

Theorem 2.6. *If $\beta'' > d$, then the stochastic heat equation* (2.3) *has a random-field solution that is jointly Hölder continuous. The critical Hölder index is $\leq (\beta'' - d)/2$ for the space variable and $\leq (\beta'' - d)/2\beta''$ for the time variable. Furthermore, if X is symmetric, then the same assertions hold for the solution to the stochastic wave equation* (2.4).

Remark 2.7. If X is in the domain of attraction of Brownian motion on $\mathbf{R}$, then $\beta'' = 2$, and the critical temporal and spatial index bounds of Theorem 2.6 are respectively 1/2 and 1/4. These numbers are well known to be the optimal Hölder indices. In fact, the relatively simple Hölder-index bounds of Theorem 2.6 are frequently sharp; see Example 5.4. □

It is well known that $0 \leq \beta'' \leq 2$ (Blumenthal and Getoor, 1961, Theorem 5.1). Thus, the preceding is inherently a one-dimensional result; this is in agreement with the result of Theorem 2.1. Alternatively, one can combine Theorems 2.1 and 2.6 to construct a proof of the fact that $\beta'' \leq 2$. [But of course the original proof of Blumenthal and Getoor is simpler.]

Several things need to be made clear here; the first being the meaning of a "solution." With this aim in mind, we treat the two equations separately.

We make precise sense of the stochastic heat equation (2.3) in much the same manner as Walsh (1986); see also Brzeźniak and van Neerven (2003), Dalang (1999, 2001), and Da Prato (2007). It is well known that it is much harder to give precise meaning to stochastic hyperbolic equations of the wave type (2.4), even when $\mathscr{L}$ is the Laplacian (Dalang and Lévêque, 2004a; 2004b; Dalang and Mueller, 2003; Quer-Sardanyons and Sanz-Solé, 2003; Mueller, 1997; Dalang, 1999; Mueller, 1993). Thus, as part of the present work, we introduce a simple and direct method that makes rigorous sense of (2.4) and other linear SPDEs of this type. We believe this method to be of some independent interest.

2.1. **The parabolic case.**

In order to describe the meaning of (2.3) we need to first introduce some notation.

Let $\{P_t\}_{t\geq 0}$ denote the semigroup of the driving Lévy process X; that is, $(P_tf)(x) := \mathrm{E}f(x + X_t)$ for all bounded Borel-measurable functions $f : \mathbf{R}^d \to \mathbf{R}$ [say], all $x \in \mathbf{R}^d$, and all $t \geq 0$. [As usual, $X_0 := 0$.] Formally speaking, $P_t = \exp(t\mathscr{L})$.

The dual semigroup is denoted by P^*, so that $(P_t^*f)(x) = \mathrm{E}f(x - X_t)$. It is easy to see that P and P^* are adjoint in $L^2(\mathbf{R}^d)$ in the sense that

$$(P_tf\,, g) = (f\,, P_t^*g) \quad \text{for all } f, g \in L^2(\mathbf{R}^d). \tag{2.7}$$

Needless to say, $(\cdot\,,\cdot)$ denotes the usual Hilbertian inner product on $L^2(\mathbf{R}^d)$.

Let $\mathscr{S}(\mathbf{R}^d)$ denote the class of all rapidly decreasing test functions on $\mathbf{R}^d$, and recall that: (i) $\mathscr{S}(\mathbf{R}^d) \subset \mathrm{Dom}(\mathscr{L}) \cap C^\infty(\mathbf{R}^d) \subset L^2(\mathbf{R}^d) \cap C^\infty(\mathbf{R}^d)$; and (ii) for all $\varphi, \psi \in \mathscr{S}(\mathbf{R}^d)$,

$$\lim_{h\downarrow 0} \left(\frac{P_{t+h}\varphi - P_t\varphi}{h}\,, \psi \right) = (\mathscr{L}P_t\varphi\,, \psi)\,. \tag{2.8}$$

Thus, $v(t\,,x) := (P_t\varphi)(x)$ solves the Kolmogorov equation $\partial_t v(t\,,x) = (\mathscr{L}v)(t\,,x)$, subject to the initial condition that $v(0\,,x) = \varphi(x)$. This identifies the Green's function for $\partial_t - \mathscr{L} = 0$. Hence, we can adapt the Green-function method of Walsh (1986, Chapter 3), without any great difficulties, to deduce that a *weak solution* to (2.3) is the Gaussian random field $\{H(t\,,\varphi);\ t \geq 0,\ \varphi \in \mathscr{S}(\mathbf{R}^d)\}$, where

$$H(t\,,\varphi) := \int_0^t \int_{\mathbf{R}^d} (P^*_{t-s}\varphi)(y)\, w(dy\, ds). \tag{2.9}$$

This is defined simply as a Wiener integral.

Proposition 2.8. *The Gaussian random field* $\{H(t\,,\varphi);\ t \geq 0,\ \varphi \in \mathscr{S}(\mathbf{R}^d)\}$ *is well defined. Moreover, the process* $\varphi \mapsto H(t\,,\varphi)$ *is a.s. linear for each* $t \geq 0$.

Proof. On one hand, the Wiener isometry tells us that

$$\mathrm{E}\left(|H(t\,,\varphi)|^2\right) = \int_0^t \|P^*_{t-s}\varphi\|^2_{L^2(\mathbf{R}^d)}\, ds. \tag{2.10}$$

On the other hand, it is known that each P^*_s is a contraction on $L^2(\mathbf{R}^d)$. Indeed, it is not hard to check directly that if ℓ_d denotes the Lebesgue measure on $\mathbf{R}^d$, then the dual Lévy process $-X$ is ℓ_d-symmetric (Fukushima et al., 1994, pp. 27–28). Therefore, the asserted contraction property of P^*_s follows from equation (1.4.13) of Fukushima, Ōshima, and Takeda (1994, p. 28). It follows then that $\mathrm{E}(|H(t\,,\varphi)|^2) \leq t\|\varphi\|^2_{L^2(\mathbf{R}^d)}$, and this is finite for all $t \geq 0$ and $\varphi \in \mathscr{S}(\mathbf{R}^d)$. This proves that u is a well-defined Gaussian random field indexed by $\mathbf{R}_+ \times \mathscr{S}(\mathbf{R}^d)$. The proof of the remaining property follows the argument of Dalang (1999, Section 4) quite closely, and is omitted. □

2.2. **The nonrandom hyperbolic case.** It has been known for some time that hyperbolic SPDEs tend to be harder to study, and even to define precisely, than their parabolic counterparts. See, for instance, Dalang and Sanz-Solé (2007) for the most recent work on the stochastic wave equation in dimension 3.

In order to define what the stochastic wave equation (2.4) means precisely, we can try to mimic the original Green-function method of Walsh (1986). But we quickly run into the technical problem of not being able to identify a suitable Green function (or even a measure) for the corresponding integral equation. In order to overcome this obstacle, one could proceed as in Dalang and Sanz-Solé (2007), but generalize the role of their fractional Laplacian. Instead, we opt for a more direct route that is particularly well suited for studying the SPDEs of the present type.

In order to understand (2.4) better, we first consider the deterministic integro-differential equation,

$$\left|\begin{aligned} &\partial_{tt} u(t\,,x) = (\mathscr{L}u)(t\,,x) + f(t\,,x),\\ &u(0\,,x) = \partial_t u(0\,,x) = 0,\end{aligned}\right. \tag{2.11}$$

where $f : \mathbf{R}_+ \times \mathbf{R}^d \to \mathbf{R}$ is a "nice" function, and the variables t and x range respectively over $\mathbf{R}_+$ and $\mathbf{R}^d$. We can study this equation only under the following symmetry condition:

$$\text{The process } X \text{ is symmetric.} \tag{2.12}$$

Equivalently, we assume that Ψ is *real and nonnegative.*

Recall that "$\hat{\ }$" denotes the Fourier transform in the x variable, and apply it informally to (2.11) to deduce that it is equivalent to the following: For all $t \geq 0$ and $\xi \in \mathbf{R}^d$,

$$\left|\begin{aligned} &\partial_{tt} \hat{u}(t\,,\xi) = -\Psi(\xi)\hat{u}(t\,,\xi) + \hat{f}(t\,,\xi),\\ &\hat{u}(0\,,\xi) = \partial_t \hat{u}(0\,,\xi) = 0.\end{aligned}\right. \tag{2.13}$$

This is an inhomogeneous second-order ordinary differential equation [in t] which can be solved explicitly, via Duhamel's principle, to produce the following "formula":

$$\hat{u}(t\,,\xi) = \frac{1}{\sqrt{\Psi(\xi)}} \int_0^t \sin\left(\sqrt{\Psi(\xi)}\,(t-s)\right) \hat{f}(s\,,\xi)\,ds. \tag{2.14}$$

We invert the preceding—informally still—to obtain

$$u(t\,,x) = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \int_0^t \frac{\sin\left(\sqrt{\Psi(\xi)}\,(t-s)\right)}{\sqrt{\Psi(\xi)}}\, e^{-i\xi\cdot x} \hat{f}(s\,,\xi)\,ds\,d\xi. \tag{2.15}$$

We can multiply this by a nice function φ, then integrate $[dx]$ to arrive at

$$u(t\,,\varphi) := \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \int_0^t \frac{\sin\left(\sqrt{\Psi(\xi)}\,(t-s)\right)}{\sqrt{\Psi(\xi)}}\, \overline{\hat{\varphi}(\xi)}\ \hat{f}(s\,,\xi)\,ds\,d\xi. \tag{2.16}$$

We may think of as this as the "weak/distributional solution" to (2.11).

2.3. **The random hyperbolic case.** We follow standard terminology and identify the white noise $\dot{w}$ with the iso-Gaussian process $\{w(h)\}_{h\in L^2(\mathbf{R}_+\times\mathbf{R}^d)}$ as follows:

$$w(h) := \int_{\mathbf{R}^d} \int_0^\infty h(s\,,x)\,w(ds\,dx). \tag{2.17}$$

Next, we define the *Fourier transform* $\hat{w}$ of white noise:

$$\hat{w}(h) := \frac{w(\hat{h})}{(2\pi)^{d/2}} = \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} \int_0^\infty \hat{h}(s\,,\xi)\, w(ds\, d\xi), \tag{2.18}$$

all the time remembering that "$\hat{h}$" refers to the Fourier transform of h in its spatial variable. Suppose $h(s\,,x) = \varphi_1(s)\varphi_2(x)$ for $t \geq 0$ and $x \in \mathbf{R}^d$, where $\varphi_1 \in L^2(\mathbf{R}_+)$ and $\varphi_2 \in L^2(\mathbf{R}^d)$. Then, it follows from the Wiener isometry and Plancherel's theorem that

$$\|\hat{w}(h)\|^2_{L^2(\mathbf{R}_+ \times \mathbf{R}^d)} = \|h\|^2_{L^2(\mathbf{R}_+ \times \mathbf{R}^d)}. \tag{2.19}$$

Because $L^2(\mathbf{R}_+) \otimes L^2(\mathbf{R}^d)$ is dense in $L^2(\mathbf{R}_+ \times \mathbf{R}^d)$, this proves that $\hat{w}$ is defined continuously on all of $L^2(\mathbf{R}_+ \times \mathbf{R}^d)$. Moreover, $\hat{w}$ corresponds to a white noise which is correlated with $\dot{w}$, as described by the following formula:

$$\mathrm{E}\left[w(h_1) \cdot \overline{\hat{w}(h_2)}\right] = \frac{1}{(2\pi)^{d/2}} \int_0^\infty \int_{\mathbf{R}^d} \overline{\hat{h}_1(s\,,\xi)}\, h_2(s\,,\xi)\, d\xi\, ds, \tag{2.20}$$

valid for all $h_1, h_2 \in L^2(\mathbf{R}_+ \times \mathbf{R}^d)$.

In light of (2.16), we define the weak solution to the stochastic wave equation (2.4) as the Wiener integral

$$W(t\,,\varphi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} \int_0^t \frac{\sin\left(\sqrt{\Psi(\xi)}\,(t-s)\right)}{\sqrt{\Psi(\xi)}} \overline{\hat{\varphi}(\xi)}\, \hat{w}(ds\, d\xi). \tag{2.21}$$

It is possible to verify that this method also works for the stochastic heat equation (2.3), and that it produces a equivalent formulation of the Walsh solution (2.9). However, in the present setting, this method saves us from having to describe the existence [and some regularity] of the Green function for the integral equation (2.11).

Proposition 2.9. *If the symmetry condition* (2.12) *holds, then the stochastic wave equation* (2.4) *has a weak solution W for all $\varphi \in \mathscr{S}(\mathbf{R}^d)$. Moreover, $\{W(t\,,\varphi);\, t \geq 0,\, \varphi \in \mathscr{S}(\mathbf{R}^d)\}$ is a well-defined Gaussian random field, and $\varphi \mapsto W(t\,,\varphi)$ is a.s. linear for all $t \geq 0$.*

Proof. We apply the Wiener isometry to obtain

$$\mathrm{E}\left(|W(t\,,\varphi)|^2\right) = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \int_0^t \frac{\sin^2\left(\sqrt{\Psi(\xi)}\,(t-s)\right)}{\Psi(\xi)} |\hat{\varphi}(\xi)|^2\, ds d\xi. \tag{2.22}$$

Because $|\sin\theta/\theta| \leq 1$,

$$\begin{aligned} \mathrm{E}\left(|W(t\,,\varphi)|^2\right) &\leq \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \int_0^t (t-s)^2 |\hat{\varphi}(\xi)|^2\, ds\, d\xi \\ &= \frac{t^3}{3} \|\varphi\|^2_{L^2(\mathbf{R}^d)}, \end{aligned} \tag{2.23}$$

thanks to Plancherel's theorem. It follows immediately from this that $\{W(t\,,\varphi);\, t\geq 0,\, \varphi\in\mathscr{S}(\mathbf{R}^d)\}$ is a well-defined Gaussian random field. The remainder of the proposition is standard. $\square$

3. Existence of functions-valued solutions: Proof of Theorem 2.1

Let $u := \{u(t\,,\varphi);\, t\geq 0,\, \varphi\in\mathscr{S}(\mathbf{R}^d)\}$ denote the weak solution to either one of (2.3) or (2.4). Our present goal is to extend uniquely the Gaussian random field u to a Gaussian random field indexed by $\mathbf{R}_+\times M$, where M is a maximal subset of $\mathscr{D}(\mathbf{R}^d)$—the space of all Schwartz distributions on $\mathbf{R}^d$. Such an M exists thanks solely to functional-analytic facts: Define, temporarily,

$$d_t(\varphi) := \sqrt{\mathrm{E}\left(|u(t\,,\varphi)|^2\right)} \qquad \text{for all } \varphi\in\mathscr{S}(\mathbf{R}^d) \text{ and } t\geq 0. \tag{3.1}$$

Then, the linearity of u in φ shows that $(\varphi\,,\psi)\mapsto d_t(\varphi-\psi)$ defines a metric for each $t\geq 0$. Let M_t denote the completion of $\mathscr{S}(\mathbf{R}^d)$ in $\mathscr{D}(\mathbf{R}^d)$ with respect to the metric induced by d_t; and define $M := \cap_{t\geq 0} M_t$. The space M can be identified with the largest possible family of candidate test functions for weak solutions to either the stochastic heat equation (2.3) or the stochastic wave equation (2.4). Standard heuristics from PDEs then tell us that (2.3) and/or (2.4) has random-field solutions if and only if $\delta_x\in M$ for all $x\in\mathbf{R}^d$; this can be interpreted as an equivalent *definition* of random-field solutions. When $\delta_x\in M$ we may write $u(t\,,x)$ in place of $u(t\,,\delta_x)$. In order to prove Theorem 2.1 we will need some a priori estimates on the weak solutions of both the stochastic equations (2.3) and (2.4). We proceed by identifying M with generalized Sobolev spaces that arise in the potential theory of symmetric Lévy processes. Now let us begin by studying the parabolic case.

3.1. The parabolic case.

Proposition 3.1. *Let H denote the weak solution* (2.9) *to the stochastic heat equation* (2.3)*. Then, for all $\varphi\in\mathscr{S}(\mathbf{R}^d)$, $\lambda>0$, and $t\geq 0$,*

$$\frac{1-e^{-2t/\lambda}}{2}\mathscr{E}(\lambda\,;\varphi)\leq \mathrm{E}\left(|H(t\,,\varphi)|^2\right)\leq \frac{e^{2t/\lambda}}{2}\mathscr{E}(\lambda\,;\varphi), \tag{3.2}$$

where

$$\mathscr{E}(\lambda\,;\varphi) := \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d}\frac{|\hat\varphi(\xi)|^2}{(1/\lambda)+\mathrm{Re}\Psi(\xi)}\,d\xi. \tag{3.3}$$

Next we record the following immediate but useful corollary; it follows from Proposition 3.1 by simply setting $\lambda := t$.

Corollary 3.2. *If H denotes the weak solution to the stochastic heat equation* (2.3)*, then for all $\varphi \in \mathscr{S}(\mathbf{R}^d)$ and $t \geq 0$,*

$$\tfrac{1}{3}\mathscr{E}(t\,;\varphi) \leq \mathrm{E}\left(|H(t\,,\varphi)|^2\right) \leq 4\mathscr{E}(t\,;\varphi), \tag{3.4}$$

The preceding upper bound for $\mathrm{E}(|H(t\,,\varphi)|^2)$ is closely tied to an energy inequality for the weakly asymmetric exclusion process. See Lemma 3.1 of Bertini and Giacomin (1999); they ascribe that lemma to H.-T. Yau.

The key step of the proof of Proposition 3.1 is an elementary real-variable result which we prove next.

Lemma 3.3. *If $g : \mathbf{R}_+ \to \mathbf{R}_+$ is Borel measurable and nonincreasing, then for all $t, \lambda > 0$,*

$$\left(1 - e^{-2t/\lambda}\right) \int_0^\infty e^{-2s/\lambda} g(s)\, ds \leq \int_0^t g(s)\, ds \leq e^{2t/\lambda} \int_0^\infty e^{-2s/\lambda} g(s)\, ds. \tag{3.5}$$

Monotonicity is not needed for the upper bound on $\int_0^t g(s)\, ds$.

Proof of Lemma 3.3. The upper bound on $\int_0^t g(s)\, ds$ follows simply because $e^{2(t-s)/\lambda} \geq 1$ whenever $t \geq s$. In order to derive the lower bound we write

$$\begin{aligned} \int_0^\infty e^{-2s/\lambda} g(s)\, ds &= \sum_{n=0}^\infty \int_{nt}^{(n+1)t} e^{-2s/\lambda} g(s)\, ds \\ &\leq \sum_{n=0}^\infty e^{-2nt/\lambda} \int_0^t g(s+nt)\, ds. \end{aligned} \tag{3.6}$$

Because g is nonincreasing we can write $g(s+nt) \leq g(s)$ to conclude the proof. □

Proof of Proposition 3.1. We know from (2.9) and the Wiener isometry that

$$\mathrm{E}\left(|H(t\,,\varphi)|^2\right) = \int_0^t \|P_s^*\varphi\|_{L^2(\mathbf{R}^d)}^2\, ds. \tag{3.7}$$

Since the Fourier multiplier of P_s^* is $\exp(-s\Psi(-\xi))$ at $\xi \in \mathbf{R}^d$, we can apply the Plancherel theorem and deduce the following formula:

$$\|P_s^*\varphi\|_{L^2(\mathbf{R}^d)}^2 = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} e^{-2s\mathrm{Re}\Psi(\xi)} |\hat{\varphi}(\xi)|^2\, d\xi. \tag{3.8}$$

Because $\mathrm{Re}\Psi(\xi) \geq 0$, Lemma 3.3 readily proves the proposition. □

Equation (3.3) can be used to define $\mathscr{E}(\lambda\,;\varphi)$ for all Schwartz distributions φ, and not only those in $\mathscr{S}(\mathbf{R}^d)$. Moreover, it is possible to verify directly that $\varphi \mapsto \mathscr{E}(\lambda\,;\varphi)^{1/2}$ defines a norm on $\mathscr{S}(\mathbf{R}^d)$. But in all but uninteresting cases, $\mathscr{S}(\mathbf{R}^d)$ is *not* complete in this norm. Let $L^2_{\mathscr{L}}(\mathbf{R}^d)$ denote the completion of $\mathscr{S}(\mathbf{R}^d)$ in the norm $\mathscr{E}(\lambda\,;\bullet)^{1/2}$. Thus the Hilbert space $L^2_{\mathscr{L}}(\mathbf{R}^d)$ can be identified with M. The following is a result about the potential theory of symmetric Lévy processes, but we present a self-contained proof that does not depend on that deep theory.

Lemma 3.4. *The space $L^2_{\mathscr{L}}(\mathbf{R}^d)$ does not depend on the value of λ. Moreover, $L^2_{\mathscr{L}}(\mathbf{R}^d)$ is a Hilbert space in norm $\mathscr{E}(\lambda\,;\bullet)^{1/2}$ for each fixed $\lambda > 0$. Finally, the quasi-isometry* (3.2) *is valid for all $t \geq 0$, $\lambda > 0$, and $\varphi \in L^2_{\mathscr{L}}(\mathbf{R}^d)$.*

Proof. We write, temporarily, $L^2_{\mathscr{L},\lambda}(\mathbf{R}^d)$ for $L^2_{\mathscr{L}}(\mathbf{R}^d)$, and seek to prove that it is independent of the choice of λ.

Define for all distributions φ and ψ,

$$\mathscr{E}(\lambda\,;\varphi\,,\psi) := \frac{1}{2(2\pi)^d}\left[\int_{\mathbf{R}^d} \frac{\hat\varphi(\xi)\,\overline{\hat\psi(\xi)}}{(1/\lambda) + \mathrm{Re}\Psi(\xi)}\,d\xi + \int_{\mathbf{R}^d} \frac{\hat\psi(\xi)\,\overline{\hat\varphi(\xi)}}{(1/\lambda) + \mathrm{Re}\Psi(\xi)}\,d\xi\right]. \tag{3.9}$$

For each $\lambda > 0$ fixed, $(\varphi\,,\psi) \mapsto \mathscr{E}(\lambda\,;\varphi\,,\psi)$ is a pre-Hilbertian inner product on $\mathscr{S}(\mathbf{R}^d)$, and $\mathscr{E}(\lambda\,;\varphi) = \mathscr{E}(\lambda\,;\varphi\,,\varphi)$.

Thanks to Proposition 3.1, for all $\alpha > 0$ there exists a finite and positive constant $c = c_{\alpha,\lambda}$ such that $c^{-1}\mathscr{E}(\alpha\,;\varphi) \leq \mathscr{E}(\lambda\,;\varphi) \leq c\mathscr{E}(\alpha\,;\varphi)$ for all $\varphi \in \mathscr{S}(\mathbf{R}^d)$. This proves that $L^2_{\mathscr{L},\lambda}(\mathbf{R}^d) = L^2_{\mathscr{L},\alpha}(\mathbf{R}^d)$, whence follows the independence of $L^2_{\mathscr{L}}(\mathbf{R}^d)$ from the value of λ. The remainder of the lemma is elementary. □

The space $L^2_{\mathscr{L}}(\mathbf{R}^d)$ is a generalized Sobolev space, and contains many classical spaces of Bessel potentials, as the following example shows.

Example 3.5. Suppose $\mathscr{L} = -(-\Delta)^{s/2}$ for $s \in (0\,,2]$. Then, $\mathscr{L}$ is the generator of an isotropic stable-s Lévy process, and $L^2_{\mathscr{L}}(\mathbf{R}^d)$ is the space $H_{-s/2}(\mathbf{R}^d)$ of Bessel potentials. For a nice pedagogic treatment see the book of Folland (1976, Chapter 6). □

3.2. **The hyperbolic case.** The main result of this section is the following quasi-isometry; it is the wave-equation analogue of Proposition 3.1.

Proposition 3.6. *Suppose the symmetry condition* (2.12) *holds, and let $W := \{W(t\,,\varphi);\, t \geq 0,\, \varphi \in \mathscr{S}(\mathbf{R}^d)\}$ denote the weak solution to the stochastic wave equation* (2.4)*. Then,*

$$\tfrac14 t\mathscr{E}\left(t^2;\varphi\right) \leq \mathrm{E}\left(|W(t\,,\varphi)|^2\right) \leq 2t\mathscr{E}\left(t^2;\varphi\right). \tag{3.10}$$

for all $t \geq 0$ and $\varphi \in \mathscr{S}(\mathbf{R}^d)$. Moreover, we can extend W by density so that the preceding display continues to remain valid when $t \geq 0$ and $\varphi \in L^2_{\mathscr{L}}(\mathbf{R}^d)$.

Proof. Although Lemma 3.3 is not applicable, we can proceed in a similar manner as we did when we proved the earlier quasi-isometry result for the heat equation (Proposition 3.1).

Namely, we begin by observing that

$$\mathrm{E}\left(|W(t\,,\varphi)|^2\right) = \frac{1}{(2\pi)^d}\int_0^t\int_{\mathbf{R}^d}\frac{\sin^2\left(\sqrt{\Psi(\xi)}\,s\right)}{\Psi(\xi)}\,|\hat\varphi(\xi)|^2\,d\xi\,ds. \tag{3.11}$$

See (2.21). If $\theta > 0$ then $\sin\theta$ is at most the minimum of one and θ. This leads to the bounds

$$\begin{aligned}\mathrm{E}\left(|W(t\,,\varphi)|^2\right) &\le \frac{1}{(2\pi)^d}\int_0^t\int_{\mathbf{R}^d}\left(s^2\wedge\frac{1}{\Psi(\xi)}\right)\,|\hat\varphi(\xi)|^2\,d\xi\,ds\\ &\le \frac{t}{(2\pi)^d}\int_{\mathbf{R}^d}\left(t^2\wedge\frac{1}{\Psi(\xi)}\right)\,|\hat\varphi(\xi)|^2\,d\xi.\end{aligned} \tag{3.12}$$

The upper bound follows from this and the elementary inequality $t^2\wedge z^{-1}\le 2/(t^{-2}+z)$, valid for all $z\ge 0$.

In order to derive the [slightly] harder lower bound we first rewrite (3.11) as follows:

$$\mathrm{E}\left(|W(t\,,\varphi)|^2\right) = \frac{t}{2(2\pi)^d}\int_{\mathbf{R}^d}\left(1-\frac{\sin\left(2\sqrt{\Psi(\xi)}\,t\right)}{2\sqrt{\Psi(\xi)}\,t}\right)\frac{|\hat\varphi(\xi)|^2}{\Psi(\xi)}\,d\xi. \tag{3.13}$$

We shall analyze the integral by splitting it according to whether or not $\Psi\le 1/t^2$.

Taylor's expansion [with remainder] reveals that if θ is nonnegative, then $\sin\theta$ is at most $\theta-(\theta^3/6)+(\theta^5/120)$. This and a little algebra together show that

$$1-\frac{\sin\theta}{\theta}\ge\frac{2\theta^2}{15}\qquad\text{if } 0\le\theta\le 2. \tag{3.14}$$

Consequently,

$$\begin{aligned}\int_{\{\Psi\le 1/t^2\}}\left(1-\frac{\sin\left(2\sqrt{\Psi(\xi)}\,t\right)}{2\sqrt{\Psi(\xi)}\,t}\right)\frac{|\hat\varphi(\xi)|^2}{\Psi(\xi)}\,d\xi &\ge \frac{8t^2}{15}\int_{\{\Psi\le 1/t^2\}}|\hat\varphi(\xi)|^2\,d\xi\\ &\ge\frac12\int_{\{\Psi\le 1/t^2\}}\left(t^2\wedge\frac{1}{\Psi(\xi)}\right)|\hat\varphi(\xi)|^2\,d\xi.\end{aligned} \tag{3.15}$$

For the remaining integral we use the elementary bound $1-(\sin\theta/\theta)\ge 1/2$, valid for all $\theta>2$. This leads to the following inequalities:

$$\begin{aligned}\int_{\{\Psi> 1/t^2\}}\left(1-\frac{\sin\left(2\sqrt{\Psi(\xi)}\,t\right)}{2\sqrt{\Psi(\xi)}\,t}\right)\frac{|\hat\varphi(\xi)|^2}{\Psi(\xi)}\,d\xi &\ge \frac12\int_{\{\Psi>1/t^2\}}\frac{|\hat\varphi(\xi)|^2}{\Psi(\xi)}\,d\xi\\ &=\frac12\int_{\{\Psi> 1/t^2\}}\left(t^2\wedge\frac{1}{\Psi(\xi)}\right)|\hat\varphi(\xi)|^2\,d\xi.\end{aligned} \tag{3.16}$$

The proof concludes from summing up equations (3.15) and (3.16), and then plugging the end result into (3.13). $\square$

We now give a proof of Theorem 2.1.

Proof. Let us begin with the proof in the case of the stochastic heat equation (2.3). Proposition 3.1 is a quasi-isometry of the maximal space M of test functions for weak solutions of (2.3) into $L^2_{\mathscr{L}}(\mathbf{R}^d)$. Thus, M can be identified with the Hilbert space $L^2_{\mathscr{L}}(\mathbf{R}^d)$, and hence (2.3) has random-field solutions if and only if $\delta_x \in L^2_{\mathscr{L}}(\mathbf{R}^d)$ for all $x \in \mathbf{R}^d$. Thanks to Lemma 3.4, the stochastic heat equation (2.3) has random-field solutions if and only if

$$\int_{\mathbf{R}^d} \frac{d\xi}{\vartheta + \mathrm{Re}\Psi(\xi)} < \infty \qquad \text{for some, and hence all, } \vartheta > 0. \tag{3.17}$$

The first part of the proof is concluded since condition (3.17) is known to be necessary as well as sufficient for $\bar{X}$ to have local times (Hawkes, 1986, Theorem 1). The remaining portion of the proof follows from the preceding in much the same way as the first portion was deduced from Proposition 3.1. □

4. Spatial continuity: Proof of Theorem 2.4

Proof. We work with the stochastic heat equation (2.3) first. Without loss of generality, we may—and will—assume that (2.3) has a random-field solution $H(t\,,x)$, and $\bar{X}$ has local times. Else, Theorem 2.1 finishes the proof.

Let $\varphi := \delta_x - \delta_y$, and note that $|\hat{\varphi}(\xi)|^2 = 2(1 - \cos(\xi(x-y)))$ is a function of $x - y$. Because $H(t\,,\varphi) = H(t\,,x) - H(t\,,y)$, equations (3.7) and (3.8) imply that $z \mapsto H(t\,,z)$ is a centered Gaussian process with stationary increments for each fixed $t \geq 0$.

Consider the function

$$h(r) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1-\cos(r\xi)}{1+\mathrm{Re}\,\Psi(\xi)}\,d\xi, \qquad \text{defined for all } r \geq 0. \tag{4.1}$$

According to Lemma 3.4, Proposition 3.1 holds for all Schwartz distributions $\varphi \in L^2_{\mathscr{L}}(\mathbf{R}^d)$. The existence of random-field solutions is equivalent to the condition that $\delta_x \in L^2_{\mathscr{L}}(\mathbf{R}^d)$ for all $x \in \mathbf{R}$. We apply Proposition 3.1 to $\varphi := \delta_x - \delta_y$, with $\lambda := 1$ [say], and find that

$$\left(1 - e^{-2t}\right) h(|x-y|) \leq \mathrm{E}\left(|H(t\,,x) - H(t\,,y)|^2\right) \leq e^{2t} h(|x-y|). \tag{4.2}$$

Define $\bar{h}$ to be the Hardy–Littlewood nondecreasing rearrangement of h. That is,

$$\bar{h}(r) := \inf\{y \geq 0 :\, g(y) > r\} \quad \text{where} \quad g(y) := \mathrm{meas}\,\{r \geq 0 :\, h(r) \leq y\}\,. \tag{4.3}$$

Then according to the proof of Corollary 6.4.4 of Marcus and Rosen (2006, p. 274), the stationary-increments Gaussian process $x \mapsto H(t\,,x)$ has a continuous modification iff

$$\int_{0+} \frac{\bar{h}(r)}{r|\log r|^{1/2}}\,dr < \infty. \tag{4.4}$$

Next we claim that (4.4) is equivalent to the continuity of the local times of the symmetrized Lévy process $\bar{X}$. We recall that the characteristic exponent of $\bar{X}$ is $2\mathrm{Re}\,\Psi$, and hence by the Lévy–Khintchine formula it can be written as

$$2\mathrm{Re}\,\Psi(\xi) = \sigma^2\xi^2 + \int_{-\infty}^{\infty} (1-\cos(\xi x))\,\nu(dx), \tag{4.5}$$

where ν is a σ-finite Borel measure on $\mathbf{R}$ with $\int_{-\infty}^{\infty}(1\wedge x^2)\,\nu(dx) < \infty$. See, for example Bertoin (1996, Theorem 1, p. 13).

Suppose, first, that

$$\text{either } \sigma^2 > 0 \ \text{ or } \int_{-\infty}^{\infty} (1\wedge |x|)\,\nu(dx) = \infty. \tag{4.6}$$

Then, (4.4) is also necessary and sufficient for the [joint] continuity of the local times of the symmetrized process $\bar{X}$; confer with Barlow (1988, Theorems B and 1). On the other hand, if (4.6) fails to hold then $\bar{X}$ is a compound Poisson process. Because $\bar{X}$ is also a symmetric process, the Lévy–Khintchine formula tells us that it has zero drift. That is, $\bar{X}_t = \sum_{j=1}^{\Pi(t)} Z_i$, where $\{Z_i\}_{i=1}^{\infty}$ are i.i.d. and symmetric, and Π is an independent Poisson process. It follows immediately from this that the range of $\bar{X}$ is a.s. countable in that case. This proves that the occupation measure for $\bar{X}$ is a.s. singular with respect to Lebesgue measure, and therefore $\bar{X}$ cannot possess local times. But that contradicts the original assumption that $\bar{X}$ has local times. Consequently, (4.4) implies (4.6), and is equivalent to the spatial continuity of [a modification of] local times of $\bar{X}$. This proves the theorem in the parabolic case.

Let us assume further the symmetry condition (2.12). Because $\mathscr{E}(t^2;\varphi)/\mathscr{E}(1\,;\varphi)$ is bounded above and below by positive finite constants that depend only on $t > 0$ (Proposition 3.1), we can apply the very same argument to the stochastic wave equation (2.4), but use Proposition 3.6 in place of Proposition 3.1. This completes our proof of Theorem 2.4. □

5. **An aside on temporal continuity**

We spend a few pages discussing matters of temporal continuity—especially temporal Hölder continuity—of weak solutions of the stochastic heat equation (2.3), as well as the stochastic wave equation (2.4).

Definition 5.1. We call a function $g(s)$ a *gauge function* if the following are satisfied:

(1) $g(s)$ is an increasing function;
(2) $g(s)$ is a slowly varying function at infinity;
(3) $g(s)$ satisfies the integrability condition,

$$\int_{0^+} \frac{ds}{s\log(1/s)g(1/s)} < \infty. \tag{5.1}$$

Next, we quote a useful property of slow varying functions (Bingham et al., 1989, p. 27).

Proposition 5.2. *If g is a slowly varying function and $\alpha > 1$, then the integral $\int_x^\infty t^{-\alpha} g(t)\,dt$ converges for every $x > 0$, and*

$$\int_x^\infty \frac{g(t)}{t^\alpha}\,dt \sim \frac{g(x)}{(\alpha-1)x^{\alpha-1}} \qquad \text{as } x \to \infty. \tag{5.2}$$

We can now state the main theorem of this section. It gives a criteria for the temporal continuity of the weak solutions of our stochastic equations.

Theorem 5.3. *Let H denote the weak solution to the stochastic heat equation (2.3). Let g be a gauge function in the sense of Definition 5.1. Choose and fix $\varphi \in L^2_{\mathscr{L}}(\mathbf{R}^d)$. Then, $t \mapsto H(t\,,\varphi)$ has a continuous modification if the following is satisfied*

$$\int_{\mathbf{R}^d} \frac{\log(1+\operatorname{Re}\Psi(\xi))g(1+|\Psi(\xi)|)}{1+\operatorname{Re}\Psi(\xi)} |\hat{\varphi}(\xi)|^2\,d\xi < \infty. \tag{5.3}$$

Moreover, the critical Hölder exponent of $t \mapsto H(t\,,\varphi)$ is precisely

$$\underline{\operatorname{ind}}\,\mathscr{E}(\bullet\,;\varphi) := \liminf_{\epsilon\downarrow 0} \frac{\log \mathscr{E}(\epsilon\,;\varphi)}{\log \epsilon}. \tag{5.4}$$

Consequently, $t \mapsto H(t\,,\varphi)$ has a Hölder-continuous modification (a.s.) iff $\underline{\operatorname{ind}}\,\mathscr{E}(\bullet\,;\varphi) > 0$.

If the symmetry condition (2.12) holds, then (5.3) guarantees the existence of a continuous modification of $t \mapsto W(t\,,\varphi)$, where W denotes the weak solution to the stochastic wave equation. Furthermore, (5.4) implies the temporal Hölder continuity of $t \mapsto W(t\,,\varphi)$ of any order $< \frac{1}{2}\,\underline{\operatorname{ind}}\,\mathscr{E}(\bullet\,;\varphi)$.

We now give two examples. The first one is about the temporal Hölder exponent while the second one provides a family of random-field solutions which are almost surely unbounded in every open space-time set.

Example 5.4. Suppose $d = 1$ and $\mathscr{L} = -(-\Delta)^{\alpha/2}$ for some $\alpha \in (1\,,2]$. Choose and fix $x \in \mathbf{R}$. According to Theorem 2.1, $\delta_x \in L^2_{\mathscr{L}}(\mathbf{R})$, so we can apply Theorem 5.3 with $\varphi := \delta_x$. In this case,

$$\begin{aligned} \mathscr{E}(\epsilon\,;\delta_x) &= \frac{1}{2\pi}\int_{-\infty}^\infty \frac{d\xi}{(1/\epsilon)+|\xi|^\alpha} \\ &= \text{const}\cdot \epsilon^{1-(1/\alpha)}. \end{aligned} \tag{5.5}$$

In particular, $\underline{\operatorname{ind}}\,\mathscr{E}(\bullet\,;\delta_x) = 1-(1/\alpha)$, whence it follows that the critical Hölder exponent of $t \mapsto H(t\,,x)$ is precisely $\frac{1}{2} - (2\alpha)^{-1}$. When $\mathscr{L} = \Delta$, we have $\alpha = 2$ and the critical temporal exponent is $1/4$, which agrees with a well-known folklore theorem. For example, Corollary 3.4 of Walsh (1986, pp. 318–320) and its proof contain this statement for the closely-related stochastic cable equation. In particular, see the last two lines on page 319 of Walsh's lectures (*loc. cit.*). □

Example 5.5. Choose and fix $\alpha \in \mathbf{R}$. According to the Lévy–Khintchine formula (Bertoin, 1996, Theorem 1, p. 13) we can find a symmetric Lévy process whose characteristic exponents satisfies $\Psi(\xi) \sim \xi(\log \xi)^\alpha$ as $\xi \to \infty$. Throughout, we assume that $\alpha > 1$. This ensures that condition (3.17) is in place; i.e., $(1+\Psi)^{-1} \in L^1(\mathbf{R})$. Equivalently, that both SPDEs (2.3) and (2.4) have a random-field solution.

A few lines of computations show that for all $x \in \mathbf{R}$,

$$\mathscr{E}(\epsilon\,;\delta_x) \asymp |\log(1/\epsilon)|^{-\alpha+1}\,, \tag{5.6}$$

where $a(\epsilon) \asymp b(\epsilon)$ means that $a(\epsilon)/b(\epsilon)$ is bounded and below by absolute constants, uniformly for all $\epsilon > 0$ sufficiently small and $x \in \mathbf{R}$. Next, consider

$$d(s\,,t) := \sqrt{\mathrm{E}\left(|H(s\,,x) - H(t\,,x)|^2\right)}. \tag{5.7}$$

According to Proposition 5.6 below, (5.6) implies that

$$d(s\,,t) \asymp \left|\log\left(\frac{1}{|s-t|}\right)\right|^{(1-\alpha)/2}, \tag{5.8}$$

uniformly for all s and t in a fixed compact subset $[0\,,T]$ of $\mathbf{R}_+$, say. Let N denote the *metric entropy* of $[0\,,T]$ in the [pseudo-] metric d (Dudley, 1967). That is, for all $\epsilon > 0$, we define $N(\epsilon)$ to be the minimum number of d-balls of radius ϵ needed to cover $[0\,,T]$. Then, it is easy to deduce from the previous display that $\log N(\epsilon) \asymp \epsilon^{-2/(\alpha-1)}$, and hence

$$\lim_{\epsilon\downarrow 0} \epsilon\sqrt{\log N(\epsilon)} = 0 \quad \text{if and only if} \quad \alpha > 2. \tag{5.9}$$

Therefore, if $1 < \alpha \le 2$ then Sudakov minorization (Marcus and Rosen, 2006, p. 250) tells us that the process $t \mapsto H(t\,,x)$—and also $t \mapsto W(t\,,x)$—does *not* have any continuous modifications, almost surely. On the other hand, if $\alpha > 2$, then the integrability condition (5.3) holds manifestly, and hence $t \mapsto H(t\,,x)$ and $t \mapsto W(t\,,x)$ both have continuous modifications. Thus, the sufficiency condition of Theorem 5.3 is also necessary for the present example. In addition, when $\alpha \le 2$, the random-field solutions H and W are both unbounded a.s. in every open set. This assertion follows from general facts about Gaussian processes; see, for example equation (6.32) of Marcus and Rosen (2006, p. 250). We skip the details. □

5.1. **Estimate in the parabolic case.**

Proposition 5.6. *For every $t, \epsilon \ge 0$ and $\varphi \in L^2_{\mathscr{L}}(\mathbf{R}^d)$,*

$$\tfrac{1}{2}\,\mathscr{E}(\epsilon\,;\varphi) \le \mathrm{E}\left(|H(t+\epsilon\,,\varphi) - H(t\,,\varphi)|^2\right) \le \mathscr{E}(\epsilon\,;\varphi) + e^{2t}\mathscr{F}(\epsilon\,;\varphi), \tag{5.10}$$

where

$$\mathscr{F}(\epsilon\,;\varphi) := \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d}\left(1\wedge \epsilon^2|\Psi(\xi)|^2\right)\frac{|\hat\varphi(\xi)|^2}{1+\operatorname{Re}\Psi(\xi)}\,d\xi. \tag{5.11}$$

Proof. Because of density it suffices to prove that the proposition holds for all functions $\varphi\in\mathscr{S}(\mathbf{R}^d)$ of rapid decrease. By (2.9) and Wiener's isometry,

$$\begin{aligned}\mathrm{E}\left(|H(t+\epsilon\,,\varphi)-H(t\,,\varphi)|^2\right) &= \int_0^t\int_{\mathbf{R}^d}\left|\left(P^*_{t-s+\epsilon}\varphi\right)(y)-\left(P^*_{t-s}\varphi\right)(y)\right|^2\,dy\,ds\\ &\quad+\int_t^{t+\epsilon}\int_{\mathbf{R}^d}\left|\left(P^*_{t-s+\epsilon}\varphi\right)(y)\right|^2\,dy\,ds.\end{aligned}\tag{5.12}$$

We apply Plancherel's theorem to find that

$$\begin{aligned}\mathrm{E}\left(|H(t+\epsilon\,,\varphi)-H(t\,,\varphi)|^2\right) &= \frac{1}{(2\pi)^d}\int_0^t\int_{\mathbf{R}^d}\left|e^{-(s+\epsilon)\Psi(-\xi)}-e^{-s\Psi(-\xi)}\right|^2\,|\hat\varphi(\xi)|^2\,d\xi\,ds\\ &\quad+\frac{1}{(2\pi)^d}\int_0^\epsilon\int_{\mathbf{R}^d}e^{-2s\operatorname{Re}\Psi(\xi)}\,|\hat\varphi(\xi)|^2\,dy\,ds.\end{aligned}\tag{5.13}$$

Thus, we can write

$$\mathrm{E}\left(|H(t+\epsilon\,,\varphi)-H(t\,,\varphi)|^2\right) := \frac{T_1+T_2}{(2\pi)^d},\tag{5.14}$$

where

$$T_1 := \int_0^t\int_{\mathbf{R}^d}e^{-2s\operatorname{Re}\Psi(\xi)}\left|1-e^{-\epsilon\Psi(\xi)}\right|^2|\hat\varphi(\xi)|^2\,d\xi\,ds,\tag{5.15}$$

and

$$T_2 := \int_0^\epsilon\int_{\mathbf{R}^d}e^{-2s\operatorname{Re}\Psi(\xi)}|\hat\varphi(\xi)|^2\,d\xi\,ds.\tag{5.16}$$

First we estimate T_2, viz.,

$$\int_0^\epsilon e^{-2s\operatorname{Re}\Psi(\xi)}\,ds = \epsilon\,\frac{1-e^{-2\epsilon\operatorname{Re}\Psi(\xi)}}{2\epsilon\operatorname{Re}\Psi(\xi)}.\tag{5.17}$$

Because

$$\frac12\,\frac{1}{1+\theta}\le\frac{1-e^{-\theta}}{\theta}\le\frac{2}{1+\theta}\quad\text{for all }\theta>0,\tag{5.18}$$

it follows that $\frac12(2\pi)^d\,\mathscr{E}(\epsilon\,;\varphi)\le T_2\le(2\pi)^d\mathscr{E}(\epsilon\,;\varphi)$. Since $T_1\ge 0$ we obtain the first inequality of the proposition.

According to Lemma 3.3 with $\lambda=1$,

$$T_1\le\frac{e^{2t}}{2}\int_{\mathbf{R}^d}\left|1-e^{-\epsilon\Psi(\xi)}\right|^2\,\frac{|\hat\varphi(\xi)|^2}{1+\operatorname{Re}\Psi(\xi)}\,d\xi.\tag{5.19}$$

Because $|1-e^{-\epsilon\Psi(\xi)}|^2 \le 1 \wedge \epsilon^2|\Psi(\xi)|^2$, it follows that $T_1 \le (2\pi)^d \exp(2t)\mathscr{F}(\epsilon\,;\varphi)$, and hence the proof is completed. □

5.2. **Estimate in the hyperbolic case.**

Proposition 5.7. *Assume the symmetry condition* (2.12)*, and let W denote the weak solution to the stochastic wave equation* (2.4)*. Then, for all $t \ge 0$, $\epsilon > 0$, and $\varphi \in L^2_{\mathscr{L}}(\mathbf{R}^d)$,*

$$\mathrm{E}\left(|W(t+\epsilon\,,\varphi) - W(t\,,\varphi)|^2\right) \le (8t+6\epsilon)\,\mathscr{E}(\epsilon^2;\varphi). \tag{5.20}$$

Proof. By density, it suffices to prove the proposition for all functions $\varphi \in \mathscr{S}(\mathbf{R}^d)$ of rapid decrease. Henceforth, we choose and fix such a function φ.

In accord with (2.21) we write

$$\mathrm{E}\left(|W(t+\epsilon\,,\varphi) - W(t\,,\varphi)|^2\right) := T_1 + T_2, \tag{5.21}$$

where

$$T_1 := \frac{1}{(2\pi)^d}\int_0^t\int_{\mathbf{R}^d} \frac{\left|\sin\left(\sqrt{\Psi(\xi)}\,(r+\epsilon)\right) - \sin\left(\sqrt{\Psi(\xi)}\,r\right)\right|^2}{\Psi(\xi)}|\hat\varphi(\xi)|^2\,d\xi\,dr, \tag{5.22}$$

and

$$T_2 := \frac{1}{(2\pi)^d}\int_t^{t+\epsilon}\int_{\mathbf{R}^d}\frac{\sin^2\left(\sqrt{\Psi(\xi)}\,r\right)}{\Psi(\xi)}\,|\hat\varphi(\xi)|^2\,d\xi\,dr. \tag{5.23}$$

We estimate T_2 first: The argument that led to (3.16) also leads to the following inequality:

$$\int_{\mathbf{R}^d}\frac{\sin^2\left(\sqrt{\Psi(\xi)}\,r\right)}{\Psi(\xi)}\,|\hat\varphi(\xi)|^2\,d\xi \le \int_{\mathbf{R}^d}\left(r^2\wedge\frac{1}{\Psi(\xi)}\right)|\hat\varphi(\xi)|^2\,d\xi. \tag{5.24}$$

Because $\int_\epsilon^{2\epsilon}(r^2\wedge a)\,dr \le 3\epsilon\,(\epsilon^2\wedge a)$ for all $a,\epsilon>0$,

$$\begin{aligned} T_2 &\le \frac{3\epsilon}{(2\pi)^d}\int_{\mathbf{R}^d}\left(\epsilon^2\wedge\frac{1}{\Psi(\xi)}\right)|\hat\varphi(\xi)|^2\,d\xi \\ &\le 6\epsilon\,\mathscr{E}(\epsilon^2;\varphi). \end{aligned} \tag{5.25}$$

The estimate for T_1 is even simpler to derive: Because $|\sin\alpha - \sin\beta|^2$ is bounded above by the minimum of 4 and $2[1-\cos(\beta-\alpha)]$,

$$\begin{aligned} &\int_{\mathbf{R}^d}\frac{\left|\sin\left(\sqrt{\Psi(\xi)}\,(r+\epsilon)\right) - \sin\left(\sqrt{\Psi(\xi)}\,r\right)\right|^2}{\Psi(\xi)}|\hat\varphi(\xi)|^2\,d\xi \\ &\qquad\qquad \le \int_{\mathbf{R}^d}\frac{1-\cos\left(\sqrt{\Psi(\xi)}\,\epsilon\right)}{\Psi(\xi)}|\hat\varphi(\xi)|^2\,d\xi. \end{aligned} \tag{5.26}$$

This and the elementary inequality $1-\cos x \le x^2/2$ together yield the bound

$$\frac{1-\cos\left(\sqrt{\Psi(\xi)}\,\epsilon\right)}{\Psi(\xi)} \le 2(\epsilon^2 \wedge \frac{1}{\Psi(\xi)}). \tag{5.27}$$

Consequently,

$$\frac{1}{(2\pi)^d}\int_{\mathbf{R}^d} \frac{\left|\sin\left(\sqrt{\Psi(\xi)}\,(r+\epsilon)\right) - \sin\left(\sqrt{\Psi(\xi)}\,r\right)\right|^2}{\Psi(\xi)} |\hat{\varphi}(\xi)|^2\,d\xi \le 8\,\mathscr{E}(\epsilon^2;\varphi), \tag{5.28}$$

whence T_1 is at most t times the right-hand side of the preceding. This and (5.25) together yield the proof. □

Proof of Theorem 5.3. We start with the weak solution H to the stochastic heat equation. Throughout, $\varphi \in L^2_{\mathscr{L}}(\mathbf{R}^d)$ is held fixed.

If the integrability condition (5.3) holds, then according to Lemma 5.6, for all $T>0$,

$$\sup_{\substack{t,\epsilon\ge 0:\\ 0\le t\le t+\epsilon\le T}} \frac{\mathrm{E}\left(|H(t+\epsilon\,,\varphi) - H(t\,,\varphi)|^2\right)}{\mathscr{E}(\epsilon\,;\varphi)} \le 4e^{2T}+1 < \infty. \tag{5.29}$$

Since $\epsilon \mapsto \mathscr{E}(\epsilon\,;\varphi)$ is nondecreasing, a direct application of Gaussian-process theory implies that $\{H(t\,,\varphi)\}_{t\in[0,T]}$ has a continuous modification provided that

$$\int_{0+} \frac{\sqrt{\mathscr{E}(\epsilon\,;\varphi) + \mathscr{F}(\epsilon\,;\varphi)}}{\epsilon\sqrt{\log(1/\epsilon)}}\,d\epsilon < \infty. \tag{5.30}$$

See Lemma 6.4.6 of Marcus and Rosen (2006, p. 275). A standard measure-theoretic argument then applies to prove that $t \mapsto H(t\,,\varphi)$ has a continuous modification. A similar argument works for the weak solution W to the stochastic wave equation (2.4), but we appeal to Proposition 5.6 in place of 5.7. Thus, the first portion of our proof will be completed, once we prove that condition (5.3) implies (5.30).

Let us write

$$\begin{aligned}\int_{0+} \frac{\sqrt{\mathscr{E}(\epsilon\,;\varphi) + \mathscr{F}(\epsilon\,;\varphi)}}{\epsilon\sqrt{\log(1/\epsilon)}}\,d\epsilon &\le \int_{0+} \frac{\sqrt{\mathscr{E}(\epsilon\,;\varphi)}}{\epsilon\sqrt{\log(1/\epsilon)}}\,d\epsilon + \int_{0+} \frac{\sqrt{\mathscr{F}(\epsilon\,;\varphi)}}{\epsilon\sqrt{\log(1/\epsilon)}}\,d\epsilon \\ &:= I_1 + I_2.\end{aligned} \tag{5.31}$$

Let us consider I_1 first. We multiply and divide the integrand of I_1 by the square root of $g(1/\epsilon)$, and then and apply the Cauchy–Schwarz inequality to obtain the following:

$$\begin{aligned} I_1 &\le \left(\int_{0+} \frac{\mathscr{E}(\epsilon\,;\varphi) g(1/\epsilon)}{\epsilon}\,d\epsilon\right)^{1/2} \left(\int_{0+} \frac{1}{\epsilon\log(1/\epsilon) g(1/\epsilon)}\,d\epsilon\right)^{1/2} \\ &= \text{const}\cdot\left(\int_{0+} \frac{\mathscr{E}(\epsilon\,;\varphi) g(1/\epsilon)}{\epsilon}\,d\epsilon\right)^{1/2}. \end{aligned} \tag{5.32}$$

Note that

$$
\begin{aligned}
\int_0^{1/e} \frac{\mathscr{E}(\epsilon\,;\varphi)g(1/\epsilon)}{\epsilon}\,d\epsilon &= \frac{1}{(2\pi)^d}\int_{R^d} |\hat\varphi(\xi)|^2\,d\xi \left(\int_0^{1/e} \frac{g(1/\epsilon)}{1+\epsilon \mathrm{Re}\,\Psi(\xi)}\,d\epsilon\right)\\
&= \frac{1}{(2\pi)^d}\int_{\mathrm{Re}\,\Psi\geq e} |\hat\varphi(\xi)|^2\,d\xi(\cdots) + \frac{1}{(2\pi)^d}\int_{\mathrm{Re}\,\Psi< e} |\hat\varphi(\xi)|^2\,d\xi(\cdots)\\
&:= \frac{I_3+I_4}{(2\pi)^d},
\end{aligned}
\tag{5.33}
$$

with the notation being clear enough. We will look at I_3 and I_4 separately. We begin with I_4 first.

$$
\begin{aligned}
I_4 &\leq \int_{\mathrm{Re}\,\Psi<e}\int_0^{1/e} \frac{|\hat\varphi(\xi)|^2 g(1/\epsilon)}{1+\epsilon\mathrm{Re}\,\Psi(\xi)}\,d\epsilon\,d\xi\\
&\leq \int_{\mathrm{Re}\,\Psi<e} |\hat\varphi(\xi)|^2\,d\xi\cdot\int_0^{1/e} g(1/\epsilon)\,d\epsilon\\
&\leq \mathrm{const}\cdot\mathscr{E}(1\,;\varphi),
\end{aligned}
\tag{5.34}
$$

where we have used Proposition 5.2 to obtain the last inequality. In order to bound I_3 we let $N>e$ and look at the following:

$$
\begin{aligned}
I_3 &= \int_0^{1/e}\frac{g(1/\epsilon)}{1+\epsilon N}\,d\epsilon\\
&= \int_0^{1/N}\frac{g(1/\epsilon)}{1+\epsilon N}\,d\epsilon + \int_{1/N}^{1/e}\frac{g(1/\epsilon)}{1+\epsilon N}\,d\epsilon\\
&= I_5+I_6.
\end{aligned}
\tag{5.35}
$$

On one hand, some calculus shows that

$$
I_6 \leq \frac{\log N g(N)}{N}. \tag{5.36}
$$

On the other hand, we can change variables and appeal to Proposition 5.2 and deduce that

$$
\begin{aligned}
I_5 &\leq \int_N^\infty \frac{g(u)}{u^2}\,du\\
&\sim \frac{g(N)}{N}\qquad \text{as } N\to\infty.
\end{aligned}
\tag{5.37}
$$

We obtain the following upon setting $N := \mathrm{Re}\,\Psi(\xi)$ and combining inequalities (5.34)–(5.36) above:

$$
(5.3)\quad\Longrightarrow\quad I_1<\infty. \tag{5.38}
$$

We now look at I_2. Let us recall the definition of $\mathscr{F}(\epsilon\,;\varphi)$ and write

$$\begin{aligned}\mathscr{F}(\epsilon\,;\varphi) &= \frac{1}{(2\pi)^d}\left[\int_{|\Psi|\le 1/e}(\cdots)\,d\xi + \int_{1/e\le|\Psi|\le 1/\epsilon}(\cdots)\,d\xi + \int_{|\Psi|>1/\epsilon}(\cdots)\,d\xi\right]\\ &= \text{const}\cdot[I_7+I_8+I_9].\end{aligned}\tag{5.39}$$

Because $I_7 \le \epsilon^2\mathscr{E}(1\,;\varphi)$ whenever $\epsilon < 1/e$,

$$\int_0^{1/e}\frac{\sqrt{I_7}}{\epsilon\sqrt{\log(1/\epsilon)}}\,d\epsilon \le \text{const}\cdot\sqrt{\mathscr{E}(1\,;\varphi)}.\tag{5.40}$$

An application of Cauchy–Schwarz inequality yields

$$\begin{aligned}\int_0^{1/e}\frac{\sqrt{I_8}}{\epsilon\sqrt{\log(1/\epsilon)}}\,d\epsilon &= \int_0^{1/e}\left(\int_{1/e<|\Psi|\le 1/\epsilon}\frac{|\Psi(\xi)|^2|\hat\varphi(\xi)|^2}{1+\operatorname{Re}\Psi(\xi)}\,d\xi\right)^{1/2}\frac{\epsilon}{\sqrt{\log(1/\epsilon)}}\,d\epsilon\\ &\le \text{const}\cdot\left(\int_{\mathbf{R}^d}\int_0^{1/\Psi(\xi)}\frac{|\Psi(\xi)|^2|\hat\varphi(\xi)|^2}{1+\operatorname{Re}\Psi(\xi)}\frac{\epsilon^2}{\log(1/\epsilon)}\,d\epsilon\,d\xi\right)^{1/2}\\ &\le \text{const}\cdot\sqrt{\mathscr{E}(1\,;\varphi)}.\end{aligned}\tag{5.41}$$

We multiply and divide the integrand below by the square root of $g(1/\epsilon)$ and apply the Cauchy–Schwarz inequality in order to obtain

$$\begin{aligned}\int_0^{1/e}\frac{\sqrt{I_9}}{\epsilon\sqrt{\log(1/\epsilon)}}\,d\epsilon &\le \left(\int_0^{1/e}\int_{|\Psi|>1/\epsilon}\frac{|\hat\varphi(\xi)|^2 g(1/\epsilon)}{\epsilon(1+\operatorname{Re}\Psi(\xi))}\,d\xi\,d\epsilon\right)^{1/2}\\ &\le \text{const}\cdot\left(\int_{\mathbf{R}^d}\frac{g(\Psi(\xi))}{(1+\operatorname{Re}\Psi(\xi))}|\hat\varphi(\xi)|^2\,d\xi\right)^{1/2}.\end{aligned}\tag{5.42}$$

In the last inequality we have changed the order of integration and used Proposition 5.2. Taking into account inequalites (5.36)–(5.42), we obtain

$$\left(\int_{\mathbf{R}^d}\frac{g(1+\Psi(\xi))}{(1+\operatorname{Re}\Psi(\xi))}|\hat\varphi(\xi)|^2\,d\xi\right)^{1/2} < \infty \quad\Longrightarrow\quad I_2<\infty.\tag{5.43}$$

Inequalities (5.38) and (5.43) together with (5.31) imply that

$$\int_{0+}\frac{\sqrt{\mathscr{E}(\epsilon\,;\varphi)+\mathscr{F}(\epsilon\,;\varphi)}}{\epsilon\sqrt{\log(1/\epsilon)}}\,d\epsilon<\infty,\tag{5.44}$$

provided that (5.3) holds. This proves the first assertion of the theorem.

Suppose $\gamma > 0$, where

$$\gamma := \underline{\text{ind}}\,\mathscr{E}(\bullet\,;\varphi),\tag{5.45}$$

for brevity. Then by definition, $\mathscr{E}(\epsilon\,;\varphi) \le \epsilon^{\gamma+o(1)}$ as $\epsilon \downarrow 0$. Another standard result from Gaussian analysis, used in conjunction with Proposition 5.6 proves that H has a Hölder-continuous modification with Hölder exponent $\le \gamma/2$ (Marcus and Rosen, 2006, Theorem 7.2.1, p. 298). The proof for W is analogous.

For the remainder of the proof we consider only the weak solution H to the stochastic heat equation (2.3), and write $H_t := H(t\,,\varphi)$ for typographical ease. If $\gamma > 0$, then, Proposition 5.6 and elementary properties of normal laws together imply that

$$\inf_{t\ge 0} \|H_{t+\epsilon} - H_t\|_{L^2(\mathrm{P})} \ge \epsilon^{\gamma+o(1)} \quad \text{for infinitely-many } \epsilon \downarrow 0. \tag{5.46}$$

Consequently, for all $\delta \in (0\,,1)$, $q > \gamma/2$, $T > S > 0$, and $t \in [S\,,T]$—all fixed—the following holds for infinitely-many values of $\epsilon \downarrow 0$:

$$\begin{aligned} \mathrm{P}\left\{\sup_{r\in[S,T]} |H_{r+\epsilon} - H_r| \ge \epsilon^q\right\} &\ge \mathrm{P}\left\{|H_{t+\epsilon} - H_t| \ge \delta\, \|H_{t+\epsilon} - H_t\|_{L^2(\mathrm{P})}\right\} \\ &= 1 - \sqrt{\frac{2}{\pi}} \int_0^\delta \exp(-x^2/2)\, dx. \end{aligned} \tag{5.47}$$

The inequality $\exp(-x^2/2) \le 1$ then implies that

$$\mathrm{P}\left\{\sup_{r\in[S,T]} |H_{r+\epsilon} - H_r| \ge (1+o(1))\epsilon^q \text{ for infinitely many } \epsilon \downarrow 0\right\} \ge 1 - \sqrt{\frac{2}{\pi}}\,\delta. \tag{5.48}$$

Since $q > \gamma/2$ is arbitrary, we can enlarge it arbitrarily to infer that

$$\mathrm{P}\left\{\limsup_{\epsilon\downarrow 0} \sup_{r\in[S,T]} \frac{|H_{r+\epsilon} - H_r|}{\epsilon^q} = \infty\right\} \ge 1 - \sqrt{\frac{2}{\pi}}\,\delta. \tag{5.49}$$

Let $\delta \downarrow 0$ to find that

$$\limsup_{\epsilon\downarrow 0} \sup_{r\in[S,T]} \frac{|H_{r+\epsilon} - H_r|}{\epsilon^q} = \infty \qquad \text{a.s.} \tag{5.50}$$

This proves that any $q > \gamma/2$ is an almost-sure lower bound for the critical Hölder exponent of H. Moreover, in the case that $\gamma = 0$, we find that (5.50) holds a.s. for all $q > 0$. Thus, it follows that with probability one, H has no Hölder-continuous modification in that case. The proof is now complete. □

6. **Spatial and joint continuity: Proofs of Theorems 2.5 and 2.6**

Proof of Theorem 2.5. We begin by proving the portion of Theorem 2.5 that relates to the stochastic heat equation and its random-field solution $\{H(t\,,x);\ t \ge 0,\ x \in \mathbf{R}\}$; Theorem 2.1 guarantees the existence of the latter process.

Throughout we can—and will—assume without loss of generality that $x \mapsto H(t\,,x)$ is continuous, and hence so is the local time of $\bar{X}$ in its spatial variable. As we saw, during the course of the proof of Theorem 2.4, this automatically implies the condition (4.6), which we are free to assume henceforth.

According to Proposition 3.1, specifically Corollary 3.2,

$$\tfrac{1}{3}\mathscr{E}(t\,;\delta_x-\delta_y) \le \mathrm{E}\left(|H(t\,,x)-H(t\,,y)|^2\right) \le 4\mathscr{E}(t\,;\delta_x-\delta_y). \tag{6.1}$$

Since $|\hat{\delta}_x(\xi)-\hat{\delta}_y(\xi)|^2 = 2[1-\cos(\xi(x-y))]$, it follows from this and (4.1) that

$$\tfrac{2}{3}h(|x-y|) \le \mathrm{E}\left(|H(t\,,x)-H(t\,,y)|^2\right) \le 8h(|x-y|). \tag{6.2}$$

This implies that the critical Hölder exponent of $z \mapsto H(t\,,z)$ is almost surely equal to one-half of the following quantity:

$$\underline{\mathrm{ind}}\, h := \liminf_{\epsilon\downarrow 0} \frac{\log h(\epsilon)}{\log \epsilon}. \tag{6.3}$$

We will not prove this here, since it is very similar to the proof of temporal Hölder continuity (Theorem 5.3). Consequently,

$$z \mapsto H(t\,,z) \text{ has a Hölder-continuous modification iff } \underline{\mathrm{ind}}\, h > 0. \tag{6.4}$$

Among other things, this implies the equivalence of parts (1) and (2) of the theorem.

Let $Z(t\,,x)$ denote the local time of $\bar{X}$ at spatial value x at time $t \ge 0$. We prove that (1), (2), and (3) are equivalent by proving that (6.4) continues to hold when H is replaced by Z. Fortunately, this can be read off the work of Barlow (1988). We explain the details briefly. Because (4.6) holds, Theorem 5.3 of Barlow (1988) implies that there exists a finite constant $c > 0$ such that for all $t \ge 0$ and finite intervals $I \subset \mathbf{R}$,

$$\lim_{\delta\downarrow 0} \sup_{\substack{a,b\in I\\ |a-b|<\delta}} \frac{|Z(t\,,a)-Z(t\,,b)|}{\sqrt{h(|a-b|)\log(1/|b-a|)}} \ge c\left(\sup_{x\in I} Z(s\,,x)\right)^{1/2} \qquad \text{a.s.} \tag{6.5}$$

Moreover, we can choose the null set to be independent of all intervals $I \subset \mathbf{R}$ with rational endpoints. In fact, we can replace I by $\mathbf{R}$, since $a \mapsto Z(t\,,a)$ is supported by the closure of the range of the process $\bar{X}$ up to time t, and the latter range is a.s. bounded since $\bar{X}$ is cadlag.

By their very definition local times satisfy $\int_{-\infty}^{\infty} Z(s\,,x)\,dx = s$ a.s. Thus, $\sup_{x\in\mathbf{R}} Z(t\,,x) > 0$ a.s., whence it follows that for all $q > \frac{1}{2}\,\underline{\mathrm{ind}}\, h$,

$$\lim_{\delta\downarrow 0} \sup_{|a-b|<\delta} \frac{|Z(t\,,a)-Z(t\,,b)|}{|a-b|^q} = \infty \qquad \text{a.s.} \tag{6.6}$$

That is, there is no Hölder-continuous modification of $a \mapsto Z(t\,,a)$ of order $> \frac{1}{2}\,\underline{\text{ind}}\,h$. In particular, if $\underline{\text{ind}}\,h = 0$, then $a \mapsto Z(t\,,a)$ does not have a Hölder-continuous modification.

Define $d(a\,,b) := \sqrt{h(|a-b|)}$; it is easy to see that d is a pseudo-metric on $\mathbf{R}$. According to Bass and Khoshnevisan (1992),

$$\limsup_{\delta\downarrow 0}\sup_{d(a,b)<\delta}\frac{|Z(t\,,a)-Z(t\,,b)|}{\int_0^{d(a,b)}(\log N(u))^{1/2}\,du} \le 2\left(\sup_x Z(t\,,x)\right)^{1/2} \qquad \text{a.s.,} \tag{6.7}$$

where $N(u)$ denotes the smallest number of d-balls of radius $\le u$ needed to cover $[-1\,,1]$. This sharpened an earlier result of Barlow (1985, Theorem 1.1). Furthermore, $\sup_x Z(t\,,x) < \infty$ a.s. (Bass and Khoshnevisan, 1992, Theorem 3.1). Consequently,

$$\sup_{d(a,b)<\delta}|Z(t\,,a)-Z(t\,,b)| = O\left(\int_0^{\delta}(\log N(u))^{1/2}\,du\right) \qquad \text{as } \delta\downarrow 0. \tag{6.8}$$

Since h is increasing,

$$\sup_{|a-b|<\delta}|Z(t\,,a)-Z(t\,,b)| = O\left(\int_0^{h^{-1}(\delta)}(\log N(u))^{1/2}\,du\right) \qquad \text{as } \delta\downarrow 0. \tag{6.9}$$

According to equations (6.128) and (6.130) of Marcus and Rosen (2006, Lemma 6.4.1, p. 271), there exists a finite constant $c>0$ such that for all $u>0$ small,

$$\begin{aligned} N(u) &\le \frac{c}{\ell_2\left\{(x\,,y)\in[-1\,,1]^2:\ h(|x-y|)<u/4\right\}} \\ &\le \frac{\text{const}}{h^{-1}(u/4)}, \end{aligned} \tag{6.10}$$

where $\ell_2(A)$ denotes the Lebesgue measure of $A\subset\mathbf{R}^2$. [Specifically, we apply Lemma 6.4.1 of that reference with their $K:=[-1\,,1]$ and their $\mu_4 := c$.] This and (6.9) together imply that with probability one the following is valid: As $\delta\downarrow 0$,

$$\begin{aligned} \sup_{|a-b|<\delta}|Z(t\,,a)-Z(t\,,b)| &= O\left(\int_0^{\delta}|\log u|^{1/2}\,h(du)\right) \\ &= O\left(h(\delta)|\log(1/\delta)|^{1/2}\right) + O\left(\int_0^{\delta}\frac{h(u)}{u|\log u|^{1/2}}\,du\right). \end{aligned} \tag{6.11}$$

The last line follows from integration by parts. If $\gamma := \underline{\text{ind}}\,h > 0$, then $h(\delta) = o(\delta^q)$ for all fixed choices of $q\in(0\,,\gamma/2)$. It follows that $a\mapsto Z(t\,,a)$ is Hölder continuous of any order $<\gamma/2$. Among other things, this implies (6.4) with Z replacing H, whence it follows that (1)–(3) of the theorem are equivalent.

The hyperbolic portion of the theorem is proved similarly, but we use Proposition 3.6 in place of Proposition 3.1 everywhere. □

Proof of Theorem 2.6. Since β'' is positive, $\operatorname{Re}\Psi(\xi) \to \infty$ as $|\xi| \to \infty$. Therefore, for all $\epsilon, \vartheta > 0$ and $x \in \mathbf{R}$,

$$\begin{aligned}\mathscr{E}(\epsilon\,;\delta_x) &= \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d} \frac{d\xi}{(1/\epsilon) + \operatorname{Re}\Psi(\xi)} \\ &\le \text{const}\cdot\epsilon + \int_{\{\operatorname{Re}\Psi > \vartheta\}} \frac{d\xi}{(1/\epsilon) + \operatorname{Re}\Psi(\xi)}.\end{aligned} \tag{6.12}$$

If ξ is sufficiently large, then for all $\gamma \in (d\,,\beta'')$, we can find a constant $C_\gamma \in (0\,,\infty)$ such that $\operatorname{Re}\Psi(\xi) \ge C_\gamma|\xi|^\gamma$ for all $\xi \in \{\operatorname{Re}\Psi > \vartheta\}$. Consequently,

$$\begin{aligned}\mathscr{E}(\epsilon\,;\delta_x) &= O\left(\epsilon + \int_{\{\operatorname{Re}\Psi > \vartheta\}} \frac{d\xi}{(1/\epsilon) + |\xi|^\gamma}\right) \\ &= O\left(\epsilon^{1-(d/\gamma)}\right),\end{aligned} \tag{6.13}$$

as $\epsilon \downarrow 0$. Thus, $\underline{\operatorname{ind}}\,\mathscr{E}(\bullet\,;\delta_x) \le 1-(d/\beta'')$, where this index was introduced in (5.4). A similar calculation shows that $\underline{\operatorname{ind}}\,h \le \beta'' - d$; confer with (6.3) for the definition of this quantity. Thus, for all fixed $T > 0$, Proposition 5.6 and (6.2) together prove the following: For all $\tau < (\beta'' - d)/\beta''$, $\zeta < \beta'' - d$, $x, y \in \mathbf{R}$, and $s, t \in [0\,,T]$,

$$\mathrm{E}\left(|H(t\,,x) - H(s\,,y)|^2\right) \le \text{const}\cdot\left(|s-t|^\tau + |x-y|^\zeta\right). \tag{6.14}$$

A two-dimensional version of Kolmogorov's continuity theorem finishes the proof; see the proof of Theorem (2.1) of Revuz and Yor (1991, p. 25).

The proof, in the case of the stochastic wave equation, is similar, but we use Propositions 3.6 and 5.6 instead of Propositions 3.1 and 5.7, respectively. □

7. Heat equation via generators of Markov processes

We now consider briefly the stochastic heat equation, where the spatial movement is governed by the generator $\mathscr{L}$ of a [weakly] Markov process $X := \{X_t\}_{t\ge 0}$ that takes values in a locally compact separable metric space F. We assume further that X admits a symmetrizing measure m that is Radon and positive. Let us emphasize that m satisfies $(P_t f, g) = (f, P_t g)$ for all $t \ge 0$ and $f, g \in L^2(m)$, where $\{P_t\}_{t\ge0}$ denotes the transition operators of X, and $(\varphi_1\,,\varphi_2) := \int \varphi_1\varphi_2\,dm$ for all $\varphi_1, \varphi_2 \in L^2(m)$.

7.1. **The general problem.** Consider the stochastic heat equation

$$\left|\begin{aligned}&\partial_t u(t\,,x) = (\mathscr{L}u)(t\,,x) + \dot{w}(t\,,x),\\ &u(0\,,x) = 0,\end{aligned}\right. \tag{7.1}$$

valid for all $t \ge 0$ and $x \in F$. Here, the underlying noise w in (7.1) is a Gaussian martingale measure on $\mathbf{R}_+ \times F$ in the sense of Walsh (1986): w is defined on the filtered probability space $(\Omega\,,\mathscr{F},\{\mathscr{F}_t\}_{t\ge0},\mathrm{P})$ and $w_t(\varphi) := \int_0^t\int_F \varphi(s\,,x)\,w(dx\,ds)$ defines an $\{\mathscr{F}_t\}_{t\ge0}$-martingale for $\varphi \in L^2(ds \times m)$; w can be characterized by the covariance functional for the corresponding Wiener integrals:

$$\mathrm{E}\left(\int f\,dw \cdot \int g\,dw\right) = \int_0^\infty \int_F \int_F f(s\,,x)g(s\,,y)\,m(dx)\,m(dy)\,ds, \tag{7.2}$$

for all $f, g \in L^2(ds \times m)$.

We can follow the description of Walsh (1986) and write the weak form of equation (7.1) as follows: For all $\varphi \in L^2(m)$ and $t \geq 0$,

$$u(t\,,\varphi) = \int_0^t \int_F (P_{t-s}\varphi)(x)\,w(dx\,ds). \tag{7.3}$$

Lemma 7.1. *The integral defined by* (7.3) *is well defined for all* $\varphi \in L^2(m)$.

Proof. We follow closely the proof of Proposition 2.8, and apply the fact that the semigroup $\{P_s\}_{s\geq 0}$ is a contraction on $L^2(m)$. □

Let Z denote the occupation measure of X; consult (1.3). The following is the key result of this section. It identifies an abstract Hilbertian quasi-isometry between the occupation-measure L^2-norm of X and a similar norm for the solution to the stochastic heat equation (7.1) for $\mathscr{L}$.

Theorem 7.2. *If u denotes the weak solution to* (7.1), *then for all* $\varphi \in L^2(m)$ *and* $t \geq 0$,

$$\tfrac{1}{8}\,t\mathrm{E}\left(|u(t\,,\varphi)|^2\right) \leq \mathrm{E}_m\left(|Z(t\,,\varphi)|^2\right) \leq 4t\mathrm{E}\left(|u(t\,,\varphi)|^2\right). \tag{7.4}$$

As usual, E_m refers to the expectation operator for the process X, started according to the measure m.

The preceding theorem follows from the next formula.

Proposition 7.3. *If u denotes the weak solution to* (7.1), *then for all* $\varphi \in L^2(m)$ *and* $t \geq 0$,

$$\mathrm{E}_m\left(|Z(t\,,\varphi)|^2\right) = 4\int_0^t \mathrm{E}\left(|u(s/2\,,\varphi)|^2\right)\,ds. \tag{7.5}$$

Proof of Proposition 7.3. Since m is a symmetrizing measure for X, the P_m-law of X_u is m for all $u \geq 0$. By the Markov property and Tonelli's theorem,

$$\mathrm{E}_m\left(|Z(t\,,\varphi)|^2\right) = 2\int_0^t \int_u^t (P_{v-u}\varphi\,,\varphi)\,dv\,du. \tag{7.6}$$

We computing the Laplace transform of both sides, viz.,

$$
\begin{aligned}
\int_0^\infty e^{-\lambda t}\mathrm{E}_m\left(|Z(t\,,\varphi)|^2\right)\,dt &= 2\int_0^\infty\int_0^t\int_u^t e^{-\lambda t}\left(P_{v-u}\varphi\,,\varphi\right)\,dv\,du\,dt\\
&= 2\int_0^\infty\int_u^\infty\left(\int_v^\infty e^{-\lambda t}\,dt\right)\left(P_{v-u}\varphi\,,\varphi\right)\,dv\,du\\
&= \frac{2}{\lambda^2}\int_0^\infty e^{-\lambda s}\left(P_s\varphi\,,\varphi\right)\,ds.
\end{aligned}\tag{7.7}
$$

The exchange of the integrals is justified because $(P_r\varphi\,,\varphi) = \|P_{r/2}\varphi\|^2_{L^2(m)}$ is positive and finite. Because $\varphi\in L^2(m)$, Fubini's theorem implies that for all $\lambda>0$,

$$
\int_0^\infty e^{-\lambda t}\mathrm{E}_m\left(|Z(t\,,\varphi)|^2\right)\,dt = \frac{2}{\lambda^2}\left(R_\lambda\varphi\,,\varphi\right),\tag{7.8}
$$

where $R_\lambda := \int_0^\infty \exp(-\lambda s)P_s\,ds$ defines the resolvent of $\{P_t\}_{t\geq 0}$.

Let T_λ denote an independent mean-$(1/\lambda)$ exponential holding time. The preceding display can be rewritten as follows:

$$
\mathrm{E}_m\left(|Z(T_\lambda\,,\varphi)|^2\right) = \frac{2}{\lambda}\left(R_\lambda\varphi\,,\varphi\right)\qquad\text{for all }\lambda>0.\tag{7.9}
$$

Next we consider the weak solution u to the stochastic heat equation (7.1) by first observing that $\mathrm{E}(|u(t\,,\varphi)|^2) = \int_0^t\|P_s\varphi\|^2_{L^2(m)}\,ds$. It follows from this, and successive applications of Tonelli's theorem, that for all $\beta>0$,

$$
\begin{aligned}
\mathrm{E}\left(|u(T_\beta\,,\varphi)|^2\right) &= \int_0^\infty \beta e^{-\beta t}\int_0^t\|P_s\varphi\|^2_{L^2(m)}\,ds\,dt\\
&= \int_0^\infty e^{-\beta s}\|P_s\varphi\|^2_{L^2(m)}\,ds.
\end{aligned}\tag{7.10}
$$

Because $\|P_s\varphi\|^2_{L^2(m)} = (P_{2s}\varphi\,,\varphi)$, we may apply Fubini's theorem once more, and select $\beta := 2\lambda$, to find that

$$
\begin{aligned}
\mathrm{E}\left(|u(T_{2\lambda}\,,\varphi)|^2\right) &= \int_0^\infty e^{-2\lambda s}\left(P_{2s}\varphi\,,\varphi\right)\,ds\\
&= \left(\varphi\,,\int_0^\infty e^{-2\lambda s}P_{2s}\varphi\,ds\right)\\
&= \frac{1}{2}\left(R_\lambda\varphi\,,\varphi\right).
\end{aligned}\tag{7.11}
$$

The condition of square integrability for φ justifies the appeal to Fubini's theorem. We can compare (7.9) and (7.11) to find that

$$
\mathrm{E}_m\left(|Z(T_\lambda\,,\varphi)|^2\right) = \frac{4}{\lambda}\mathrm{E}\left(|u(T_{2\lambda}\,,\varphi)|^2\right)\qquad\text{for all }\lambda>0.\tag{7.12}
$$

Define $q(t) := \mathrm{E}_m(|Z(t\,,\varphi)|^2)$, $\rho(t) := \mathrm{E}(|u(t/2\,,\varphi)|^2)$ and $\mathbf{1}(t) := 1$ for all $t \geq 0$. The preceding shows that the Laplace transform of q is equal to 4 times the product of the respective Laplace transforms of ρ and $\mathbf{1}$. Thus, we can invert to find that $q = 4\rho * \mathbf{1}$, which is another way to state the theorem. □

Proof of Theorem 7.2. Let us choose and fix a measurable function $\varphi : F \to \mathbf{R}$ such that $|\varphi| \in L^2(m)$. The defining isometry for Wiener integrals yields the following identity, where both sides are convergent: $\mathrm{E}(|u(t\,,\varphi)|^2) = \int_0^t \|P_s\varphi\|_{L^2(m)}^2\,ds$. Proposition 7.3 implies that

$$2t\mathrm{E}\left(|u(t/4\,,\varphi)|^2\right) \leq \mathrm{E}_m\left(|Z(t\,,\varphi)|^2\right) \leq 4t\mathrm{E}\left(|u(t/2\,,\varphi)|^2\right). \tag{7.13}$$

[For the lower bound, we use the bound $\int_0^t \mathrm{E}(|u(s/2\,,\varphi)|^2)\,ds \geq \int_{t/2}^t \mathrm{E}(|u(s/2\,,\varphi)|^2)\,ds$.] By monotonicity, $\mathrm{E}(|u(t/2\,,\varphi)|^2) \leq \mathrm{E}(|u(t\,,\varphi)|^2)$, whence follows the announced upper bound for $\mathrm{E}_m(|Z(t\,,\varphi)|^2)$.

In order to prove the other bound we first write

$$Z(t\,,\varphi) = Z(t/2\,,\varphi) + Z(t/2\,,\varphi) \circ \theta_{t/2}, \tag{7.14}$$

where $\{\theta_s\}_{s\geq 0}$ denotes the collection of all shifts on the paths of X. We care only about distributional properties. Therefore, by working on an appropriate probability space, we can always insure that these shifts can be constructed; see Blumenthal and Getoor (1968).

We apply the Markov property at time $t/2$. Since $\mathrm{P}_m \circ X_{t/2}^{-1} = m$, it follows that

$$\mathrm{E}_m\left(\left|Z(t/2\,,\varphi) \circ \theta_{t/2}\right|^2\right) = \mathrm{E}_m\left(|Z(t/2\,,\varphi)|^2 \circ \theta_{t/2}\right) = \mathrm{E}_m\left(|Z(t/2\,,\varphi)|^2\right), \tag{7.15}$$

and hence $\mathrm{E}_m(|Z(t\,,\varphi)|^2) \leq 4\mathrm{E}_m(|Z(t/2\,,\varphi)|^2)$. Consequently,

$$\mathrm{E}_m\left(|Z(t\,,\varphi)|^2\right) \leq 16\mathrm{E}_m\left(|Z(t/4\,,\varphi)|^2\right) \qquad \text{for all } t \geq 0. \tag{7.16}$$

This and the first inequality of (7.13) together imply the remaining bound in the statement of the theorem. □

7.2. **The stochastic heat equation in dimension $2-\epsilon$.** We now specialize the setup of the preceding subsection to produce an interesting family of examples: We suppose that F is a locally compact subset of $\mathbf{R}^d$ for some integer $d \geq 1$, and m is a positive Radon measure on F, as before. Let $\{R_\lambda\}_{\lambda>0}$ denote the resolvent of X, and suppose that X has jointly continuous and uniformly bounded resolvent densities $\{r_\lambda\}_{\lambda>0}$. In particular, $r_\lambda(x\,,y) \geq 0$ for all $x, y \in F$ and

$$(R_\lambda f)(x) = \int_F r_\lambda(x\,,y) f(y)\,m(dy), \tag{7.17}$$

for all measurable functions $f : F \to \mathbf{R}_+$. Recall that X has local times $\{Z(t\,,x)\}_{t\geq 0, x\in F}$ if and only if for all measurable functions $f : F \to \mathbf{R}_+$, and every $t \geq 0$,

$$Z(t\,,f) = \int_F Z(t\,,z)f(z)\,m(dz), \tag{7.18}$$

valid P_x-a.s. for all $x \in F$. Choose and fix some point $a \in F$, and define

$$f_\epsilon^a(z) := \frac{\mathbf{1}_{B(a,\epsilon)}(z)}{m(B(a\,,\epsilon))} \qquad \text{for all } z \in F \text{ and } \epsilon > 0. \tag{7.19}$$

Of course, $B(a\,,\epsilon)$ denotes the ball of radius ϵ about a, measured in the natural metric of F. Because r_λ is jointly continuous, $\lim_{\epsilon\downarrow 0}(R_\lambda f_\epsilon^a)(x) = r_\lambda(x\,,a)$, uniformly for x-compacta. Define $\varphi_{\epsilon,\delta} := f_\epsilon^a - f_\delta^a$, and observe that $\varphi \in L^2(m)$. Furthermore,

$$\lim_{\epsilon,\delta\downarrow 0} (R_\lambda\varphi_{\epsilon,\delta}\,,\varphi_{\epsilon,\delta}) = \lim_{\epsilon,\delta\downarrow 0} \{(R_\lambda f_\epsilon^a\,,f_\epsilon^a) - 2\,(R_\lambda f_\epsilon^a\,,f_\delta^a) + (R_\lambda f_\delta^a\,,f_\delta^a)\} = 0. \tag{7.20}$$

If $h \in L^2(m)$, then the weak solution h to (7.1)—where $\mathscr{L}$ denotes the L^2-generator of X—satisfies

$$\begin{aligned} \mathrm{E}\left(|u(t\,,h)|^2\right) &= \int_0^t \|P_s h\|^2_{L^2(m)}\,ds \\ &\leq e^{2\lambda t}\int_0^\infty e^{-2\lambda s}\|P_s h\|^2_{L^2(m)}\,ds \\ &= \frac{e^{2\lambda t}}{2}(R_\lambda h\,,h). \end{aligned} \tag{7.21}$$

Therefore, Theorem 7.2 and (7.20) together imply that $\{u(t\,,f_\epsilon^a)\}_{\epsilon>0}$ is a Cauchy sequence in $L^2(\mathrm{P})$ for all $t \geq 0$. In other words, we have shown that the stochastic heat equation (7.1) has a "random-field solution."

Example 7.4. It is now easy to check, using the heat-kernel estimates of Barlow (1998, Theorems 8.1.5 and 8.1.6), that for all $d \in (0\,,2)$ there exists a compact "fractal" $F \subset \mathbf{R}^2$ of Hausdorff dimension d such that Brownian motion on F satisfies the bounded/continuous resolvent-density properties here. The preceding proves that if we replace $\mathscr{L}$ by the Laplacian on F, then the stochastic heat equation (7.1) has a "random-field solution." Specifically, the latter means that for all $t \geq 0$ and $a \in F$, $u(t\,,f_\epsilon^a)$ converges in $L^2(\mathrm{P})$ as $\epsilon \downarrow 0$, where f_ϵ^a is defined in (7.19). This example comes about, because the fractional diffusions of Barlow (1998) have local times when the dimension d of the fractal on which they live satisfies $d < 2$. See Barlow (1998, Theorem 3.32), for instance. □

8. A semilinear parabolic problem

We consider the semilinear problems that correspond to the stochastic heat equation (2.3). At this point in the development of SPDEs, we can make general sense of nonlinear stochastic PDEs only when the linearized SPDE is sensible. Thus, we assume henceforth that $d = 1$.

We investigate the semilinear stochastic heat equation. Let $b : \mathbf{R} \to \mathbf{R}$ be a measurable function, and consider the solution H_b to the following SPDE:

$$\left| \begin{aligned} &\partial_t H_b(t\,,x) = (\mathscr{L} H_b)(t\,,x) + b(H_b(t\,,x)) + \dot{w}(t\,,x), \\ &H_b(0\,,x) = 0, \end{aligned} \right. \tag{8.1}$$

where $\mathscr{L}$ denotes the generator of the Lévy process X, as before.

Equation (8.1) has a chance of making sense only if the linearized problem (2.3) has a random-field solution H, in which case we follow Walsh (1986) and write the solution H_b as the solution to the following:

$$H_b(t\,,x) = H(t\,,x) + \int_0^t \int_{-\infty}^{\infty} b(H_b(s\,,x-y))\, P_{t-s}(dy)\, ds, \tag{8.2}$$

where the measures $\{P_t\}_{t\ge 0}$ are determined from the semigroup of X by $P_t(E) := (P_t \mathbf{1}_E)(0)$ for all Borel sets $E \subset \mathbf{R}$. [This is standard notation.] We will soon see that this random integral equation has a "good solution" H_b under more or less standard conditions on the function b. But first, let us make an observation.

Lemma 8.1. *If* (2.3) *has a random-field solution, then the process X has a jointly measurable transition density $\{p_t(x)\}_{t>0, x\in\mathbf{R}}$ that satisfies the following: For all $\eta > 0$ there exists a constant $C := C_\eta \in (0\,,\infty)$ such that for all $t > 0$,*

$$\int_0^t \|p_s\|_{L^2(\mathbf{R})}^2\, ds \le C e^{\eta t}. \tag{8.3}$$

Finally, $(t\,,x) \mapsto p_t(x)$ is uniformly continuous on $[\epsilon\,,T] \times \mathbf{R}$ for all fixed $\epsilon, T > 0$.

Proof. We can inspect the function $y = x\exp(-x)$ to find that $\exp(-x) \le (1+x)^{-1}$ for all $x \ge 0$. Consequently,

$$\int_{-\infty}^{\infty} e^{-\epsilon \mathrm{Re}\,\Psi(\xi)}\, d\xi \le \int_{-\infty}^{\infty} \frac{d\xi}{1+\epsilon \mathrm{Re}\,\Psi(\xi)}. \tag{8.4}$$

The second integral, however, has been shown to be equivalent to the existence of random-field solutions to (2.3); see (3.17). It follows that the first integral in (8.4) is convergent. We apply the inversion theorem to deduce from this that the transition densities of X are given by $p_t(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} \exp\{-ix\xi - t\Psi(\xi)\} d\xi$, where the integral is absolutely convergent for all $t > 0$ and $x \in \mathbf{R}$. Among other things, this formula implies the uniform continuity of

$p_t(x)$ away from $t = 0$. In addition, by Plancherel's theorem, for all $s > 0$,

$$\begin{aligned} \|p_s\|^2_{L^2(\mathbf{R})} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|e^{-s\Psi(\xi)}\right|^2 d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-2s\mathrm{Re}\,\Psi(\xi)}\, d\xi. \end{aligned} \tag{8.5}$$

Therefore, Lemma 3.3 and Tonelli's theorem together imply that for all $\lambda > 0$,

$$\int_0^t \|p_s\|^2_{L^2(\mathbf{R})}\, ds \le \frac{e^{2t/\lambda}}{4\pi}\int_{-\infty}^{\infty} \frac{d\xi}{(1/\lambda) + \mathrm{Re}\,\Psi(\xi)} < \infty. \tag{8.6}$$

This completes the proof. □

Thanks to the preceding lemma, by (8.1) we mean a solution to the following:

$$H_b(t\,,x) = H(t\,,x) + \int_0^t \int_{-\infty}^{\infty} b(H_b(s\,,x-y))\, p_{t-s}(y)\, dy\, ds. \tag{8.7}$$

For the following we assume that the underlying probability space $(\Omega\,,\mathscr{F},\mathrm{P})$ is complete.

Theorem 8.2. *Suppose b is bounded and globally Lipschitz, and the stochastic heat equation* (2.3) *has a random-field solution H; thus, in particular, $d = 1$. Then, there exists a modification of H, denoted still by H, and a process H_b with the following properties:*

1. *$H_b \in L^p_{loc}(\mathbf{R}_+ \times \mathbf{R})$ for all $p \in [1\,,\infty)$.*
2. *With probability one,* (8.7) *holds for all $(t\,,x) \in \mathbf{R}_+ \times \mathbf{R}$.*
3. *For all $T > 0$, J is a.s. bounded and continuous on $[0\,,T] \times \mathbf{R}$, where*

$$J(t\,,x) := \int_0^t \int_{-\infty}^{\infty} b(H_b(s\,,x-y))p_{t-s}(y)\, dy\, ds.$$

Remark 8.3. Before we proceed with a proof, we make two remarks:

1. It is possible to adapt the argument of Nualart and Pardoux (1994, Proposition 1.6) to deduce that the laws of H and H_b are mutually absolutely continuous with respect to one another; see also Dalang and Nualart (2004, Corollary 5.3) and Dalang, Khoshnevisan, and Nualart (2007, Equations (5.2) and (5.3)). A consequence of this mutual absolute continuity is that H_b is [Hölder] continuous iff H is.
2. Theorem 8.2 implies facts that are cannot be described by change-of-measure methods. For instance, it has the striking consequence that with probability one, H_b *and H blow up in exactly the same points*! [This is simply so, because $H_b - H$ is locally bounded.] For an example, we mention that the operators considered Example 5.5, when the parameter α there is ≤ 2, lead to discontinuous solutions H that blow up (a.s.) in every open subset of $\mathbf{R}_+ \times \mathbf{R}$ (Marcus and Rosen, 2006, Section 5.3). In those cases, H_b inherits this property as well.

Proof. We will need the following fact:

$$H \text{ is continuous in probability.} \tag{8.8}$$

In fact, we prove that H is continuous in $L^2(\mathrm{P})$.

Owing to (4.2), for all $t \geq 0$ and $x, y \in \mathbf{R}$,

$$\mathrm{E}\left(|H(t\,,x) - H(t\,,y)|^2\right) \leq \frac{e^{2t}}{2\pi}\int_{-\infty}^{\infty} \frac{1-\cos(\xi|x-y|)}{1+\mathrm{Re}\,\Psi(\xi)}\,d\xi. \tag{8.9}$$

Moreover, according to (3.17), $(1+\mathrm{Re}\,\Psi)^{-1} \in L^1(\mathbf{R})$. Therefore, the dominated convergence theorem implies that for all $T > 0$ and $y \in \mathbf{R}$,

$$\sup_{t\in[0,T]} \|H(t\,,x) - H(t\,,y)\|_{L^2(\mathrm{P})} \to 0 \quad \text{as } x \to y. \tag{8.10}$$

Similarly, Theorem 5.3 implies that for all $t \geq 0$,

$$\lim_{s\to t}\sup_{x\in\mathbf{R}} \|H(t\,,x) - H(s\,,x)\|_{L^2(\mathrm{P})}^2 \leq \lim_{s\to t}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d\xi}{(1/|t-s|) + \mathrm{Re}\,\Psi(\xi)}, \tag{8.11}$$

which is zero by the dominated convergence theorem. This and (8.10) together imply (8.8). Now we begin the proof in earnest.

Throughout, we fix

$$\mathrm{Lip}_b := \sup_{x\neq y}\frac{|b(x)-b(y)|}{|x-y|} \quad \text{and} \quad \lambda := 2\mathrm{Lip}_b. \tag{8.12}$$

The condition that b is globally Lipschitz tells precisely that Lip_b and/or λ are finite.

Now we begin with a fixed-point scheme: Set $u_0(t\,,x) := 0$, and define, iteratively for all integers $n \geq 0$,

$$u_{n+1}(t\,,x) := H(t\,,x) + \int_0^t\int_{-\infty}^{\infty} b(u_n(s\,,x-y))p_{t-s}(y)\,dy\,ds. \tag{8.13}$$

Consider the processes

$$D_{n+1}(x) := \int_0^\infty e^{-\lambda t}\,|u_{n+1}(t\,,x) - u_n(t\,,x)|\;dt \qquad \text{for all } n \geq 0 \text{ and } x \in \mathbf{R}. \tag{8.14}$$

Also, define r_λ to be the λ-potential density of X, given by

$$r_\lambda(z) := \int_0^\infty p_s(z)e^{-\lambda s}\,ds \qquad \text{for all } z \in \mathbf{R}. \tag{8.15}$$

According to Lemma 8.1, this is well defined.

The processes $D_1, D_2, \ldots$ satisfy the following recursion:

$$\begin{aligned} D_{n+1}(x) &\le \frac{\mathrm{Lip}_b}{\lambda}(D_n * r_\lambda)(x) \\ &= \frac{1}{2}(D_n * r_\lambda)(x), \end{aligned} \tag{8.16}$$

as can be seen by directly manipulating (8.13). We can iterate this to its natural end, and deduce that

$$D_{n+1}(x) \le 2^{-n-1}(D_0 * r_\lambda)(x) \qquad \text{for all } n \ge 0. \tag{8.17}$$

Since $u_1(t\,,x) - u_0(t\,,x) = H(t\,,x) + tb(0)$,

$$D_0(x) \le \int_0^\infty e^{-\lambda t}|H(t\,,x)|\,dt + \frac{|b(0)|}{\lambda}. \tag{8.18}$$

Thanks to (8.8), we can always select a measurable modification of $(\omega\,,t\,,x) \mapsto H(t\,,x)(\omega)$, by the separability theory of Doob (1953, Theorem 2.6, p. 61). Therefore, we can apply the preceding to a Lebesgue-measurable modification of $t \mapsto H(t\,,x)$ to avoid technical problems (Doob, 1953, Theorem 2.7, p. 62). In addition, it follows from this and convexity that

$$\int_{-\infty}^\infty \|D_0(x)\|_{L^2(\mathrm{P})}\, e^{-|x|}\,dx \le \int_{-\infty}^\infty \int_0^\infty e^{-\lambda t - |x|}\, \|H(t\,,x)\|_{L^2(\mathrm{P})}\,dt\,dx + \frac{2|b(0)|}{\lambda}. \tag{8.19}$$

We can apply Proposition 3.1, with its λ replaced by $(4\lambda)^{-1}$ here, to find that

$$\begin{aligned} \|H(t\,,x)\|_{L^2(\mathrm{P})}^2 &\le \frac{e^{\lambda t/2}}{4\pi}\int_{-\infty}^\infty \frac{d\xi}{(4\lambda)^{-1} + \mathrm{Re}\,\Psi(\xi)} \\ &:= \mathrm{const}\cdot e^{\lambda t/2}, \end{aligned} \tag{8.20}$$

where the constant depends neither on t nor on n. Consequently,

$$\int_{-\infty}^\infty \|D_0(x)\|_{L^2(\mathrm{P})} e^{-|x|}\,dx < \infty. \tag{8.21}$$

Because λr_λ is a probability density on $\mathbf{R}$, it follows that for all integers $n \ge 0$,

$$\int_{-\infty}^\infty \|D_{n+1}(x)\|_{L^2(\mathrm{P})}\, e^{-|x|}\,dx \le \mathrm{const}\cdot 2^{-n}. \tag{8.22}$$

Therefore, $\sum_{n=0}^\infty \int_{-\infty}^\infty \|D_n(x)\|_{L^2(\mathrm{P})} \exp(-|x|)\,dx < \infty$. Furthermore, for all $T, k > 0$,

$$\begin{aligned} \lim_{n\to\infty} \int_0^T \int_{-k}^k |u_{n+1}(t\,,x) - u_n(t\,,x)|\;dx\,dt &\le e^{\lambda T + k} \lim_{n\to\infty}\int_{-k}^k D_n(x) e^{-|x|}\,dx \\ &= 0 \qquad \text{a.s.} \end{aligned} \tag{8.23}$$

Because $L^1([0\,,T]\times[-k\,,k])$ is complete, and since $(\Omega\,,\mathscr{F},\mathrm{P})$ is complete, standard arguments show that there exists a process $u_\infty\in L^1_{loc}(\mathbf{R}_+\times\mathbf{R})$ such that

$$\lim_{n\to\infty}\int_0^T\int_{-k}^k|u_n(t\,,x)-u_\infty(t\,,x)|\,dx\,dt=0, \tag{8.24}$$

almost surely for all $T,k>0$. Since b is globally Lipschitz, it follows easily from this that outside a single set of P-measure zero,

$$u_\infty(t\,,x)=H(t\,,x)+\int_0^t\int_{-\infty}^\infty b(u_\infty(s\,,x-y))\,p_{t-s}(y)\,dy\,ds, \tag{8.25}$$

simultaneously for almost all $(t\,,x)\in\mathbf{R}_+\times\mathbf{R}$. An application of Fubini's theorem implies then that the assertion (8.25) holds a.s. for almost all $(t\,,x)\in\mathbf{R}_+\times\mathbf{R}$. [Observe the order of the quantifiers.]

Consider the finite Borel measure

$$\Upsilon(dt\,dx):=e^{-\lambda t-|x|}\,dt\,dx, \tag{8.26}$$

defined on $\mathbf{R}_+\times\mathbf{R}$. Our proof, so far, contains the fact that $u_\infty\in L^1(\Upsilon)$ almost surely. Moreover, thanks to (8.25), if $p\in[1\,,\infty)$ then

$$\|u_\infty(t\,,x)\|_{L^p(\mathrm{P})}\le\|H(t\,,x)\|_{L^p(\mathrm{P})}+\sup_z|b(z)|t\qquad\text{for all }(t\,,x)\in\mathbf{R}_+\times\mathbf{R}. \tag{8.27}$$

One of the basic properties of centered Gaussian random variables is that their pth moment is proportional to their second moment. Consequently,

$$\|u_\infty(t\,,x)\|_{L^p(\mathrm{P})}\le\text{const}\cdot\Big(\|H(t\,,x)\|_{L^2(\mathrm{P})}+1\Big), \tag{8.28}$$

which we know to be locally bounded. This and the Tonelli theorem together prove that $u_\infty\in L^p_{loc}(\mathbf{R}_+\times\mathbf{R})$ a.s.

We recall a standard fact from classical analysis: *If $f\in L^1(\Upsilon)$, then f is continuous in the measure Υ.* This means that for all $\delta>0$,

$$\lim_{\epsilon,\eta\to 0}\Upsilon\left\{(t\,,x):\ \Big|f(t+\eta\,,x+\epsilon)-f(t\,,x)\Big|>\delta\right\}=0, \tag{8.29}$$

and follows immediately from standard approximation arguments; see the original classic book of Zygmund (1935, §2.201, p. 17), for instance. Consequently, Lemma 8.1 and the already-proved fact that $u_\infty\in L^1(\Upsilon)$ a.s. together imply that u_∞ is continuous in measure a.s. Because b is bounded and Lipschitz, the integrability/continuity properties of $p_t(x)$, as explained in Lemma 8.1, together imply that $(t\,,x)\mapsto\int_0^t\int_{-\infty}^\infty b(u_\infty(s\,,x-y))\,p_{t-s}(y)\,dy\,ds=\int_0^t\int_{-\infty}^\infty b(u_\infty(s\,,z))\,p_{t-s}(x-z)\,dz\,ds$ is a.s. continuous, and bounded on $[0\,,T]\times\mathbf{R}$, for every nonrandom and fixed $T > 0$. Now let us *define*

$$H_b(t\,,x) := H(t\,,x) + \int_0^t \int_{-\infty}^{\infty} b(u_\infty(s\,,x-y))\, p_{t-s}(y)\, dy\, ds. \tag{8.30}$$

Thanks to (8.25),

$$\mathrm{P}\left\{H_b(t\,,x) = u_\infty(t\,,x)\right\} = 1 \qquad \text{for all } (t\,,x) \in \mathbf{R}_+ \times \mathbf{R}. \tag{8.31}$$

Thus, H_b is a modification of u_∞. In addition, the Tonelli theorem applies to tell us that H_b inherits the almost-sure local integrability property of u_∞. That is, $H_b \in L^p_{loc}(\mathbf{R}_+ \times \mathbf{R})$ a.s. Moreover, outside a single null set we have

$$J(t\,,x) = \int_0^t \int_{-\infty}^{\infty} b(u_\infty(s\,,x-y))\, p_{t-s}(y)\, dy\, ds \qquad \text{for all } (t\,,x) \in \mathbf{R}_+ \times \mathbf{R}. \tag{8.32}$$

This proves the theorem. □

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