

# Dynkin's isomorphism theorem and the stochastic heat equation\*

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## Abstract

Consider the stochastic heat equation  $\partial_t u = \mathcal{L}u + \dot{W}$ , where  $\mathcal{L}$  is the generator of a [Borel right] Markov process in duality. We show that the solution is locally mutually absolutely continuous with respect to a smooth perturbation of the Gaussian process that is associated, via Dynkin's isomorphism theorem, to the local times of the replica-symmetric process that corresponds to  $\mathcal{L}$ . In the case that  $\mathcal{L}$  is the generator of a Lévy process on  $\mathbf{R}^d$ , our result gives a probabilistic explanation of the recent findings of Foondun et al. [6].

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# 1 Introduction and main results

The purpose of this article is to give some probabilistic insight into the structure of the linear stochastic heat equation

$$\begin{cases} \frac{\partial}{\partial t} U(t, x) = (\mathcal{L}U)(t, x) + \dot{W}(t, x), \\ U(0, x) = 0, \end{cases} \quad (1.1)$$

where  $\mathcal{L}$ —the generator of a nice Markov process with values on a nice space  $E$ —acts on the variable  $x \in E$ ,  $t$  is strictly positive, and  $\dot{W}$  is a suitable space-time white noise on  $\mathbf{R}_+ \times E$ .

A typical example is when  $E = \mathbf{R}^d$ , and  $\mathcal{L}$  is the  $L^2$ -generator of a Lévy process  $\{X_t; t \geq 0\}$  on  $\mathbf{R}^d$ . Let  $X^*$  denote an independent copy of the Lévy process  $-X$  and consider the symmetric Lévy process  $\bar{X}$  defined by

$$\bar{X}_t := X_t + X_t^* \quad \text{for all } t \geq 0. \quad (1.2)$$

It has been shown recently in [6] that, under these conditions on  $\mathcal{L}$  and  $E$ , (1.1) has a random-field solution  $U$  if and only if  $\bar{X}$  has local times  $\{L_t^x\}_{t \geq 0, x \in \mathbf{R}^d}$ . Moreover, when the local times exist, many of the local features of  $x \mapsto U(t, x)$  are precisely the same as the corresponding features of  $x \mapsto L_t^x$ . Most notably,  $x \mapsto U(t, x)$  is [Hölder] continuous if and only if  $x \mapsto L_t^x$  is [Hölder] continuous. And the critical Hölder exponent of  $x \mapsto U(t, x)$  is the same as that of  $x \mapsto L_t^x$ .

The approach taken in [6] is a purely analytic one: One derives necessary and sufficient analytic conditions for the desired local properties of  $x \mapsto U(t, x)$  and/or  $x \mapsto L_t^x$ , and checks that the analytic conditions are the same.

The purpose of the present paper is to give an abstract probabilistic explanation for the many connections that exist between the solution to (1.1) and local times of the symmetrized process  $\bar{X}$ . Our explanation does not require us to study special local properties, and, moreover, allows us to study a much more general family of operators  $\mathcal{L}$  than those that correspond

to Lévy processes.

We close the Introduction by describing our main findings. Before we do that we identify precisely the family operators  $\mathcal{L}$  with which we are concerned, since this requires some careful development. We refer the reader to the recent monograph by Marcus and Rosen [10], which contains a wealth of information on Markov processes, local times, and their deep connections to Gaussian processes. Our notation for Markov processes is standard and follows [10] as well.

Let  $X := \{X_t\}_{t \geq 0}$  denote a Borel right process with values on a locally compact, separable metric space  $E$ , and let  $\{P_t\}_{t \geq 0}$  denote the semigroup of  $X$ . We assume that there exists a Radon measure  $m$  on  $E$  with respect to which  $X$  has regular transition functions  $p_t(x, y)$ .

Let  $L^2(m)$  denote the collection of all Borel-measurable functions  $f : E \rightarrow \mathbf{R}$  such that  $\|f\| < \infty$ , where

$$\|f\| := (f, f)^{1/2} \quad \text{and} \quad (g, h) := \int gh \, dm. \quad (1.3)$$

As usual, we define the  $L^2$ -generator  $\mathcal{L}$  and its domain as follows:

$$\text{Dom}[\mathcal{L}] := \left\{ \phi \in L^2(m) : \mathcal{L}\phi := \lim_{t \rightarrow 0^+} \frac{P_t\phi - \phi}{t} \text{ exists in } L^2(m) \right\}. \quad (1.4)$$

We assume that the process  $X$  has a dual process  $X^*$  under  $m$ , so that the adjoint  $P_t^*$  of  $P_t$  is itself a Markov semigroup on  $L^2(m)$ . We emphasize that our assumptions imply that each  $P_t$  [and also  $P_t^*$ ] is a contraction on  $L^2(m)$ .

Next, define

$$\bar{P}_t := P_t^* P_t \quad \text{for all } t \geq 0. \quad (1.5)$$

A moment's thought shows that  $\{\bar{P}_t\}_{t \geq 0}$  is a symmetric Markovian semigroup on  $L^2(m)$ ;  $\{\bar{P}_t\}_{t \geq 0}$  is the *replica semigroup* associated to the process  $X$ , and appears prominently in the work of Kardar [9], for example.

Also, consider the corresponding replica generator and its domain, viz.,

$$\text{Dom}[\bar{\mathcal{L}}] := \left\{ \phi \in L^2(m) : \bar{\mathcal{L}}\phi := \lim_{t \rightarrow 0^+} \frac{\bar{P}_t\phi - \phi}{t} \text{ exists in } L^2(m) \right\}, \quad (1.6)$$

as well as the  $\alpha$ -potentials  $\bar{U}_\alpha := \int_0^\infty e^{-\alpha s} \bar{P}_s ds$  for  $\alpha > 0$ .

Throughout, we assume that the semigroup  $\{\bar{P}_t\}_{t \geq 0}$  corresponds to a Borel right Markov process  $\{\bar{X}_t\}_{t \geq 0}$ , and  $\text{Dom}[\bar{\mathcal{L}}]$  is dense in  $L^2(m)$ . A simple computation shows that  $\{\bar{P}_t\}_{t \geq 0}$  has regular [and *symmetric*] transition functions that are denoted by  $\bar{p}_t(x, y)$ .

The process  $\{\bar{X}_t\}_{t \geq 0}$  has  $\alpha$ -potential densities that are described as follows: For all  $\alpha > 0$  and  $x, y \in E$ ,

$$\bar{u}_\alpha(x, y) = \int_0^\infty e^{-\alpha s} \bar{p}_s(x, y) ds. \quad (1.7)$$

Since  $\bar{p}_s(x, y) = \bar{p}_s(y, x)$ ,  $\{\bar{X}_t\}_{t \geq 0}$  is a strongly symmetric Markov process.

We are interested mainly in the case that  $\bar{u}_\alpha(x, x) < \infty$  for all  $x$  because that is precisely the condition that guarantees that  $\bar{X}$  has local times. As we shall see [Theorem 3.1], this condition is equivalent to the existence of an a.s.-unique mild solution to (1.1). When this condition is satisfied, we choose to normalize the local times so that for all  $\alpha > 0$  and  $x, y \in E$ ,

$$\bar{u}_\alpha(x, y) = \alpha \int_0^\infty e^{-\alpha s} \mathbf{E}^x L_s^y ds. \quad (1.8)$$

In broad terms, the Dynkin isomorphism theorem [10, Chapter 8] tells us that many of the local properties of the local-time process  $x \mapsto L_t^x$ , where  $t > 0$  is fixed, are the same as those of the process  $\eta_\alpha$ , where  $\eta_\alpha$  is a centered Gaussian process, indexed by  $E$ , with covariance

$$\text{Cov}(\eta_\alpha(x), \eta_\alpha(y)) = u_\alpha(x, y) \quad \text{for all } x, y \in E. \quad (1.9)$$

Here,  $\alpha > 0$  is a fixed but arbitrary.

The following is the main result of this paper. Its proof is a combination of results proved in Sections 2 and 3.

**Theorem 1.1.** *Assume that  $\bar{u}_\alpha$  is finite and continuous on  $E \times E$  for some (equivalently for all)  $\alpha > 0$ . Let  $U$  denote the unique mild solution to (1.1), where the white noise  $\dot{W}$  is chosen so that its control measure is  $dt \times dm$ . Choose and fix  $\alpha > 0$  and  $t > 0$ . Then, there exists a space-time process  $V_\alpha$  with the following properties:*

1. *For every compact set  $A \subset E$ , the law of  $\{V_\alpha(t, x)\}_{x \in A}$  is mutually absolutely continuous with respect to the law of  $\{U(t, x)\}_{x \in A}$ ;*
2. *There exists a process  $\{S_\alpha(t, x)\}_{x \in E}$ , independent of  $V_\alpha(t, \cdot)$ , such that  $S_\alpha(t, \cdot)$  is “smoother than”  $V_\alpha(t, \cdot)$  and  $S_\alpha(t, \cdot) + V_\alpha(t, \cdot)$  has the same law as Dynkin’s Gaussian process  $\eta^\alpha$  that is associated to the local times of  $\bar{X}$ .*

We will define “smoother than” more precisely in due time. But suffice it to say that because  $S_\alpha(t, \cdot)$  is “smoother than”  $V_\alpha(t, \cdot)$ , many of the local properties of  $S_\alpha(t, \cdot)$  follow from those of  $V_\alpha(t, \cdot)$ . For instance, the following properties hold [and many more]:

- If  $V_\alpha(t, \cdot)$  is [Hölder] continuous up to a modification, then so is  $S_\alpha(t, \cdot)$ ;
- The critical [global/local] Hölder exponent of  $S_\alpha(t, \cdot)$  is at least that of  $V_\alpha(t, \cdot)$ , etc.

By mutual absolute continuity, and thanks to Dynkin’s isomorphism theorem [10, Chapter 8], it follows that many of the local features of  $L_t$  and  $U(t, \cdot)$  are shared. This explains the aforementioned connections between (1.1) and local times in the case that  $\mathcal{L}$  is the generator of a Lévy process.

We will also prove [Section 4] that when  $\mathcal{L}$  is the generator of a nice Lévy process, then we can select a  $C^\infty$  version of  $S_\alpha(t, \cdot)$ . Thus, in such cases, “smooth” has the usual meaning.

Note that all the required assumptions for Theorem 1.1 are satisfied in case  $\mathcal{L}$  is the generator of a strongly symmetric Markov process  $X$  with finite continuous  $\alpha$ -potential densities ( $\alpha > 0$ ) and  $\text{Dom}[\mathcal{L}]$  is dense in  $L^2(m)$ . Indeed in that case  $\{\bar{X}_t\}_{t \geq 0}$  has the same law as  $\{X_{2t}\}_{t \geq 0}$ .

## 2 Preliminaries

### 2.1 Markov processes

We begin by making some remarks on the underlying Markov processes  $X$  and  $\bar{X}$ . The process  $X$  is chosen so that it has the following properties: First of all, we have the identity  $(P_t f)(x) = \int_E p_t(x, y) f(y) m(dy)$ , valid for all Borel functions  $f : E \rightarrow \mathbf{R}_+$ ,  $t > 0$ , and  $x \in E$ . And the Chapman–Kolmogorov equation holds pointwise:

$$p_{t+s}(x, y) = \int p_t(x, z) p_s(z, y) m(dz). \quad (2.1)$$

As was pointed out in the Introduction, a simple computation shows that  $\{\bar{P}_t\}_{t \geq 0}$  has regular [and *symmetric*] transition functions. In fact, they are described as follows: for all  $t > 0$  and  $x, y \in E$ ,

$$\bar{p}_t(x, y) = \int p_t(y, z) p_t(x, z) m(dz). \quad (2.2)$$

We establish in this subsection two technical lemmas. Here and throughout, we denote by  $M(E)$  the space of finite Borel-measurable signed measures on  $E$ .

**Lemma 2.1.** *If  $\|P_r^* \mu\| < \infty$  for some  $\mu \in M(E)$  and  $r > 0$ , then  $t \mapsto \|P_{t+r}^* \mu\|$  is a nonincreasing function on  $[0, \infty)$ . In particular, the function  $t \mapsto \bar{p}_t(x, x)$  is nonincreasing on  $(0, \infty)$  for all  $x \in E$ .*

*Proof.* Let us choose and fix the  $r > 0$  as given, and observe that because  $P_{t+r} = P_t P_r$  and  $P_t$  is a contraction on  $L^2(m)$ , it follows that  $\|P_{t+r}^* \mu\| < \infty$  for all  $t \geq 0$ .

Next, let us consider only  $\mu \in \text{Dom}[\bar{\mathcal{L}}]$ , so that  $\mu$  is for the time being a function. Because  $\bar{P}_t$  is a contraction on  $L^2(m)$  for all  $t \geq 0$ , it follows that  $\bar{P}_t \mu \in \text{Dom}[\bar{\mathcal{L}}]$  for all  $t \geq 0$ , and therefore

$$\begin{aligned} \frac{d}{ds} \|P_s^* \mu\|^2 &= \frac{d}{ds} (P_s^* \mu, P_s^* \mu) = \frac{d}{ds} (\bar{P}_s \mu, \mu) = (\bar{\mathcal{L}} \bar{P}_s \mu, \mu) \\ &= (\bar{\mathcal{L}} P_s \mu, P_s \mu), \end{aligned} \quad (2.3)$$

where  $d/ds$  denotes the right derivative at  $s$ . It is well known that  $\bar{\mathcal{L}}$  is a negative-definite operator. That is,

$$(\bar{\mathcal{L}}\phi, \phi) \leq 0 \quad \text{for all } \phi \in \text{Dom}[\bar{\mathcal{L}}]. \quad (2.4)$$

Indeed, because every  $P_t$  is a contraction on  $L^2(m)$ , it follows that  $(\bar{P}_t\phi, \phi) = \|P_t\phi\|^2 \leq (\phi, \phi)$ . Take the difference, divide by  $t$ , and then let  $t \downarrow 0$  to deduce (2.4). In turn, (2.4) and (2.3) together imply that for all  $\mu \in \text{Dom}[\bar{\mathcal{L}}]$ ,

$$\|P_{s+t}^*\mu\|^2 \leq \|P_s^*\mu\|^2 \quad \text{for all } s, t \geq 0. \quad (2.5)$$

Since every  $P_t^*$  is a contraction on  $L^2(m)$ , the assumed density of the domain of  $\bar{\mathcal{L}}$  implies that (2.5) continues to hold for all  $\mu \in L^2(m)$  and  $s, t > 0$ .

Now let  $\mu$  be a finite signed Borel measure on  $E$  such that  $\|P_r^*\mu\| < \infty$ ; we can apply (2.5), with  $\mu$  replaced by  $P_r^*\mu \in L^2(m)$ , and this leads us to the following inequality:

$$\|P_{s+t}^*P_r^*\mu\| \leq \|P_s^*P_r^*\mu\| \quad \text{for all } s, t > 0. \quad (2.6)$$

Since  $P_u^*P_v^*\mu = P_{u+v}^*\mu$ , the preceding shows that (2.5) holds for all  $s \geq r$  and all  $t \geq 0$ .

In order to conclude, we choose  $\mu := \delta_x$ . In that case, the Chapman–Kolmogorov equations imply that  $\|P_r^*\mu\| = \bar{p}_r(x, x)^{1/2} < \infty$  for all  $r > 0$ ; therefore, (2.5) implies the announced result.  $\square$

Let us end this subsection by introducing an estimate on  $\alpha$ -potentials.

**Lemma 2.2.** *Assume that  $\bar{u}_\alpha$  is finite on  $E \times E$  for  $\alpha > 0$ . Then for all  $x, y \in E$  and  $\alpha > 0$ ,*

$$\bar{u}_\alpha(x, y) \leq c_\alpha \bar{u}_1(y, y), \quad (2.7)$$

where  $c_\alpha = e(\alpha + 2\alpha^{-1})$ .

*Proof.* We begin by proving that for all  $x, y \in E$  and  $t \geq 1$ ,

$$E^x L_t^y \leq 2tE^y L_1^y. \quad (2.8)$$

In order to prove this we recall that if  $\theta$  denotes the shifts on the path of the underlying Markov process, then  $L_{t+s}^y = L_t^y + L_s^y \circ \theta_t$ ,  $\mathbb{P}^x$ -almost surely. Therefore,

$$\mathbb{E}^x L_{t+s}^y = \mathbb{E}^x L_t^y + \mathbb{E}^x \mathbb{E}^{X_t} L_s^y. \quad (2.9)$$

We can apply the strong Markov property to the first hitting time of  $y$  to find that  $\mathbb{E}^x L_v^y \leq \mathbb{E}^y L_v^y$  for all  $x, y \in E$  and  $v \geq 0$ . Consequently,

$$\mathbb{E}^x L_{t+s}^y \leq \mathbb{E}^y L_t^y + \mathbb{E}^y L_s^y \quad \text{for all } x, y \in E \text{ and } s, t \geq 0; \quad (2.10)$$

and therefore,  $\mathbb{E}^x L_n^y \leq n \mathbb{E}^y L_1^y$  for all integers  $n \geq 1$  and  $x, y \in E$ . If  $t \geq 1$ , then we can find an integer  $n \geq 2$  such that  $n - 1 \leq t < n$ , whence

$$\mathbb{E}^x L_t^y \leq \mathbb{E}^x L_n^y \leq n \mathbb{E}^y L_1^y \leq (t + 1) \mathbb{E}^y L_1^y \leq 2t \mathbb{E}^y L_1^y. \quad (2.11)$$

This establishes (2.8) for  $t \geq 1$ .

Next, we note that since  $t \mapsto L_t^y$  is nondecreasing a.s. [ $\mathbb{P}^a$  for all  $a \in E$ ],

$$\begin{aligned} \bar{u}_\alpha(x, y) &= \int_0^\alpha e^{-r} \mathbb{E}^x L_{r/\alpha}^y dr + \int_\alpha^\infty e^{-r} \mathbb{E}^x L_{r/\alpha}^y dr \\ &\leq \alpha \mathbb{E}^x L_1^y + \frac{2}{\alpha} \mathbb{E}^y L_1^y \cdot \int_\alpha^\infty r e^{-r} dr \\ &\leq (\alpha + 2\alpha^{-1}) \mathbb{E}^y L_1^y, \end{aligned} \quad (2.12)$$

thanks to (2.8) and the strong Markov property. On the other hand,

$$\bar{u}_1(y, y) = \int_0^\infty e^{-r} \mathbb{E}^y L_r^y dr \geq \mathbb{E}^y L_1^y \cdot \int_1^\infty e^{-r} dr. \quad (2.13)$$

The lemma follows from the preceding and (2.12).  $\square$

## 2.2 Gaussian random fields

Suppose  $\{G(x)\}_{x \in E}$  is a centered Gaussian process that is continuous in  $L^2(\mathbb{P})$ . The latter means that  $\mathbb{E}(|G(x) - G(y)|^2) \rightarrow 0$  as  $x \rightarrow y$ . It follows

that  $G$  has a separable version for which we can define

$$G(\mu) := \int_E G \, d\mu, \quad (2.14)$$

for all  $\mu$  in the space  $M(E)$  of finite Borel-measurable signed measures on  $E$ . Moreover,  $\{G(\mu)\}_{\mu \in M(E)}$  is a Gaussian random field with mean process zero and

$$\text{Cov}(G(\mu), G(\nu)) = \iint E[G(x)G(y)] \mu(dx) \nu(dy). \quad (2.15)$$

**Definition 2.3.** Let  $G$  and  $G_*$  denote two  $L^2(\mathbb{P})$ -continuous Gaussian processes indexed by  $E$ . We say that  $G$  is *smoother than*  $G_*$  if there exists a finite constant  $c$  such that

$$E\left(|G(x) - G(y)|^2\right) \leq cE\left(|G_*(x) - G_*(y)|^2\right) \quad \text{for all } x, y \in E. \quad (2.16)$$

We say that  $G$  is *as smooth as*  $G^*$  when  $G$  is smoother than  $G^*$  and  $G^*$  is smoother than  $G$ .

It is easy to deduce from general theory [10, Chapters 5–7] that if  $G$  is smoother than  $G_*$ , then the continuity of  $x \mapsto G_*(x)$  implies the continuity of  $x \mapsto G(x)$ . Similar remarks can be made about Hölder continuity and existence of nontrivial  $p$ -variations, in case the latter properties hold and/or make sense.

### 3 The heat and cable equations

Let  $\dot{W} := \{\dot{W}(t, x); t \geq 0, x \in E\}$  denote white noise on  $\mathbf{R}_+ \times E$  with control measure  $dt \times dm$ , and consider the stochastic heat equation (1.1), where  $t > 0$  and  $x \in E$ . Also, consider the stochastic cable equation with parameter  $\alpha > 0$ :

$$\begin{cases} \frac{\partial}{\partial t} V_\alpha(t, x) = (\mathcal{L}V_\alpha)(t, x) - \frac{\alpha}{2} V_\alpha(t, x) + \dot{W}(t, x), \\ V_\alpha(0, x) = 0, \end{cases} \quad (3.1)$$

where  $t > 0$  and  $x \in E$ .

First we establish the existence of a mild solution to (3.1) and to (1.1) [Theorem 3.1]. Since these are linear equations, issues of uniqueness do not arise.

Then, we return to the first goal of this section and prove that all local properties of  $U(t, \cdot)$  and  $V_\alpha(t, \cdot)$  are the same; see Proposition 3.3. We do this in two steps. First, we study the case that  $\alpha$  is sufficiently small. In that case, we follow a change of measure method that is completely analogous to Proposition 1.6 of Nualart and Pardoux [11]; see also Dalang and Nualart [5, Theorem 5.2]. In a second step, we bootstrap from small values of  $\alpha$  to large values of  $\alpha$  by an argument that might have other applications as well.

Next, we prove that  $V_\alpha(t, \cdot)$  is equal to a smooth perturbation of the associated Gaussian process that arises in the Dynkin isomorphism theorem; see Proposition 3.4. The key estimate is a peculiar inequality [Proposition 3.4] that is nontrivial even when  $X$  is Brownian motion.

As a consequence of all this, and thanks to the Dynkin isomorphism theorem, many of the local properties of the solution to (1.1) are the same as those of the local times of the process  $\bar{X}$ . We refer the reader to Chapter 8 of the book by Marcus and Rosen [10] for details on Dynkin's isomorphism theorem and its applications to the analysis of local properties of local times.

Let us concentrate first on the cable equation.

The weak solution to the Kolmogorov equation

$$\begin{cases} \frac{\partial}{\partial t} f(t, x, y) = (\mathcal{L}_y f)(t, x, y) - \frac{\alpha}{2} f(t, x, y), \\ f(0, x, \cdot) = \delta_x, \end{cases} \quad (3.2)$$

is the function  $f(t, x, y) := e^{-\alpha t/2} p_t(x, y)$ . Therefore, we can use the theory of Walsh [13, Chapter 3], and write the solution to (3.1) as

$$V_\alpha(t, x) = \int_0^t \int_E e^{-\alpha(t-s)/2} p_{t-s}(x, y) W(dy ds). \quad (3.3)$$

This is a well-defined Gaussian process if and only if the stochastic integral

has two finite moments. But then, Wiener's isometry tells us that

$$\begin{aligned} \mathbb{E} \left( |V_\alpha(t, x)|^2 \right) &= \int_0^t e^{-\alpha s} ds \int_E m(dy) |p_s(x, y)|^2 \\ &= \int_0^t e^{-\alpha s} \bar{p}_s(x, x) ds. \end{aligned} \quad (3.4)$$

Similarly, the stochastic heat equation (1.1) has the following solution:

$$U(t, x) = \int_0^t \int_E p_{t-s}(x, y) W(dy ds), \quad (3.5)$$

which is a well-defined Gaussian process if and only if its second moment is finite. Note that

$$\mathbb{E} \left( |U(t, x)|^2 \right) = \int_0^t \bar{p}_s(x, x) ds. \quad (3.6)$$

**Theorem 3.1.** *1. The stochastic cable equation (3.1) has an a.s.-unique mild solution if and only if the Markov process  $\bar{X}$  has local times.*

*2. The stochastic heat equation (1.1) has an a.s.-unique mild solution if and only if the Markov process  $\bar{X}$  has local times.*

*Proof.* According to [6, Lemma 3.5], the following holds for every nonincreasing measurable function  $g : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ , and  $t, \lambda > 0$ :

$$(1 - e^{-2t/\lambda}) \int_0^\infty e^{-2s/\lambda} g(s) ds \leq \int_0^t g(s) ds \leq e^{2t/\lambda} \int_0^\infty e^{-2s/\lambda} g(s) ds. \quad (3.7)$$

We apply Lemma 2.1 and (3.7), with  $\lambda := 2/\alpha$  and  $g(s) := e^{-\alpha s} \bar{p}_s(x, x)$  to find that

$$(1 - e^{-t\alpha}) \bar{u}_{2\alpha}(x, x) \leq \mathbb{E} \left( |V_\alpha(t, x)|^2 \right) \leq e^{t\alpha} \bar{u}_{2\alpha}(x, x) \quad (3.8)$$

Similarly, we apply (3.7) with  $\lambda := 1/\alpha$  and  $g(s) := \bar{p}_s(x, x)$  to obtain

$$(1 - e^{-2t\alpha}) \bar{u}_{2\alpha}(x, x) \leq \mathbb{E} \left( |U(t, x)|^2 \right) \leq e^{2t\alpha} \bar{u}_{2\alpha}(x, x). \quad (3.9)$$

This proves that the processes  $V_\alpha$  and  $U$  are well defined if and only if  $\bar{u}_{2\alpha}(x, x) < \infty$  for all  $x \in E$  [and some—hence all— $\alpha > 0$ ]. And the latter is equivalent to the existence of local times; see Blumenthal and Gettoor [2, Theorem 3.13 and (3.15), pp. 216–217].  $\square$

From now on, we assume that all the  $\alpha$ -potentials are finite and continuous on  $E \times E$ . Hence the stochastic cable equation (3.1) has an a.s.-unique mild solution such that  $\sup_{t \geq 0} \sup_{x \in A} \mathbb{E}(|V_\alpha(t, x)|^2) < \infty$  for every compact set  $A \subset E$ . It follows easily from this discussion that  $x \mapsto V_\alpha(t, x)$  is continuous in  $L^2(\mathbb{P})$ , and hence in probability as well.

Since  $x \mapsto V_\alpha(t, x)$  is continuous in probability, Doob's separability theorem implies that  $V_\alpha(t, \mu) := \int_E V_\alpha(t, x) \mu(dx)$  is well-defined for all finite signed Borel measures  $\mu$  on  $E$ . We will be particularly interested in the two examples,  $\mu := \delta_a$  and  $\mu := \delta_a - \delta_b$  for fixed  $a, b \in E$ . In those two cases,  $V_\alpha(t, \mu) = V_\alpha(t, a)$  and  $V_\alpha(t, \mu) = V_\alpha(t, a) - V_\alpha(t, b)$ , respectively. One can prove quite easily the following: For all finite signed Borel measures  $\mu$  on  $E$ ,

$$V_\alpha(t, \mu) = \int_0^t \int_E e^{-\alpha(t-s)/2} (P_{t-s}^* \mu)(y) W(dy ds), \quad (3.10)$$

provided that  $(|\mu|, \bar{U}_\alpha |\mu|) < \infty$ . This can be derived by showing that the second moment of the difference of the two quantities is zero. It also follows from the stochastic Fubini theorem of Walsh [13, Theorem 2.6, p. 296].

Using the same methods as before, one shows that  $U(t, \mu)$  is well defined if and only if  $\mu \in M(E)$  satisfies  $(|\mu|, \bar{U}_\alpha |\mu|) < \infty$ . The following result shows that each  $V_\alpha(t, \cdot)$  is as smooth as  $U(t, \cdot)$ . A much better result will be proved subsequently.

**Lemma 3.2.** *For all  $t, \alpha > 0$  and  $\mu \in M(E)$  such that  $(|\mu|, \bar{U}_\alpha |\mu|) < \infty$ ,*

$$\mathbb{E} \left( |V_\alpha(t, \mu)|^2 \right) \leq \mathbb{E} \left( |U(t, \mu)|^2 \right) \leq 3e^{\alpha t} \mathbb{E} \left( |V_\alpha(t, \mu)|^2 \right). \quad (3.11)$$

*Thus,  $\{V_\alpha(t, x); x \in E\}$  is as smooth as  $\{U(t, x); x \in E\}$  for every  $t, \alpha > 0$ .*

*Proof.* The first inequality follows from a direct computation. For the second bound we note that if  $0 \leq s \leq t$ , then  $(1 - e^{-\alpha s/2})^2 \leq (e^{\alpha t/2} - 1)^2 e^{-\alpha s} \leq$

$e^{\alpha(t-s)}$ . Therefore,

$$\begin{aligned} \mathbf{E} \left( |U(t, \mu) - V_\alpha(t, \mu)|^2 \right) &= \int_0^t \int_E (1 - e^{-\alpha s})^2 |(P_s^* \mu)(y)|^2 m(dy) ds \\ &\leq e^{\alpha t} \int_0^t e^{-\alpha s} \|P_s^* \mu\|^2 ds \\ &= e^{\alpha t} \mathbf{E} \left( |V_\alpha(t, \mu)|^2 \right). \end{aligned} \quad (3.12)$$

Because

$$\begin{aligned} \mathbf{E} [U(t, \mu) V_\alpha(t, \mu)] &= \int_0^t e^{-\alpha s} \|P_s^* \mu\|^2 ds \\ &\leq e^{\alpha t} \int_0^t e^{-\alpha s} \|P_s^* \mu\|^2 ds \\ &= e^{\alpha t} \mathbf{E} \left( |V_\alpha(t, \mu)|^2 \right), \end{aligned} \quad (3.13)$$

the second inequality of the lemma follows.  $\square$

We propose to prove Proposition 3.3 which is a better version of Lemma 3.2. But first recall that laws of two real-valued random fields  $\{A_v\}_{v \in \Gamma}$  and  $\{B_v\}_{v \in \Gamma}$  are said to be mutually absolutely continuous if there exists an almost surely strictly-positive mean-one random variable  $D$  such that for every  $v_1, \dots, v_n \in \Gamma$  and Borel sets  $\Sigma_1, \dots, \Sigma_n \in \mathbf{R}$ ,

$$\mathbf{P} \left( \bigcap_{j=1}^n \{A_{v_j} \in \Sigma_j\} \right) = \mathbf{E} \left( D; \bigcap_{j=1}^n \{B_{v_j} \in \Sigma_j\} \right). \quad (3.14)$$

**Proposition 3.3.** *Choose and fix  $T > 0$  and a compact set  $A \subset E$ . Then, then for any  $\alpha > 0$ , the law of the random field  $\{V_\alpha(t, x)\}_{t \in [0, T], x \in A}$  is mutually absolutely continuous with respect to the law of the random field  $\{U(t, x)\}_{t \in [0, T], x \in A}$ .*

*Proof.* Throughout this proof define for all  $t \in [0, T]$ ,

$$Q_\alpha(t, A) := \int_0^t ds \int_A m(dy) |V_\alpha(x, y)|^2. \quad (3.15)$$

Minkowski's inequality implies that for all integers  $k \geq 0$ ,

$$\|Q_\alpha(T, A)\|_{L^k(\mathbb{P})} \leq \int_0^T ds \int_A m(dy) \|V_\alpha(s, y)\|_{L^{2k}(\mathbb{P})}^2. \quad (3.16)$$

Because of (3.3), each  $V_\alpha(s, y)$  is a centered Gaussian random variable. Therefore,

$$\|V_\alpha(s, y)\|_{L^{2k}(\mathbb{P})}^2 = \|V_\alpha(s, y)\|_{L^2(\mathbb{P})}^2 \cdot \|Z\|_{L^{2k}(\mathbb{P})}^2, \quad (3.17)$$

where  $Z$  is a standard-normal random variable. Thanks to (3.4),

$$\mathbb{E} \left( |Q_\alpha(T, A)|^k \right) \leq C_\alpha^k \cdot \mathbb{E}(Z^{2k}), \quad (3.18)$$

where  $C_\alpha := C_\alpha(T, A) := Tm(A) \sup_{x \in A} \bar{u}_\alpha(x, x)$ . Consequently, for any positive integer  $l$ ,

$$\mathbb{E} \left[ \exp \left( \frac{\alpha^2}{8l^2} Q_\alpha(T, A) \right) \right] \leq \mathbb{E} \left[ \exp \left( \frac{\alpha^2 C_\alpha Z^2}{8l^2} \right) \right], \quad (3.19)$$

and this is finite if and only if  $\alpha^2 C_\alpha / l^2 < 4$ . By continuity,  $\bar{u}_1$  is bounded uniformly on  $A \times A$ . Therefore, Lemma 2.2 tells us that there exist a large  $l$  such that  $\alpha^2 C_\alpha / l^2 < 4$ . Hence for such an  $l$ ,

$$\mathbb{E} \left[ \exp \left( \frac{\alpha^2}{8l^2} Q_\alpha(T, A) \right) \right] < \infty. \quad (3.20)$$

Consequently, we can apply the criteria of Novikov and Kazamaki (see Revuz and Yor [12, pp. 307–308]) to conclude that

$$\exp \left( \frac{\alpha}{2l} \int_0^t \int_A V_\alpha(s, y) W(dy ds) - \frac{\alpha^2}{8l^2} Q_\alpha(t, A) \right) \quad (3.21)$$

defines a mean-one martingale indexed by  $t \in [0, T]$ . Define

$$\dot{W}^{(1)}(t, x) := \dot{W}(t, x) - \frac{\alpha}{2l} V_\alpha(t, x) \quad \text{for } t \in [0, T] \text{ and } x \in A. \quad (3.22)$$

Recall that  $\mathbb{P}$  denotes the measure under which  $\{\dot{W}(t, x)\}_{t \in [0, T], x \in A}$  is a

white noise. The preceding and Girsanov's theorem together imply that  $\{\dot{W}^{(1)}(t, x)\}_{t \in [0, T], x \in A}$  is a white noise under a different probability measure  $P_1$  which is mutually absolutely continuous with respect to  $P$ ; see Da Prato and Zabczyk [3, p. 290]. Next, we define iteratively,

$$\dot{W}^{(n+1)}(t, x) := \dot{W}^{(n)}(t, x) - \frac{\alpha}{l} V_\alpha(t, x) \quad \text{for } n = 1, \dots, l-1. \quad (3.23)$$

A second application of Girsanov's theorem allows us to conclude that  $\{\dot{W}^{(2)}(t, x)\}_{t \in [0, T], x \in A}$  is a white noise under a certain probability measure  $P_2$  which is mutually absolutely continuous with respect to  $P_1$ .

In fact, the very same argument implies existence of a finite sequence of measures  $\{P_n\}_{n=1}^l$  such that  $\{\dot{W}^{(n)}(t, x)\}_{t \in [0, T], x \in A}$  is a white noise under  $P_n$  for  $1 \leq n \leq l$ . We can now conclude that  $\dot{W}^{(l)}(t, x) = \dot{W}(t, x) - (\alpha/2)V_\alpha(t, x)$  defines a white noise [indexed by  $t \in [0, T]$  and  $x \in A$ ] under the measure  $P_l$ , and that  $P_l$  is mutually absolutely continuous with respect to  $P$ . The latter fact follows from the transitivity property of absolute continuity of measures; this is the property that asserts that whenever  $Q_1$  and  $Q_2$  are mutually absolutely continuous probability measures, and  $Q_2$  and  $Q_3$  are mutually absolutely continuous probability measures, then so are  $Q_1$  and  $Q_3$ .

The result follows from the strong existence of solutions to (1.1) and (3.1).  $\square$

Consider the Gaussian random field

$$S_\alpha(t, x) := \int_t^\infty \int_E e^{-\alpha s/2} p_s(x, y) W(dy ds). \quad (3.24)$$

[This process was suggested to us by Zhan Shi.] One verifies, just as one does for  $V_\alpha(t, \cdot)$ , that for all finite Borel signed measures  $\mu$  on  $E$ ,

$$S_\alpha(t, \mu) = \int_t^\infty \int_E e^{-\alpha s/2} (P_s^* \mu)(y) W(dy ds), \quad (3.25)$$

indexed by  $\mu \in L^2(m)$  and  $t > 0$ . Elementary properties of the processes  $S_\alpha$  and  $V_\alpha$  show that they are independent mean-zero Gaussian processes.

Consider the Gaussian random field

$$\eta_\alpha(t, \cdot) := V_\alpha(t, \cdot) + S_\alpha(t, \cdot). \quad (3.26)$$

Because  $\mathbb{E}[\eta_\alpha(t, \mu)\eta_\alpha(t, \nu)] = \mathbb{E}[V_\alpha(t, \mu)V_\alpha(t, \nu)] + \mathbb{E}[S_\alpha(t, \mu)S_\alpha(t, \nu)]$ , a direct computation shows that for all  $t > 0$  and finite Borel measures  $\mu$  and  $\nu$  on  $E$ ,

$$\begin{aligned} \text{Cov}(\eta_\alpha(t, \mu), \eta_\alpha(t, \nu)) &= \int_0^\infty e^{-\alpha s} (P_s^* \mu, P_s^* \nu) \, ds \\ &= \int_0^\infty e^{-\alpha s} (\mu, \bar{P}_s \nu) \, ds \\ &= (\mu, \bar{U}_\alpha \nu). \end{aligned} \quad (3.27)$$

In other words, the law of  $\eta_\alpha(t, \cdot)$  does not depend on  $t > 0$ , and

$$\text{Cov}(\eta_\alpha(t, x), \eta_\alpha(t, z)) = \bar{u}_\alpha(x, z). \quad (3.28)$$

Thus,  $\eta_\alpha(t, \cdot)$  is precisely the associated Gaussian process that arises in Dynkin's isomorphism theorem [10, Chapter 8].

It is easy to see that since the law of  $\eta_\alpha(t, \cdot)$  is independent of  $t > 0$ ,  $\eta_\alpha(t, \cdot)$  is the [weak] steady-state solution to (3.1), in the sense that  $S_\alpha(t, x) \rightarrow 0$  in  $L^2(\mathbb{P})$  as  $t \rightarrow \infty$  for all  $x \in E$ ; this follows directly from the definition of  $S_\alpha$ .

Our next result implies that many of the local regularity properties of  $V_\alpha(t, \cdot)$  and  $\eta_\alpha(t, \cdot) = V_\alpha(t, \cdot) + S_\alpha(t, \cdot)$  are shared.

**Proposition 3.4.** *For every fixed  $t, \alpha > 0$ ,  $S_\alpha(t, \cdot)$  is smoother than  $V_\alpha(t, \cdot)$ . In fact, for all  $\mu \in M(E)$  such that  $(|\mu|, \bar{U}_\alpha |\mu|) < \infty$ ,*

$$\mathbb{E}(|S_\alpha(t, \mu)|^2) \leq \left[ \frac{1}{e^{t\alpha} - 1} \right] \cdot \mathbb{E}(|V_\alpha(t, \mu)|^2). \quad (3.29)$$

*Proof.* Suppose  $\mu$  is a finite signed Borel measure on  $E$  with  $(|\mu|, \bar{U}_\alpha |\mu|) < \infty$ . Then,  $\int_0^\infty e^{-\alpha s} \|P_s^* \mu\|^2 \, ds = (\mu, \bar{U}_\alpha \mu) < \infty$ , and hence  $\|P_r^* \mu\|$  is finite for almost all  $r > 0$ .

We first note the following:

$$\int_t^\infty e^{-\alpha s} \|P_s^* \mu\|^2 ds = \sum_{n=1}^\infty e^{-n\alpha t} \cdot \int_0^t e^{-\alpha s} \|P_{s+nt}^* \mu\|^2 ds. \quad (3.30)$$

But thanks to Lemma 2.1, we have

$$\|P_{s+nt}^* \mu\| \leq \|P_s^* \mu\| \quad \text{for all } s, t > 0 \text{ and } n \geq 1, \quad (3.31)$$

which together with (3.30) implies that

$$\int_t^\infty e^{-\alpha s} \|P_s^* \mu\|^2 ds \leq \left[ \frac{e^{-t\alpha}}{1 - e^{-t\alpha}} \right] \cdot \int_0^t e^{-\alpha s} \|P_s^* \mu\|^2 ds. \quad (3.32)$$

This is another way of stating the Proposition.  $\square$

We mention, in passing, a nontrivial consequence of Proposition 3.4: Consider the special case that  $\mu := \delta_a - \delta_b$  for fixed  $a, b \in E$ . In that case,  $(|\mu|, \bar{U}_\alpha |\mu|)$  is finite. Indeed,  $(|\mu|, \bar{U}_\alpha |\mu|) = \bar{u}_\alpha(a, a) + \bar{u}_\alpha(b, b) - 2\bar{u}_\alpha(a, b)$ , from which it follows that  $\|P_s^* \mu\|^2 = \bar{p}_s(a, a) + \bar{p}_s(b, b) - 2\bar{p}_s(a, b)$ . Therefore, time reversal and Proposition 3.4—specifically in the form given by (3.32)—together assert the following somewhat unusual inequality.

**Corollary 3.5.** *Let  $S(\alpha)$  denote an independent exponentially-distributed random variable with mean  $1/\alpha$ . Then, for all  $t, \alpha > 0$  and  $a, b \in E$ ,*

$$\mathbb{E}^a \left[ L_{S(\alpha)}^a - L_{S(\alpha)}^b \mid S(\alpha) \geq t \right] \leq \mathbb{E}^a \left[ L_{S(\alpha)}^a - L_{S(\alpha)}^b \mid S(\alpha) < t \right]. \quad (3.33)$$

This appears to be novel even when  $X$  is a linear Brownian motion.

## 4 Lévy processes

Next we study the special case that  $\mathcal{L}$  is the generator of a Lévy process  $X$  on  $\mathbf{R}$ . Recall that  $X$  is best understood via its characteristic exponent  $\Psi$  [1]; we normalize that exponent as follows:  $\mathbb{E} \exp(i\xi X_t) = \exp(-t\Psi(\xi))$ .

Let  $m$  denote the Lebesgue measure on  $E := \mathbf{R}$ ; then  $X$  is in duality with  $-X$  under  $m$ , and the replica semigroup  $\{P_t^*\}_{t \geq 0}$  is the semigroup associated to the Lévy process  $\bar{X}$  defined by (1.2).

We know from the general theory of Dalang [4] that (1.1) has a random-field solution if and only if

$$\int_{-\infty}^{\infty} \frac{d\xi}{\alpha + 2\operatorname{Re} \Psi(\xi)} < \infty, \quad (4.1)$$

for one, and hence all,  $\alpha > 0$ . This condition implies the existence of jointly-continuous transition functions for both  $X$  and  $\bar{X}$  [6, Lemma 8.1], and

$$\bar{u}_\alpha(x, y) = \frac{1}{\pi} \int_0^\infty \frac{\cos(\xi(x-y))}{\alpha + 2\operatorname{Re} \Psi(\xi)} d\xi. \quad (4.2)$$

In particular,  $\bar{u}_\alpha$  is continuous on  $\mathbf{R} \times \mathbf{R}$ . Finally, one checks that the domain of  $\bar{\mathcal{L}}$  is precisely the collection of all  $f \in L^2(m)$  such that  $\operatorname{Re} \Psi \cdot |\hat{f}|^2 \in L^1(\mathbf{R})$ . The well-known fact that  $\operatorname{Re} \Psi(\xi) = O(\xi^2)$  as  $|\xi| \rightarrow \infty$  tells us that all rapidly-decreasing test functions are in  $\operatorname{Dom}[\bar{\mathcal{L}}]$ , and therefore  $\operatorname{Dom}[\bar{\mathcal{L}}]$  is dense in  $L^2(m)$ . Thus, all the conditions of Theorem 1.1 are verified in this case.

Choose and fix some  $t > 0$ . According to Proposition 3.4, and thanks to the general theory of Gaussian processes, the process  $S_\alpha(t, \cdot)$  is at least as smooth as the process  $V_\alpha(t, \cdot)$ ; the latter solves the stochastic cable equation. Next we prove that under a mild condition on  $\Psi$ ,  $S_\alpha(t, \cdot)$  is in fact extremely smooth.

**Proposition 4.1.** *Suppose, in addition to (4.1), that*

$$\lim_{|\xi| \rightarrow \infty} \frac{\operatorname{Re} \Psi(\xi)}{\log |\xi|} = \infty. \quad (4.3)$$

*Then, for each fixed  $t, \alpha > 0$ , the process  $S_\alpha(t, \cdot)$  has a modification that is in  $C^\infty(\mathbf{R})$ .*

*Proof.* By Plancherel's theorem, if  $\mu$  is a finite signed measure on  $\mathbf{R}$ , then

$$\mathbb{E} (|S_\alpha(t, \mu)|^2) = \int_t^\infty e^{-\alpha s} \|P_s^* \mu\|^2 ds = \frac{1}{2\pi} \int_t^\infty e^{-\alpha s} \left\| \widehat{P_s^* \mu} \right\|^2 ds, \quad (4.4)$$

where “ $\widehat{\cdot}$ ” denotes the Fourier transform, normalized so that

$$\hat{g}(\xi) = \int_{-\infty}^\infty e^{i\xi z} g(z) dz \quad \text{for every } g \in L^1(\mathbf{R}). \quad (4.5)$$

But  $\{P_s^*\}_{s \geq 0}$  is a convolution semigroup in the present setting, and has Fourier multiplier  $\exp(-s\Psi(-\xi))$ . That is,

$$\widehat{P_s^* \mu}(\xi) = e^{-s\Psi(-\xi)} \hat{\mu}(\xi) \quad \text{for all } \xi \in \mathbf{R} \text{ and } s \geq 0. \quad (4.6)$$

Therefore,  $\|\widehat{P_s^* \mu}\|^2 = \int_{-\infty}^\infty e^{-2s\operatorname{Re} \Psi(\xi)} |\hat{\mu}(\xi)|^2 d\xi$ . From this and the Tonelli theorem we deduce the following:

$$\mathbb{E} (|S_\alpha(t, \mu)|^2) = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{|\hat{\mu}(\xi)|^2}{\alpha + 2\operatorname{Re} \Psi(\xi)} e^{-(\alpha + 2\operatorname{Re} \Psi(\xi))t} d\xi. \quad (4.7)$$

Recall [7] that the generalized  $n$ th derivative of  $S_\alpha(t, \cdot)$  is defined as the following random field:

$$S_\alpha^{(n)}(t, \phi) := (-1)^n S_\alpha(t, \phi^{(n)}), \quad (4.8)$$

for all rapidly-decreasing test functions  $\phi$  on  $\mathbf{R}$ . Here,  $\phi^{(n)}$  denotes the  $n$ th derivative of  $\phi$ . Since the Fourier transform of  $\phi^{(n)}$  is  $i^n \xi^n \hat{\phi}(\xi)$ , (4.7) implies that the following holds for all rapidly-decreasing test functions  $\phi$  on  $\mathbf{R}$ :

$$\begin{aligned} \mathbb{E} \left( \left| S_\alpha^{(n)}(t, \phi) \right|^2 \right) &\leq \frac{1}{2\pi} \int_{-\infty}^\infty \frac{|\xi|^n \cdot |\hat{\phi}(\xi)|^2}{\alpha + 2\operatorname{Re} \Psi(\xi)} e^{-(\alpha + 2\operatorname{Re} \Psi(\xi))t} d\xi \\ &\leq \frac{\|\phi\|_{L^1(\mathbf{R})}^2}{2\pi\alpha} \int_{-\infty}^\infty |\xi|^n e^{-2t\operatorname{Re} \Psi(\xi)} d\xi. \end{aligned} \quad (4.9)$$

The growth condition on  $\operatorname{Re} \Psi$  ensures that the integral is finite regardless of the value of  $n$  and  $t$ . Therefore, a density argument shows that  $Z_\alpha^{(n)}(t, \mu)$

can be defined as a  $L^2(\mathbf{P})$ -continuous Gaussian random field indexed by all finite signed Borel measures  $\mu$  on  $\mathbf{R}$ , and

$$\mathbb{E} \left( \left| S_\alpha^{(n)}(t, \mu) \right|^2 \right) \leq \frac{(|\mu|(\mathbf{R}))^2}{2\pi\alpha} \int_{-\infty}^{\infty} |\xi|^n e^{-2t\operatorname{Re} \Psi(\xi)} d\xi. \quad (4.10)$$

This estimate and the Kolmogorov continuity theorem together imply that  $\mathbf{R} \ni x \mapsto S_\alpha^{(n)}(t, x) := S_\alpha(t, \delta_x)$  has a modification that is a bona fide continuous Gaussian process such that  $S_\alpha^{(n)}(t, \mu) = \int S_\alpha^{(n)}(t, x) \mu(dx)$ . Apply this with  $\mu$  replaced by a rapidly-decreasing test function  $\phi$  and apply integration by parts to deduce that  $S_\alpha$  has a  $C^\infty$  modification.  $\square$

It would be interesting to know when (4.1) implies (4.3). This turns out to be a perplexing problem, about which we next say a few words.

**Remark 4.2.** 1. It is easy to see that Condition (4.1) always implies that  $\limsup_{|\xi| \rightarrow \infty} \operatorname{Re} \Psi(\xi)/|\xi| = \infty$ . Therefore, in order to understand when (4.1) implies (4.3), we need to know the lim inf behavior of  $\operatorname{Re} \Psi(\xi)/\log |\xi|$  for large values of  $|\xi|$ .

2. It is shown in [6, Lemma 8.1] that (4.1) implies the existence of transition densities  $\{p_t(x)\}_{t>0, x \in \mathbf{R}}$  for  $X$  such that  $p_t$  is a bounded function for each fixed  $t > 0$ . Theorem 6 of Hawkes [8] then tells us that

$$\lim_{|\xi| \rightarrow \infty} \frac{H(\xi)}{\log |\xi|} = \infty, \quad (4.11)$$

where  $H$  denotes Hardy–Littlewood’s monotone increasing rearrangement of  $\operatorname{Re} \Psi$ . Condition (4.11) is tantalizingly close to (4.3).

3. It is easy to show that (4.1) implies (4.3) if  $\operatorname{Re} \Psi$  is nondecreasing on  $[q, \infty)$  for  $q$  sufficiently large. More generally, suppose  $\operatorname{Re} \Psi$  is “quasi-increasing” on  $\mathbf{R}_+$  in the sense that there exist  $C, q > 0$  such that

$$C\operatorname{Re} \Psi(2z) \geq \sup_{u \in [z, 2z]} \operatorname{Re} \Psi(u) \quad \text{for all } z > q. \quad (4.12)$$

Then (4.1) implies (4.3). Here is the [abelian] proof: For all  $z > q$ ,

$$\int_z^{2z} \frac{d\xi}{1 + 2\operatorname{Re} \Psi(\xi)} = 2 \int_{z/2}^z \frac{d\xi}{1 + 2\operatorname{Re} \Psi(2\xi)} \geq \frac{z}{1 + 2\operatorname{Re} \Psi(2z)} \quad (4.13)$$

Let  $z$  tend to infinity and apply symmetry to find that

$$\lim_{|z| \rightarrow \infty} \frac{\operatorname{Re} \Psi(z)}{|z|} = \infty. \quad (4.14)$$

Clearly, this implies [more than] (4.3).  $\square$

The preceding remarks suggest that, quite frequently, (4.1) implies (4.3). We do not know whether or not (4.1) always implies (4.3). But we are aware of some easy-to-check conditions that guarantee this property. Let us state two such conditions next. First, we recall the following two of the three well-known functions of Feller, both defined in terms of the Lévy measure  $\nu$  of the process  $X$ : For all  $\epsilon > 0$ ,

$$K(\epsilon) := \epsilon^{-2} \int_{|z| \leq \epsilon} z^2 \nu(dz) \quad \text{and} \quad G(\epsilon) := \nu\{x \in \mathbf{R} : |x| > \epsilon\}. \quad (4.15)$$

Then we have the following:

**Lemma 4.3.** *Suppose that at least one of the following two conditions holds: (i)  $X$  has a nontrivial gaussian component; or (ii)*

$$\limsup_{\epsilon \rightarrow 0^+} \frac{G(\epsilon)}{K(\epsilon)} < \infty. \quad (4.16)$$

*Then (4.1) implies (4.14), whence (4.3).*

Conditions (4.16) and (4.1) are not contradictory. For a simple example, one can consider  $X$  to be a symmetric stable process of index  $\alpha \in (0, 2]$ . Then, (4.1) holds if and only if  $\alpha > 1$ . And, when  $\alpha \in (1, 2)$ , (4.16) holds automatically; in fact, in this case,  $G(\epsilon)$  and  $K(\epsilon)$  both grow within constant multiples of  $\epsilon^{-\alpha}$  as  $\epsilon \rightarrow 0^+$ .

*Proof.* In the case that  $X$  contains a nontrivial gaussian component, the

Lévy–Khintchine formula [1, p. 13] implies that there exists  $\sigma > 0$  such that  $\operatorname{Re} \Psi(\xi) \geq \sigma^2 \xi^2$  for all  $\xi \in \mathbf{R}$ . Therefore, the result holds in this case. Thus, let us consider the case where there is no gaussian component in  $X$  and (4.16) holds.

Because  $1 - \cos \theta \geq \theta^2/3$  for  $\theta \in (0, 1)$ , we may apply the Lévy–Khintchine formula to find the following well-known bound: For all  $\xi > 0$ ,

$$\operatorname{Re} \Psi(\xi) = \int_{-\infty}^{\infty} (1 - \cos(|x|\xi)) \nu(dx) \geq \frac{1}{3} K(1/\xi). \quad (4.17)$$

Now we apply an averaging argument from harmonic analysis; define

$$\mathcal{R}(\xi) := \frac{1}{\xi} \int_0^\xi \operatorname{Re} \Psi(z) dz \quad \text{for all } \xi > 0. \quad (4.18)$$

By the Lévy–Khintchine formula,

$$\mathcal{R}(\xi) = \int_{-\infty}^{\infty} (1 - \operatorname{sinc}(|x|\xi)) \nu(dx), \quad (4.19)$$

where  $\operatorname{sinc} \theta := \sin \theta / \theta$ , as usual. Because  $1 - \operatorname{sinc} \theta \leq \min(\theta^2/2, 1)$  for all  $\theta > 0$ , it follows that for all  $\xi > 0$ ,

$$\mathcal{R}(\xi) \leq \frac{1}{2} K(1/\xi) + G(1/\xi). \quad (4.20)$$

Because of (4.16), (4.17), and (4.20),  $\mathcal{R}(\xi) = O(\operatorname{Re} \Psi(\xi))$  as  $\xi \rightarrow \infty$ . Thanks to symmetry, it suffices to prove that

$$\lim_{\xi \rightarrow \infty} \frac{\mathcal{R}(\xi)}{\xi} = \infty. \quad (4.21)$$

To this end, we observe that for all  $\alpha, \xi > 0$ ,

$$\begin{aligned} \alpha + \mathcal{R}(\xi) &= \frac{1}{\xi} \int_0^\xi (\alpha + \operatorname{Re} \Psi(z)) dz \\ &\geq \left( \frac{1}{\xi} \int_0^\xi \frac{dz}{\alpha + 2\operatorname{Re} \Psi(z)} \right)^{-1}; \end{aligned} \quad (4.22)$$

this follows from the Cauchy–Schwarz inequality. Consequently,

$$\liminf_{\xi \rightarrow \infty} \frac{\mathcal{R}(\xi)}{\xi} \geq \sup_{\alpha > 0} \left( \int_0^\infty \frac{dz}{\alpha + 2\operatorname{Re} \Psi(z)} \right)^{-1} = \infty. \quad (4.23)$$

Because of this and symmetry we obtain (4.21), and hence the lemma.  $\square$

**Remark 4.4.** Our proof was based on the general idea that, under (4.16), Condition (4.21) implies (4.14). The converse holds even without (4.16). In fact, because  $1 - \cos \theta \leq \text{const} \cdot (1 - \operatorname{sinc} \theta)$  for a universal constant, the Lévy–Khintchine formula implies that  $\operatorname{Re} \Psi(\xi) \leq \text{const} \cdot \mathcal{R}(\xi)$  for the same universal constant.  $\square$

We conclude this paper by presenting an example.

**Example 4.5.** Consider the following stochastic cable equation: For  $t > 0$  and  $x \in \mathbf{R}$ ,

$$\frac{\partial}{\partial t} V_\alpha(t, x) = (\Delta_{\beta/2} V_\alpha)(t, x) - \frac{\alpha}{2} V_\alpha(t, x) + \dot{W}(t, x), \quad (4.24)$$

with  $V_\alpha(0, x) = 0$ . Here,  $\Delta_{\beta/2}$  denotes the fractional Laplacian of index  $\beta/2$ , and  $\beta \in (1, 2]$ , normalized so that  $\Delta_1 f = f''$ . Because  $\beta > 1$ , (4.1) is verified. It is well known that  $\Delta_{\beta/2}$  is the generator of  $\{cX_t\}_{t \geq 0}$ , where  $X$  denote the symmetric stable process of index  $\beta$  and  $c = c(\beta)$  is a certain positive constant. In this case,  $\Psi(\xi) = \text{const} \cdot |\xi|^\beta$  is real, and (4.3) holds. According to Proposition 4.1,  $S_\alpha(t, \cdot) \in C^\infty(\mathbf{R})$  [up to a modification] for every fixed  $t > 0$ . Therefore, the solution  $V_\alpha(t, \cdot)$  to the stochastic cable equation (3.1) is a  $C^\infty$  perturbation of the Gaussian process  $\eta_\alpha(t, \cdot)$  that is associated to the local times of a symmetric stable process of index  $\beta$ . But  $\eta_\alpha(t, \cdot)$  is fractional Brownian motion of index  $\beta - 1 \in (0, 1]$ ; see Marcus and Rosen [10, p. 498]. Recall that  $\eta_\alpha(t, \cdot)$  is a continuous centered Gaussian process with

$$\mathbb{E} (|\eta_\alpha(t, x) - \eta_\alpha(t, y)|^2) = \text{const} \cdot |x - y|^{\beta-1} \quad \text{for } x, y \in \mathbf{R}. \quad (4.25)$$

As a consequence, we find that local regularity of the solution to the cable equation (4.24) [in the space variable] is the same as local regularity of fractional Brownian motion of index  $\beta - 1$ . A similar phenomenon has been observed by Walsh [13, Exercise 3.10, p. 326] in the case that we solve (4.24) for  $\beta = 2$  on a finite  $x$ -interval. Dynkin's isomorphism theorem [10, Chapter 8] then implies that the local regularity, in the space variable, of the solution to (4.24) is the same as the local regularity of the local time of a symmetric stable process of index  $\beta$  in the space variable. And thanks to a change of measure (Proposition 3.3), the same is true of local regularity of the solution to the heat equation,

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} U(t, x) = (\Delta_{\beta/2} U)(t, x) + \dot{W}(t, x), \\ U(0, x) = 0. \end{array} \right. \quad (4.26)$$

One can consult other local-time examples of this general type by drawing upon the examples from the book of Marcus and Rosen [10].  $\square$

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