

On dynamical bit sequences

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Abstract

Let $\mathbf{X}_k(t) := (X_1(t), \dots, X_k(t))$ denote a k -vector of i.i.d. random variables, each taking the values 1 or 0 with respective probabilities $p \in (0, 1)$ and $1 - p$. As a process indexed by $t \geq 0$, \mathbf{X}_k is constructed—following Benjamini, Häggström, Peres, and Steif [3]—so that it is strong Markov with invariant measure $((1 - p)\delta_0 + p\delta_1)^k$. We derive sharp estimates for the probability that “ $X_1(t) + \dots + X_k(t) = k - \ell$ for some $t \in F$,” where $F \subseteq [0, 1]$ is nonrandom and compact. We do this in two very different settings: (i) Where ℓ is a constant; and (ii) Where $\ell = k/2$, k is even, and $p = q = 1/2$. We prove that the probability is described by the Kolmogorov capacitance of F for case (i) and Howroyd’s $\frac{1}{2}$ -dimensional box-dimension profiles for case (ii). We also present sample-path consequences, and a connection to capacities that answers a question of [3].

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1 Introduction and main results

Choose and fix some $p \in (0, 1)$. By a *bit sequence* we mean simply a k -vector (X_1, \dots, X_k) —where $k \geq 1$ is fixed—of independent, identically-distributed random variables with $\mathbb{P}\{X_1 = 1\} = 1 - \mathbb{P}\{X_1 = 0\} = p$. For simplicity, we write \mathbb{P}_p in order to keep track of all dependencies on the parameter p .

A *dynamical bit sequence* is the process $\mathbf{X}_k := \{(X_1(t), \dots, X_k(t))\}_{t \geq 0}$ that is constructed as follows: We begin with a bit sequence (X_1, \dots, X_k) at time zero.

Then, to every index $j \in \{1, \dots, k\}$ we associate a rate-one Poisson process; all Poisson processes being independent of all X variables. And whenever the j th Poisson process jumps, we replace the corresponding variable X_j by a copy, independent of all else, in order to obtain a time-dependent family of random variables. More precisely, let $\{X_i^{(j)}\}_{1 \leq i, j < \infty}$ be an i.i.d. array of random bits, each satisfying $\mathbb{P}_p\{X_i^{(j)} = 1\} = p$, and consider also a sequence of independent rate-one Poisson processes, independent of the $\{X_i^{(j)}\}_{1 \leq i, j < \infty}$. Denote the jump times of the i -th Poisson process by $J_i^{(0)} := 0 < J_i^{(1)} < J_i^{(2)} < \dots$, and define

$$X_i(t) := \sum_{j=1}^{\infty} X_i^{(j)} \mathbf{1}_{\{J_i^{(j)} \leq t < J_i^{(j-1)}\}} \quad \text{for } t \geq 0 \text{ and } 1 \leq i \leq k. \quad (1.1)$$

Dynamical bit sequences were introduced by Benjamini, Häggström, Peres, and Steif in 2003. They can be used to model how bit sequences can get corrupted over time, for instance, see [3] and [8, 9]. A closely-related variant of dynamical bit sequences was introduced earlier by Rusakov [16, 17, 18]; see also Rusakov and Chuprunov [15].

A key feature of \mathbf{X}_k is that it is a strong Markov process on $\{0, 1\}^k$ whose invariant measure is the law $(p\delta_1 + (1-p)\delta_0)^k$ of the bit sequence (X_1, \dots, X_k) ; in particular, $(X_1(t), \dots, X_k(t))$ and (X_1, \dots, X_k) have the same distribution for every fixed $t \geq 0$.

The main goal of the present work is to estimate a family of probabilities for dynamical bit sequences. Among other things, these estimates can be used to describe almost-sure properties of runs [i.e., contiguous sequences of ones and/or zeros]; see [3] and Section 4 below for examples.

Let $S_k := X_1 + \dots + X_k$. We note the classical binomial identity

$$\mathbb{P}_p\{S_k = k - \ell\} = \binom{k}{\ell} p^{k-\ell} (1-p)^\ell, \quad (1.2)$$

valid for all integers $\ell = 0, \dots, k$. Consequently,

$$\mathbb{P}_p\{S_k = k - \ell\} \asymp k^\ell p^k \quad \text{as } k \rightarrow \infty, \quad (1.3)$$

for every fixed integer $\ell \geq 0$, where “ $a_k \asymp b_k$ for a given set of values of k ’s” means that “ (a_k/b_k) is bounded above and below by positive and finite constants, uniformly for all mentioned values of k .”

In contrast to (1.3), one can verify that for every fixed integer $\ell \in \{0, \dots, k\}$,

$$\mathbb{P}_p \{S_k(t) = k - \ell \text{ for some } t \in [0, 1]\} \asymp k^{\ell+1} p^\ell, \quad (1.4)$$

as $k \rightarrow \infty$, where $S_k(t) := X_1(t) + \dots + X_k(t)$. Benjamini et al [3] have proved (1.4) in the case that $\ell = 0$, and the more general case follows from their methods as well. Moreover, the time set $[0, 1]$ can be replaced by any other time interval without affecting the form of (1.4).

Our main goal is to describe the effect of the geometry of the time variable t on the bounds in (1.4). In order to describe our first main estimate, choose and fix a compact set $F \subseteq \mathbf{R}_+$. Then for all $\epsilon > 0$ define $K_F(\epsilon)$ to be the largest integer $m \geq 1$ for which there exist points $t_1, \dots, t_m \in F$ such that $|t_i - t_j| \geq \epsilon$ for all $i \neq j$. The function K_F is known as the *Kolmogorov ϵ -capacity* [or *capacitance*, or *ϵ -packing*] of the set F . Now we can describe our first result.

Theorem 1.1. *Choose and fix an integer $\ell \geq 1$ and a nonrandom compact set $F \subset \mathbf{R}_+$. Then as $k \rightarrow \infty$,*

$$\mathbb{P}_p \{\exists t \in F : S_k(t) = k - \ell\} \asymp K_F(1/k) k^\ell p^k. \quad (1.5)$$

When $F = \{0\}$ this recovers (1.3) because $K_{\{0\}}(1/k) = 1$. And when $F = [0, 1]$, it recovers (1.4) since $kK_{[0,1]}(1/k) \rightarrow 1$ as $k \rightarrow \infty$. The following implies more interesting behavior when F is less regular.

Remark 1.2. Let

$$\alpha := \liminf_{k \rightarrow \infty} \frac{\log K_F(1/k)}{\log k} \quad \text{and} \quad \beta := \limsup_{k \rightarrow \infty} \frac{\log K_F(1/k)}{\log k} \quad (1.6)$$

respectively denote the lower, and the upper, Minkowski dimension of F . Then it follows from Theorem 1.1 that

$$k^{\ell+\alpha+o(1)} p^k \leq \mathbb{P}_p \{\exists t \in F : S_k(t) = k - \ell\} \leq k^{\ell+\beta+o(1)} p^k. \quad (1.7)$$

And each bound is achieved along a suitable subsequence of k 's. \square

The probability in Theorem 1.1 has even more interesting behavior in the cases that $\ell \rightarrow \infty$ at a prescribed rate. We investigate one critical case in this paper. First, let us apply Stirling's formula in (1.2) to recover the well-known local limit theorem $\mathbb{P}_{1/2}\{S_k = k/2\} \asymp k^{-1/2}$, valid for all sufficiently-large

even integers k . Our next result describes the significantly-different dynamical version of this estimate, and is motivated by sequence-matching estimates of Arratia and Waterman [1, 2] in the what turns out to be the critical case where $p = 1/2$ and $\ell = k/2$. Let

$$\gamma := \text{B-dim}_{1/2} F \quad \text{and} \quad \delta := \text{B-}\overline{\text{dim}}_{1/2} F \quad (1.8)$$

respectively denote Howroyd's $\frac{1}{2}$ -dimensional lower, and upper, box-dimension profile of F [10]. [These dimension numbers will be recalled later on during the course of the proof of Theorem 1.3 below; see §3.] Then we have the following.

Theorem 1.3. *If $k \rightarrow \infty$ along the even integers, then for every nonrandom bounded set $F \subset \mathbf{R}_+$,*

$$k^{-(1-\gamma)/2+o(1)} \leq \mathbb{P}_{1/2} \{\exists t \in F : S_k(t) = k/2\} \leq k^{-(1-\delta)/2+o(1)}. \quad (1.9)$$

Moreover, each bound is achieved along a suitable subsequence of k 's.

Thus, in the cases where $\gamma = \text{B-dim}_{1/2} F < \text{B-}\overline{\text{dim}}_{1/2} F = \delta$, we can deduce that the probability in (1.9) decays roughly like a power of k , and nevertheless there are no power-law asymptotics.

Theorems 1.1 and 1.3 are proved respectively in §2 and §3. We collect some consequences of Theorem 1.1 in §4. Finally, we describe a connection to Riesz capacity in §5, and use it to verify a conjecture of Benjamini et al [3].

2 Proof of Theorem 1.1

Lemma 2.1. *For every integers $\ell = 0, \dots, k$ and real number $t \geq 0$,*

$$\begin{aligned} & \mathbb{P}_p (S_k(t) = k - \ell \mid S_k(0) = k - \ell) \\ &= \sum_{i=0}^{\min(\ell, k-\ell)} \binom{\ell}{i} \theta_t^i (1 - \theta_t)^{\ell-i} \binom{k-\ell}{i} \kappa_t^{k-\ell-i} (1 - \kappa_t)^i, \end{aligned} \quad (2.1)$$

where $\theta_t := p(1 - e^{-t})$ and $\kappa_t := p + e^{-t}q$.

Proof. We first find the transition probabilities for the process $\{X_1(t)\}_{t \geq 0}$. Let $\{P_t\}_{t \geq 0}$ define the transition matrices defined by

$$P_t(i, j) := \mathbb{P}_p \{X_1(t) = j \mid X_1(0) = i\} \quad \text{for all } i, j \in \{0, 1\}. \quad (2.2)$$

Recall that $\{J_1^{(k)}\}_{k=1}^\infty$ are the jump times of the rate-one Poisson clock associated to X_1 . Then

$$\begin{aligned}
& \mathbb{P}_p \{ X_1(t) = 1 \mid X_1(0) = 1 \} \\
&= \frac{1}{p} \mathbb{P}_p \{ X_1(t) = 1, X_1(0) = 1, t < J_1^1 \} \\
&+ \frac{1}{p} \mathbb{P}_p \{ X_1(t) = 1, X_1(0) = 1, t \geq J_1^1 \} \\
&= \frac{1}{p} [pe^{-t} + p^2(1 - e^{-t})],
\end{aligned} \tag{2.3}$$

and this quantity is manifestly equal to κ_t .

In fact, we can follow the same argument to conclude that

$$P_t = \begin{bmatrix} pe^{-t} + q & p(1 - e^{-t}) \\ q(1 - e^{-t}) & qe^{-t} + p \end{bmatrix} = \begin{bmatrix} 1 - \theta_t & \theta_t \\ 1 - \kappa_t & \kappa_t \end{bmatrix} \quad \text{for all } t \geq 0. \tag{2.4}$$

For all $u, v \in \{0, 1\}$ define

$$N_{u \rightarrow v} := \sum_{j=1}^{k-1} \mathbf{1}_{\{X_j(t)=v, X_j(0)=u\}}. \tag{2.5}$$

This quantity denotes the number of integers $j \in \{1, \dots, k-1\}$ such that $X_j(0) = u$ and yet $X_j(t) = v$. It follows from the strong Markov property and (2.4) that the following properties are valid under the conditional measure $\mathbb{P}_p(\dots \mid S_k(0) = k - \ell)$:

- $N_{0 \rightarrow 1}$ has the binomial distribution with parameters ℓ and θ_t ;
- $N_{1 \rightarrow 1}$ has the binomial distribution with parameters $k - \ell$ and κ_t ;
- $N_{0 \rightarrow 1}$ and $N_{1 \rightarrow 1}$ are independent.

Additionally, the conditional probability in the statement of the lemma is

$$\begin{aligned}
& \mathbb{P}_p \{ N_{0 \rightarrow 1} + N_{1 \rightarrow 1} = k - \ell \mid S_k(0) = k - \ell \} \\
&= \sum_{i=0}^{\min(\ell, k-\ell)} \mathbb{P}_p \{ N_{0 \rightarrow 1} = i \mid S_k(0) = k - \ell \} \\
&\quad \times \mathbb{P}_p \{ N_{1 \rightarrow 1} = k - \ell - i \mid S_k(0) = k - \ell \}.
\end{aligned} \tag{2.6}$$

The lemma follows from these observations. \square

The proof of Theorem 1.1 uses a 2-scale argument that is borrowed from our earlier work on dynamical random walks [12]; it can be outlined as follows: First we prove that if I is a “small” closed interval, then the two events $\{S_k(t) = k - \ell$ for some $t \in I\}$ and $\{S_k(0) = k - \ell\}$ have more or less the same chances of occurring (Proposition 2.2). We will also demonstrate that “small” means “whose length is of sharp order $1/k$ ”; this length scale—or correlation length—was found earlier in [3]. Then we cover our set F with closed intervals of length $1/k$, and apply a covering argument. Finally, we show that this covering argument produces sharp answers. With this outline in mind, we begin with our first step.

Proposition 2.2. *As $k \rightarrow \infty$,*

$$\mathbb{P}_p \{S_k(t) = k - \ell \text{ for some } t \in [0, 1/k]\} \asymp k^\ell p^k. \quad (2.7)$$

Proof. Let $\mathcal{F} := \{\mathcal{F}_t\}_{t \geq 0}$ denote the filtration generated by the strong Markov process \mathbf{X}_k . We can assume, without loss of generality, that \mathcal{F} is augmented in the usual manner. Define

$$L_k := \int_0^{2/k} \mathbf{1}_{\{S_k(t) = k - \ell\}} dt. \quad (2.8)$$

According to (1.3),

$$\mathbb{E}_p(L_k) \asymp k^{\ell-1} p^k \quad \text{as } k \rightarrow \infty. \quad (2.9)$$

Next, we consider the stopping times, $\sigma := \inf\{t \in [0, 1/k] : S_k(t) = k - \ell\}$, where $\inf \emptyset := \infty$. By the strong Markov property, the following holds almost sure $[\mathbb{P}_p]$ on $\{\sigma < \infty\}$:

$$\begin{aligned} \mathbb{E}_p(L_k | \mathcal{F}_\sigma) &= \mathbb{E}_p \left(\int_\sigma^{2/k} \mathbf{1}_{\{S_k(t) = k - \ell\}} dt \mid \mathcal{F}_\sigma \right) \\ &\geq \int_0^{1/k} \mathbb{P}_p \{S_k(t + \sigma) = k - \ell \mid \mathcal{F}_\sigma\} dt \\ &= \int_0^{1/k} \mathbb{P}_p \{S_k(t) = k - \ell \mid S_k(0) = k - \ell\} dt. \end{aligned} \quad (2.10)$$

We apply Lemma 2.1 to find that \mathbb{P}_p -almost surely on $\{\sigma < \infty\}$,

$$\mathbb{E}_p(L_k | \mathcal{F}_\sigma) \geq \int_0^{1/k} (1 - \theta_t)^a \kappa_t^{k-\ell} dt \geq \frac{1}{k} \int_0^1 (1 - \theta_u)^k \kappa_{u/k}^k du. \quad (2.11)$$

The integrand converges to e^{-u} as $k \rightarrow \infty$. Consequently, thanks to the bounded convergence theorem,

$$\mathbb{E}_p(L_k | \mathcal{F}_\sigma) \geq \frac{\text{const}}{k} \cdot \mathbf{1}_{\{\sigma < \infty\}} \quad \mathbb{P}_p\text{-almost surely.} \quad (2.12)$$

Here, the implied constant does not depend on either k or ℓ . We can take expectations of both side, solve, and apply the optional stopping theorem of Doob to find that $\mathbb{P}_p\{\sigma < \infty\} \leq \text{const} \cdot k\mathbb{E}_p(L_k)$. The proposition follows from this and (2.9). \square

Proof of Theorem 1.1

We divide the proof in to two parts. The first part (the upper bound) is a relatively simple covering argument. In the second part, we show that the covering argument produces the correct answer. For that we use a second-moment argument.

Proof of the upper bound in (1.5). Let \mathcal{J}_k denote the collection of all closed subintervals I of $[0, 1]$ that have the form $I = [i/k, (i+1)/k]$, where $i \in \{0, \dots, k-1\}$. According to Proposition 2.2,

$$\begin{aligned} \mathbb{P}_p\{\exists t \in F : S_k(t) = k - \ell\} &\leq \sum_{\substack{I \in \mathcal{J}_k \\ I \cap F \neq \emptyset}} \mathbb{P}_p\{\exists t \in I : S_k(t) = k - \ell\} \\ &\leq \text{const} \cdot |\{I \in \mathcal{J}_k : I \cap F \neq \emptyset\}| \cdot k^\ell p^k, \end{aligned} \quad (2.13)$$

where $|\dots|$ denotes cardinality. The lemma follows since the preceding cardinality is $\leq 3K_F(1/k)$ [12, Proposition 2.7]. The upper bound in (1.5) follows readily from this. \square

Proof of the lower bound in (1.5). According to the definition of $K_F(1/k)$, we can find $t_1 < \dots < t_{K_F(1/k)}$ in F such that $t_{i+1} - t_i \geq k^{-1}$ for $1 \leq i \leq$

$K_F(1/k) - 1$. It follows easily that

$$t_j - t_i \geq \frac{j-i}{k} \quad \text{for } 1 \leq i < j \leq K_F(1/k). \quad (2.14)$$

Define

$$N_k := \sum_{j=1}^{K_F(1/k)} \mathbf{1}_{\{S_k(t_j)=k-\ell\}}. \quad (2.15)$$

According to (1.3),

$$\mathbb{E}_p(N_k) \asymp K_F(1/k) k^\ell p^\ell \quad \text{as } k \rightarrow \infty. \quad (2.16)$$

Observe that $\mathbb{E}_p(N_k^2)$ is equal to

$$\mathbb{E}_p(N_k) + \sum_{1 \leq i \neq j \leq K_F(1/k)} \sum_{j=1}^{K_F(1/k)} \mathbb{P}_p \{S_k(|t_j - t_i|) = k - \ell, S_k(0) = k - \ell\}. \quad (2.17)$$

Define

$$\mathcal{T} := \mathbb{P}_p \{S_k(h) = k - \ell \mid S_k(0) = k - \ell\}. \quad (2.18)$$

By the Markov property and Lemma 2.1,

$$\mathcal{T} = \sum_{i=0}^{\min(\ell, k-\ell)} Q_i^{(0)}(h), \quad (2.19)$$

where for all $h > 0$ and $i \in \{0, \dots, \min(\ell, k - \ell)\}$,

$$\begin{aligned} Q_i^{(0)}(h) &:= \binom{\ell}{i} \binom{k-\ell}{i} \theta_h^i (1-\theta_h)^{\ell-i} \kappa_h^{k-\ell-i} (1-\kappa_h)^i \\ &\leq \ell^{(\ell)} k^i \theta_h^i (1-\theta_h)^{\ell-i} \kappa_h^{k-\ell-i} (1-\kappa_h)^i. \end{aligned} \quad (2.20)$$

In order to simplify this object, let us first define

$$a := \max(p, 1-p) \quad \text{and} \quad b := \min(p, 1-p). \quad (2.21)$$

Let us also observe that

$$\theta_h(1-\kappa_h) = ab(1-e^{-h})^2 \leq \frac{a}{b} [1 - (a + be^{-h})]^2. \quad (2.22)$$

And

$$(1-\theta_h)^{\ell-i} \kappa_h^{k-\ell-i} \leq (a + be^{-h})^{k-2i}. \quad (2.23)$$

It follows that uniformly for all $\ell \in \{0, \dots, k\}$ and $i \in \{0, \dots, \min(\ell, k - \ell)\}$,

$$Q_i^{(0)}(h) \leq \text{const} \cdot k^i Q_i^{(1)}(h), \quad (2.24)$$

where

$$Q_i^{(1)}(h) := [1 - (a + be^{-h})]^{2i} (a + be^{-h})^{k-2i}. \quad (2.25)$$

The function $[0, 1] \ni \lambda \mapsto (1 - \lambda)^{k-2i} \lambda^{2i}$ is maximized at $\lambda := 2i/k$, and the value of the maximum is at most k^{-2i} . And therefore, by (2.24),

$$Q_i^{(0)}(h) \leq \text{const} \cdot k^{-i}. \quad (2.26)$$

The preceding is a good estimate when $i \geq 1$. In the case that $i = 0$, we merely use

$$Q_0^{(0)}(h) = (1 - \theta_h)^\ell \kappa_h^{k-\ell} \leq (a + be^{-h})^k. \quad (2.27)$$

We plug this and (2.26) in (2.19), using (2.18), to find

$$\begin{aligned} \mathcal{T} &\leq \text{const} \cdot \left((a + be^{-h})^k + \sum_{i=1}^{\min(\ell, k-\ell)} k^{-i} \right) \\ &\leq \text{const} \cdot ((a + be^{-h})^k + k^{-1}). \end{aligned} \quad (2.28)$$

Therefore, it follows from (1.3), (2.14), and (2.17) that $\mathbb{E}_p(N_k^2)$ is at most

$$\begin{aligned} \mathbb{E}_p(N_k) + \text{const} \cdot k^\ell p^k \sum_{1 \leq i \neq j \leq K_F(1/k)} \left((a + be^{-|t_j - t_i|})^k + k^{-1} \right) \\ \leq \mathbb{E}_p(N_k) + \text{const} \cdot k^\ell p^k \sum_{1 \leq i \neq j \leq K_F(1/k)} \left((a + be^{-|j-i|/k})^k + k^{-1} \right). \end{aligned} \quad (2.29)$$

Consequently, $\mathbb{E}_p(N_k^2)$ is bounded from above by

$$\begin{aligned} \mathbb{E}_p(N_k) + \text{const} \cdot K_F(1/k) \cdot k^\ell p^k \sum_{j=1}^{K_F(1/k)} \left((a + be^{-j/k})^k + k^{-1} \right) \\ \leq \mathbb{E}_p(N_k) + \text{const} \cdot K_F(1/k) \cdot k^\ell p^k \left\{ k \int_0^1 (a + be^{-u})^k du + \frac{K_F(1/k)}{k} \right\}. \end{aligned} \quad (2.30)$$

[The final term is at most one because $K_F(1/k) \leq K_{[0,1]}(1/k) = k$.] A direct

computation of the preceding integral reveals that

$$\mathbb{E}_p(N_k^2) \leq \mathbb{E}_p(N_k) + \text{const} \cdot K_F(1/k) k^\ell p^k \leq \text{const} \cdot \mathbb{E}_p(N_k); \quad (2.31)$$

see (2.16). Therefore, (2.16) and the Paley–Zygmund inequality [11, p. 72] together imply that $\mathbb{P}_p\{N_k > 0\} \geq \text{const} \cdot K_F(1/k) k^\ell p^k$. The theorem follows because $\{N_k > 0\} \subseteq \{\exists t \in F : S_k(t) = k - \ell\}$. \square

3 Proof of Theorem 1.3

Let $\mathcal{P}(F) :=$ all Borel probability measures μ such that $\mu(F) = 1$.

Theorem 3.1. *If k is an even integer, then for every measurable and nonrandom set $F \subseteq [0, 1]$, the following holds as $k \rightarrow \infty$:*

$$\begin{aligned} & \mathbb{P}_{1/2}\{\exists t \in F : S_k(t) = k/2\} \\ & \asymp \frac{1}{\sqrt{k}} \cdot \left[\inf_{\mu \in \mathcal{P}(F)} \iint \min\left(\frac{1}{\sqrt{k|t-s|}}, 1\right) \mu(ds) \mu(dt) \right]^{-1}. \end{aligned} \quad (3.1)$$

As a first key step, we simplify the expression in Lemma 2.1.

Lemma 3.2. *Uniformly for every even integer $k \geq 1$ and $t \in [0, 1]$:*

$$\mathbb{P}_{1/2}(S_k(t) = k/2 \mid S_k(0) = k/2) \asymp \min\left(\frac{1}{\sqrt{kt}}, 1\right). \quad (3.2)$$

Proof. Throughout, we will use the following simple fact:

$$\frac{t}{2} \leq 1 - e^{-2t} \leq 2t \quad \text{for all } t \in [0, 1]. \quad (3.3)$$

We appeal to the result, and notation, of Lemma 2.1. Notice that in the present case, $\theta_t = 1 - \kappa_t = (1 - e^{-t})/2$, and moreover,

$$\mathbb{P}_{1/2}(S_k(t) = k/2 \mid S_k(0) = k/2) = \mathbb{P}\{X = Y\}, \quad (3.4)$$

where X and Y are two independent binomial random variables with common

parameters $k/2$ and $(1 - e^{-t})/2$. We can apply the Plancherel formula:

$$\mathbb{P}\{X = Y\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\phi(z)|^2 dz, \quad (3.5)$$

where ϕ is the characteristic function of X ; i.e.,

$$\phi(z) = \left(\frac{1 - e^{-t}}{2} + \frac{1 + e^{-t}}{2} e^{iz} \right)^{k/2}. \quad (3.6)$$

Of course,

$$\left| \frac{1 - e^{-t}}{2} + \frac{1 + e^{-t}}{2} e^{iz} \right|^2 = \frac{1 + e^{-2t}}{2} + \frac{1 - e^{-2t}}{2} \cos z. \quad (3.7)$$

Taylor's theorem with remainder shows that $1 - (z^2/2) \leq \cos z \leq 1 - (z^2/20)$ for all $z \in [-\pi, \pi]$. Therefore,

$$1 - \frac{z^2}{4} (1 - e^{-2t}) \leq \left| \frac{1 - e^{-t}}{2} + \frac{1 + e^{-t}}{2} e^{iz} \right|^2 \leq 1 - \frac{z^2}{40} (1 - e^{-2t}). \quad (3.8)$$

Now, (3.3) implies that for all $t \in [0, 1]$ and $z \in [-\pi, \pi]$,

$$1 - \frac{z^2 t}{2} \leq \left| \frac{1 - e^{-t}}{2} + \frac{1 + e^{-t}}{2} e^{iz} \right|^2 \leq 1 - \frac{z^2 t}{80}; \quad (3.9)$$

and therefore,

$$\int_{-1}^1 \left(1 - \frac{z^2 t}{2} \right)^{k/2} dz \leq \int_{-\pi}^{\pi} |\phi(z)|^2 dz \leq \int_{-\pi}^{\pi} \left(1 - \frac{z^2 t}{80} \right)^{k/2} dz. \quad (3.10)$$

Since $1 - u \leq e^{-u}$ for all $u \geq 0$, we find that uniformly for all $t \in [0, 1]$,

$$\mathbb{P}\{X = Y\} \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-z^2 tk/160} dz \leq \frac{\text{const}}{\sqrt{tk}}. \quad (3.11)$$

This implies the upper bound of the lemma. For the lower bound follows from the fact $1 - u/2 \geq e^{-u}$, for all $0 \leq u \leq 1$, and $\int_{-1}^1 e^{-z^2 tk/2} dz \geq \text{const} \cdot \min((tk)^{-1/2}, 1)$, uniformly for all $t \in [0, 1]$. \square

Next we prove Theorem 3.1.

Proof of Theorem 3.1. For all $\mu \in \mathcal{P}(F)$ and even integers $k \geq 1$ define

$$L_k^\mu := \int \mathbf{1}_{\{S_k(t)=k/2\}} \mu(dt). \quad (3.12)$$

In accord with (2.9), the following is valid: As $k \rightarrow \infty$,

$$E_{1/2}(L_k^\mu) \asymp k^{-1/2}, \quad (3.13)$$

where the approximation holds uniformly for all $\mu \in \mathcal{P}(F)$. Next we estimate the second moment of L_k^μ , using Lemma 3.2:

$$\begin{aligned} E_{1/2} \left[(L_k^\mu)^2 \right] &\leq 2 \iint_{t \geq s} P_{1/2} \{S_k(t) = S_k(s) = k/2\} \mu(ds) \mu(dt) \\ &\leq \frac{\text{const}}{\sqrt{k}} \cdot \iint \min \left(\frac{1}{\sqrt{k|t-s|}}, 1 \right) \mu(ds) \mu(dt). \end{aligned} \quad (3.14)$$

This, and the Paley–Zygmund inequality, together imply that

$$P_{1/2} \{L_k^\mu > 0\} \geq \frac{\text{const}}{\sqrt{k}} \cdot \left[\iint \min \left(\frac{1}{\sqrt{k|t-s|}}, 1 \right) \mu(ds) \mu(dt) \right]^{-1}. \quad (3.15)$$

The preceding probability is at most $P_{1/2} \{\exists t \in F : S_k(t) = k/2\}$. Since the latter does not depend on $\mu \in \mathcal{P}(F)$,

$$\begin{aligned} &P_{1/2} \{\exists t \in F : S_k(t) = k/2\} \\ &\geq \frac{\text{const}}{\sqrt{k}} \cdot \left[\inf_{\mu \in \mathcal{P}(F)} \iint \min \left(\frac{1}{\sqrt{k|t-s|}}, 1 \right) \mu(ds) \mu(dt) \right]^{-1}. \end{aligned} \quad (3.16)$$

This proves half of the theorem.

For a converse bound, let us consider the stopping time $\sigma := \inf\{t \in F : S_k(t) = k/2\}$, where $\inf \emptyset := 2$. We apply our existing computation with the following special choice of μ : $\mu(\bullet) := P_{1/2}(\sigma \in \bullet \mid \sigma < 1)$. By the strong Markov property and Lemma 3.2,

$$\begin{aligned} E_{1/2} (L_k^\mu \mid \mathcal{F}_\sigma) &= \int_{s \geq \sigma} P_{1/2} (S_k(s) = k/2 \mid \mathcal{F}_\sigma) \mu(ds) \\ &\geq \text{const} \cdot \int_{s \geq \sigma} \min \left(\frac{1}{\sqrt{k(s-\sigma)}}, 1 \right) \mu(ds), \end{aligned} \quad (3.17)$$

where the implied constant does not depend on μ , and is nonrandom. Since σ is a bounded stopping time, we can apply the optional stopping theorem to deduce from the preceding that for all probability measures μ on F ,

$$E_{1/2}(L_k^\mu) \geq \text{const} \cdot E_{1/2} \left[\int_{s \geq \sigma} \min \left(\frac{1}{\sqrt{k(s-\sigma)}}, 1 \right) \mu(ds) \right]. \quad (3.18)$$

According to (3.13),

$$E_{1/2} \left[\int_{s \geq \sigma} \min \left(\frac{1}{\sqrt{k(s-\sigma)}}, 1 \right) \mu(ds) \right] \leq \frac{\text{const}}{\sqrt{k}}. \quad (3.19)$$

On the other hand, the definition of μ implies that the left-hand side is equal to

$$\begin{aligned} & \iint_{s \geq t} \min \left(\frac{1}{\sqrt{k(s-t)}}, 1 \right) \mu(ds) \mu(dt) \cdot P_{1/2}\{\sigma < 1\} \\ & \geq \frac{1}{2} \iint \min \left(\frac{1}{\sqrt{k|t-s|}}, 1 \right) \mu(ds) \mu(dt) \cdot P_{1/2}\{\sigma < 1\}. \end{aligned} \quad (3.20)$$

It follows from the preceding two displays that

$$P_{1/2}\{\sigma < 1\} \leq \frac{\text{const}}{\sqrt{k}} \cdot \left[\iint \min \left(\frac{1}{\sqrt{k|t-s|}}, 1 \right) \mu(ds) \mu(dt) \right]^{-1}. \quad (3.21)$$

This proves the theorem. \square

Finally, we prove Theorem 1.3.

Proof of Theorem 1.3. We recall Howroyd's theory of box-dimension profiles [10]: For every $s > 0$ and $x \in \mathbf{R}$ define $\psi_s(x) := \min(1, |x|^{-s})$. Then, given $r > 0$, a sequence of pairs $(w_i, x_i)_{i=1}^n$ is a *size- r weighted ψ_s -packing of F* if: (i) $x_i \in F$ for all i ; (ii) $w_i \geq 0$ for all i ; and (iii) $\sum_{j=1}^n w_j \psi_s((x_i - x_j)/r) \leq 1$, uniformly for all $i = 1, \dots, n$. Define $N_r(F; \psi_s) := \sup \sum_{i=1}^n w_i$, where the supremum is taken over all size- r weighted ψ_s -packings $(w_i, x_i)_{i=1}^n$ of F . The *s -dimensional upper box-dimension profile* $B\text{-}\overline{\text{dim}}_s F$ of F is then defined as

$$B\text{-}\overline{\text{dim}}_s F := \limsup_{r \downarrow 0} \frac{\log N_r(F; \psi_s)}{\log(1/r)}, \quad (3.22)$$

where $\log 0 := -\infty$. The *s -dimensional lower box-dimension profile* $B\text{-}\underline{\text{dim}}_s F$ of

F is defined as above, but with a \liminf in place of the \limsup .

According to Khoshnevisan and Xiao [13, Theorem 4.1],

$$\text{B-}\overline{\text{dim}}_s F = \limsup_{n \rightarrow \infty} \frac{1}{\log(1/n)} \log \inf_{\mu \in \mathcal{P}(F)} \iint \psi_s \left(\frac{u-v}{1/n} \right) \mu(du) \mu(dv). \quad (3.23)$$

That proof shows also that $\text{B-}\underline{\text{dim}}_s F$ is equal to the same quantity, but with \limsup replaced by \liminf . In light of these two facts, Theorem 1.3 follows from Theorem 3.1. \square

4 Some applications to runs

Throughout this section we will be studying \mathbf{X}_∞ and its dynamical version; both can be defined in the usual way via infinite product spaces, as respective “limits” of \mathbf{X}_k and its dynamical version, as $k \rightarrow \infty$. We skip the details, as they are outlined nicely in [3].

As was pointed out in [3], one can use an estimate such as (1.4) to study the behavior of the longest runs in a dynamical bit sequence, as the sequence length tends to infinity. We describe this work next. For every two integers $n, \ell \geq 1$, define $Z_n^{(\ell)}$ to be the largest integer $j \geq \ell + 1$ such that $X_m = 1$ for all but ℓ values of $m \in \{n, \dots, n+j-1\}$. If such a j does not exist, then $Z_n^{(\ell)} := 0$. One defines the dynamical version similarly: $Z_n^{(\ell)}(t)$ is defined as above, by X_m is replaced by $X_m(t)$.

According to the Erdős–Rényi theorem [6], for every $\ell \geq 1$,

$$\lim_{n \rightarrow \infty} \frac{Z_n^{(\ell)}}{\log_{1/p} n} = 1 \quad \text{P}_p\text{-almost surely,} \quad (4.1)$$

where $\log_{1/p}$ denotes the base- $(1/p)$ logarithm. Erdős and Révész [7] improved this statement by showing that the following holds for every nonrandom sequence $\mathbf{a} := \{a_j\}_{j=1}^\infty$ of positive integers that tend to infinity:

$$\text{P}_p \left\{ \limsup_{n \rightarrow \infty} Z_n^{(\ell)} \geq a_n \text{ i.o.} \right\} = \begin{cases} 0 & \text{if } \sum_{n=1}^\infty a_n^\ell p^{a_n} < \infty, \\ 1 & \text{otherwise.} \end{cases} \quad (4.2)$$

This particular formulation appears explicitly, for example, in the book by Révész [14, p. 60] in the case that $p = 1/2$. To be more precise, Révész (*loc. cit.*) defines $Z_n^{(\ell)}$ as the longest run having *at most* ℓ defects in the first n bits.

But a real-variable comparison argument reveals that our definition and that of Révész have the same asymptotic behavior [7].

Of course, $Z_n^{(\ell)}$ can be replaced by $Z_n^{(\ell)}(t)$ for a fixed t , and (4.2) continues to hold. By contrast, it is shown in [3, Theorem 1.4] that

$$\mathbb{P}_p \left\{ \exists t \in [0, 1] : Z_n^{(\ell)}(t) \geq a_n \text{ i.o.} \right\} = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} a_n^{\ell+1} p^{a_n} < \infty, \\ 1 & \text{otherwise.} \end{cases} \quad (4.3)$$

To be precise, Benjamini et al [3] prove this for $\ell = 0$; but one can apply their method, using (1.4), to produce (4.3). We learned about (4.3) from the monograph by Révész [14, p. 61], who conjectured (4.3) in the case that $p = 1/2$. It is possible to study the size of the set of times t at which $Z_n^{(\ell)}(t) \geq a_n$ infinitely often. Define

$$\mathcal{E}^{(\ell)}(\mathbf{a}) := \left\{ t \in [0, 1] : Z_n^{(\ell)}(t) \geq a_n \text{ i.o.} \right\}. \quad (4.4)$$

In light of (4.3), $\mathcal{E}^{(\ell)}(\mathbf{a})$ is nonempty if and only if $\sum_{n=1}^{\infty} a_n^{\ell+1} p^{a_n} = \infty$. Then, it is possible to adapt the argument of [3, Theorem 1.5], using (1.4) of the present paper, to derive the following:

$$\dim_{\mathbb{H}} \mathcal{E}^{(\ell)}(\mathbf{a}) = \sup \left\{ s \in (0, 1) : \sum_{n=1}^{\infty} a_n^{\ell+1-s} p^{a_n} = \infty \right\}, \quad (4.5)$$

where $\dim_{\mathbb{H}}$ denotes Hausdorff dimension.

Equation (4.2) is a statement about whether or not $\mathcal{E}^{(\ell)}(\mathbf{a})$ is void. Our next result describes all nonrandom compact sets $F \subset [0, 1]$ that have positive probability of intersecting $\mathcal{E}^{(\ell)}(\mathbf{a})$. First we need some notation from [12].

We say that an interval is *rational* if its endpoints are rational numbers. From here on, we choose and fix a nonrandom set F and a nonrandom sequence $\mathbf{a} := \{a_j\}_{j=1}^{\infty}$ of positive integers that tend to infinity. It turns out that the following definition [12] defines the correct intersection property for the random set $\mathcal{E}^{(\ell)}(\mathbf{a})$:

Definition 4.1. We write “ $\Psi(\mathbf{a}; F) < \infty$ ” if and only if we can find closed rational intervals F_1, F_2, \dots such that: (i) $\cup_{n=1}^{\infty} F_n \supseteq F$; and (ii)

$$\sum_{j=1}^{\infty} K_{F_n}(1/a_j) a_j^{\ell} p^{a_j} < \infty \quad \text{for all } n \geq 1. \quad (4.6)$$

Theorem 4.2. For every nonrandom compact set $F \subseteq [0, 1]$,

$$\mathbb{P}_p \left\{ \mathcal{E}^{(\ell)}(\mathbf{a}) \cap F \neq \emptyset \right\} = \begin{cases} 0 & \text{if } \Psi(\mathbf{a}; F) < \infty, \\ 1 & \text{otherwise.} \end{cases} \quad (4.7)$$

One can adapt the proof of [12, Theorem 2.5] to the present setting to see that the following implies Theorem 4.2. We will not prove Theorem 4.2, since its proof does not require new ideas. However, we will prove the following crucial step, whose proof borrows ideas from the proof of Theorem 1.4 of [3].

Proposition 4.3. For all nonrandom compact sets $F \subseteq [0, 1]$,

$$\mathbb{P}_p \left\{ \sup_{t \in F} Z_n^{(\ell)}(t) \geq a_n \text{ i.o.} \right\} = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \mathbb{K}_F(1/a_n) a_n^\ell p^{a_n} < \infty, \\ 1 & \text{otherwise.} \end{cases} \quad (4.8)$$

Proof. According to Theorem 1.1, $\sum_{n=1}^{\infty} \mathbb{K}_F(1/a_n) a_n^\ell p^{a_n} < \infty$ if and only if

$$\sum_{n=1}^{\infty} \mathbb{P} \left\{ \sup_{t \in F} Z_n^{(\ell)}(t) \geq a_n \right\} < \infty. \quad (4.9)$$

Therefore, whenever $\sum_{n=1}^{\infty} \mathbb{K}_F(1/a_n) a_n^\ell p^{a_n}$ is finite, $\sup_{t \in F} Z_n^{(\ell)}(t) \geq a_n$ only occurs finitely often. For the converse let us suppose $\sum_{n=1}^{\infty} \mathbb{K}_F(1/a_n) a_n^\ell p^{a_n} = \infty$, and define the events

$$F_n := \{X_{n-j}(t) = 0 \text{ for every } j = 1, 2, \dots, \ell + 1 \text{ and } t \in [0, 1]\}. \quad (4.10)$$

Note that F_n is independent of the event

$$\begin{aligned} \tilde{F}_n &:= \left\{ \sup_{t \in F} Z_n^{(\ell)}(t) \geq a_n \right\} \\ &= \{ \exists t \in F : S_{n+a_n-1}(t) - S_n(t) = a_n - \ell \}, \end{aligned} \quad (4.11)$$

and $\mathbb{P}_p(F_n) \geq (q/e)^{\ell+1}$ with $q := 1 - p$. Since $\sum_{n=1}^{\infty} \mathbb{K}_F(1/a_n) a_n^\ell p^{a_n} = \infty$,

$$\sum_{j=1}^{\infty} \mathbb{P}_p(G_j) = \infty \quad \text{where} \quad G_n := F_n \cap \tilde{F}_n. \quad (4.12)$$

Note that for all integers $N \geq 1$,

$$\sum_{n,m=1}^N \mathbb{P}_p(G_n \cap G_m) = \sum_{n=1}^N \mathbb{P}_p(G_n) + 2 \sum_{n,m=1}^N \mathbb{P}_p(G_n \cap G_m). \quad (4.13)$$

Suppose $n < m$. If $m \leq n + a_n - 1$, then the events G_n and G_m are disjoint. If $m \geq n + a_n + \ell + 1$, then G_n and G_m are independent. In the remaining $O(1)$ cases we can use the elementary bound $\mathbb{P}_p(G_n \cap G_m) \leq \mathbb{P}_p(G_n)$. Since $\sum_n \mathbb{P}_p(G_n) = \infty$, it follows that $\sum_{n,m=1}^N \mathbb{P}_p(G_n \cap G_m)$ is bounded above by a constant times $(\sum_{j=1}^N \mathbb{P}_p(G_j))^2$, where the implied constant does not depend on $N \geq 1$. The Borel–Cantelli lemma for dependent event [4] implies that infinitely-many of the G_n 's—and hence the events \tilde{F}_n —occur almost surely. \square

Let us end this section with the following example: Let $\theta > 0$ be fixed, and consider the sequence $\mathbf{a}(\theta)$ given by

$$a_n = a_n(\theta) := l_p n + \theta l_p l_p n, \quad (4.14)$$

where $l_p x := \log_{1/p}(\max(x, 100))$. It is possible to check that

$$\sum_{n=1}^{\infty} \mathbb{K}_F(1/a_n(\theta)) [a_n(\theta)]^\ell p^{a_n(\theta)} < \infty \quad \text{iff} \quad \sum_{n=100}^{\infty} \frac{\mathbb{K}_F(1/\log n)}{n(\log n)^{\theta-\ell}} ds < \infty. \quad (4.15)$$

The doubling property, $\mathbb{K}_F(2\epsilon) \asymp \mathbb{K}_F(\epsilon)$, valid for all $\epsilon > 0$ sufficiently small [12, eq. (2.8)], implies that

$$\sum_{n=1}^{\infty} \mathbb{K}_F(1/a_n(\theta)) [a_n(\theta)]^\ell p^{a_n(\theta)} < \infty \quad \text{iff} \quad \int_1^{\infty} \frac{\mathbb{K}_F(1/s)}{s^{\theta-\ell}} ds < \infty. \quad (4.16)$$

According to Proposition 2.9 of [12],

$$\dim_p F + \ell + 1 = \inf \{ \theta : \Psi(\mathbf{a}(\theta); F) < \infty \}, \quad (4.17)$$

where \dim_p denotes packing dimension. Therefore, we can combine the preceding facts to deduce the following:

$$\sup_{t \in F} \limsup_{n \rightarrow \infty} \frac{Z_n^{(\ell)}(t) - l_p n}{l_p l_p n} = \dim_p F + \ell + 1 \quad \mathbb{P}_p\text{-a.s.} \quad (4.18)$$

When $F := \{0\}$, this is due to Erdős and Rényi [6]; and when $F := [0, 1]$ it is

due to Benjamini et al [3, Theorem 3.1].

5 A sharp capacity criterion

Let $m : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ be a strictly increasing function so that $m(\mathbf{N}) \subset \mathbf{N}$. Benjamini et al [3] have proposed the following “bit process” as part of their parity test that is motivated by complexity theory: Define

$$B_k(t) := \bigoplus_{j=m(k)}^{m(k+1)} X_j(t), \quad (5.1)$$

where $t \geq 0$, $k \in \mathbf{N}_+$, and \oplus denotes addition mod 2. Of course, $B_k(t)$ is either zero or one. It is proved in [3, Lemma 4.1] that

$$P_{1/2} \{ \exists t \in [0, 1] : B_k(t) = 0 \text{ for all } k \in \mathbf{N} \} > 0 \quad \text{iff} \quad \sum_{k=1}^{\infty} \frac{2^k}{m(k)} < \infty, \quad (5.2)$$

provided that m satisfies the Hadamard gap condition,

$$\inf_{k \geq 1} \frac{m(k+1)}{m(k)} > 1. \quad (5.3)$$

Consider the random set

$$T_m := \{ t \in [0, 1] : B_k(t) = 0 \text{ for all } k \in \mathbf{N} \}. \quad (5.4)$$

Consider the special case that m is the function $m_q(x) := \lceil 2^{x/q} \rceil$ for some fixed $q > 0$. Lemma 4.1 of [3] shows that for all nonrandom compact sets $E \subset [0, 1]$,

$$\text{Cap}_q(E) > 0 \quad \Rightarrow \quad P_{1/2} \{ T_{m_q} \cap E \neq \emptyset \} > 0 \quad \Rightarrow \quad \mathcal{H}^q(E) > 0, \quad (5.5)$$

where Cap_q denotes the q -dimensional Riesz capacity and \mathcal{H}^q the q -dimensional Hausdorff measure [11, App. C & D]. In their Remark 4.5, Benjamini et al (*loc. cit.*) point out that there is a [small] gap between the conditions of positive Hausdorff measure versus positive capacity. If we specialize the next theorem to $m := m_q$ then we obtain a verification of their conjecture.

Theorem 5.1. *Suppose, in addition, that $t \mapsto 2^{-t}m(t)$ and m are both strictly increasing. Additionally, choose and fix a nonrandom compact set $E \subset [0, 1]$. Then, $P_{1/2} \{ T_m \cap E \neq \emptyset \} > 0$ iff there exists a probability measure ρ on E such*

that $J(\rho) < \infty$, where

$$J(\rho) := \iint (\mathcal{L}g)(|t-s|) \rho(ds) \rho(dt); \quad (5.6)$$

g is defined by $\log_2 g(t) = m^{-1}(t)$, and $(\mathcal{L}g)(\lambda) := \int_0^\infty e^{-\lambda s} g(ds)$ denotes the Laplace transform of the Stieltjes measure dg .

Remark 5.2. The monotonicity condition on $t \mapsto 2^{-t}m(t)$ implies the gap condition (5.3). Indeed, under the monotonicity condition we have $m(k+1)/m(k) \geq 2$ for all $k \geq 1$. \square

Remark 5.3. One can inspect the proof to see that the monotonicity of $2^{-t}m(t)$ can be generalized to the condition that $c^t m(t)$ is strictly decreasing for some $c \in (0, 1)$. \square

In order to prove the conjecture of Benjamini et al from Theorem 5.1, we first define $m_q(k) := \lceil 2^{k/q} \rceil$ and interpolate linearly to obtain a strictly increasing function [also called m_q] on \mathbf{R}_+ . If $2^{-t}m(t)$ fails to be increasing near zero, then any reasonable alteration near zero works because the conditions of Theorem 5.1 only restrict the behavior of m near infinity. It is not hard to check that in this case, $(\mathcal{L}g)(\lambda) \asymp \lambda^{-q}$ as $\lambda \downarrow 0$. From this it follows that $J(\rho) < \infty$ for some $\rho \in \mathcal{P}(E)$ if and only if $\iint |x-y|^{-q} \rho(dx) \rho(dy) < \infty$. This shows that the capacity criterion in (5.5) is the sharp necessary and sufficient condition.

We conclude by proving Theorem 5.1

Proof of Theorem 5.1. Let $\mathcal{F} := \{\mathcal{F}_t\}_{t \geq 0}$ denote the filtration such that each \mathcal{F}_t is generated by all variables $X_j(r)$, where $j \geq 1$ and $r \in [0, t]$. If it is not already augmented in the usual way, then we need to augment \mathcal{F} so that it satisfies the ‘‘usual conditions’’ of Dellacherie and Meyer [5].

Consider the events

$$U_n(t) := \{B_k(t) = 0 \text{ for all } k \in \{1, \dots, n\}\}. \quad (5.7)$$

A first-passage time argument shows that for all \mathcal{F} -stopping times τ ,

$$P_{1/2}(U_n(t+\tau) \mid \mathcal{F}_\tau) = 2^{-n} f_n(\tau), \quad (5.8)$$

where for all $n \in \mathbf{N} \cup \{\infty\}$ and $\lambda \in \mathbf{R}$,

$$f_n(\lambda) := \prod_{k=1}^n \left(1 - e^{-m(k)|\lambda|}\right). \quad (5.9)$$

Indeed, by the strong Markov property, it suffices to prove this for $\tau := 0$; and that is what we do next: Because $\{B_k(t)\}_{k=1}^\infty$ are conditionally independent given \mathcal{F}_0 ,

$$\mathbb{P}_{1/2}(U_n(t) \mid \mathcal{F}_0) = \prod_{k=1}^n \mathbb{P}_{1/2}(B_k(t) = 0 \mid \mathcal{F}_0). \quad (5.10)$$

Let τ_k denote the first jump-time of the Markov process

$$s \mapsto (X_{m(k)}(s), \dots, X_{m(k+1)}(s)). \quad (5.11)$$

The $\mathbb{P}_{1/2}$ -law of τ_k is exponential with mean $1/m(k)$. Therefore, we obtain the following by splitting the probability according to whether or not $\tau_k > t$:

$$\mathbb{P}_{1/2}(B_k(t) = 0 \mid \mathcal{F}_0) = e^{-m(k)t} \mathbf{1}_{\{B_k(0)=0\}} + \frac{1}{2} \left[1 - e^{-m(k)t}\right]. \quad (5.12)$$

This and (5.10) together imply (5.8). Next we begin by recalling the argument of [3, Theorem 4.3] for the necessity of the positive-capacity condition.

Choose and fix some $\rho \in \mathcal{P}(E)$, and define

$$Z_n(\omega) := \int \mathbf{1}_{U_n(t)}(\omega) \rho(dt). \quad (5.13)$$

It is easy to see that $\mathbb{P}_{1/2}(U_n(0)) = 2^{-n}$. Moreover, stationarity and (5.8) together imply

$$\mathbb{E}_{1/2}(Z_n^2) = \frac{1}{4^n} \iint f_n(t-s) \rho(dt) \rho(ds) = \frac{1}{4^n} \int (f_n * \rho) d\rho. \quad (5.14)$$

Therefore, the Paley–Zygmund lemma implies that

$$\liminf_{n \rightarrow \infty} \mathbb{P}_{1/2}\{Z_n > 0\} \geq \left[\int (f_\infty * \rho) d\rho \right]^{-1}. \quad (5.15)$$

From this it follows readily that

$$\mathbb{P}_{1/2} \left\{ \exists t \in E : \sup_{k \geq 1} B_k(t) = 0 \right\} \geq \left[\inf_{\rho \in \mathcal{P}(E)} \int (f_\infty * \rho) d\rho \right]^{-1}. \quad (5.16)$$

For the converse bound we use a first-passage argument. Define $\tau(n)(\omega) := \inf\{t \in E : \omega \in U_n(t)\}$ where $\inf \emptyset := \infty$; $\tau(n)$ is an \mathcal{F} -stopping time. Define $\rho_n \in \mathcal{P}(E)$ via $\rho(\bullet) := \rho_n(\bullet) := \mathbb{P}_{1/2}(\tau(n) \in \bullet \mid \tau(n) < \infty)$. And consider the martingales $\{M_n(t)\}_{n=0}^\infty$, defined via

$$M_n := \mathbb{E}_{1/2}(Z_n \mid \mathcal{F}_t). \quad (5.17)$$

Equation (5.8) implies that $M_n(\tau(n) \wedge 1) \geq 2^{-n} \int_{\tau(n)}^1 f_n(t - (\tau(n) \wedge 1)) \rho_n(dt)$. Because $\tau(n) \wedge 1 = \tau(n)$, $\mathbb{P}_{1/2}$ -a.s. on $\{\tau(n) < \infty\}$, it follows that

$$M_n(\tau(n)) \geq 2^{-n} \int_{\tau(n)}^1 f_n(t - \tau(n)) \rho_n(dt), \quad (5.18)$$

$\mathbb{P}_{1/2}$ -a.s. on $\{\tau(n) < \infty\}$. The trivial estimate

$$\mathbb{E}_{1/2}(M_n(\tau(n) \wedge 1)) \geq \mathbb{E}_{1/2}(M_n(\tau(n)); \tau(n) < \infty), \quad (5.19)$$

together with the definition of ρ_n , impose the following:

$$\begin{aligned} & 2^n \mathbb{E}_{1/2}(M_n(\tau(n) \wedge 1)) \\ & \geq \mathbb{E}_{1/2} \left(\int_{\tau(n)}^1 f_n(t - \tau(n)) \rho_n(dt) \mid \tau(n) < \infty \right) \times \mathbb{P}_{1/2}\{\tau(n) < \infty\} \quad (5.20) \\ & \geq \frac{1}{2} \int (f_n * \rho_n) d\rho_n \times \mathbb{P}_{1/2}\{\tau(n) < \infty\}. \end{aligned}$$

[The last inequality is an identity when ρ_n is atomless.] By the optional stopping theorem, the left-most term is equal to $\mathbb{E}_{1/2}(M_n(1)) = \mathbb{E}_{1/2}(Z_n) = 2^{-n}$. Therefore, $\mathbb{P}_{1/2}\{\exists t \in E : \max_{1 \leq k \leq n} B_k(t) = 0\} \leq 2[\int (f_n * \rho_n) d\rho]^{-1}$.

We obtain an even smaller quantity if we replace f_n by f_ℓ , whenever $n \geq \ell$. By Prohorov's theorem, there is a subsequence of $\{\rho_n\}$ and a probability measure ρ on E such that the subsequence converges weakly to ρ . It follows from the preceding that $\mathbb{P}_{1/2}\{\exists t \in E : \sup_{k \geq 1} B_k(t) = 0\} \leq 2[\int (f_\ell * \rho) d\rho]^{-1}$. Let $\ell \uparrow \infty$ and apply (5.16) and the monotone convergence theorem to find

$$\frac{1}{\inf_{\rho \in \mathcal{P}(E)} I(\rho)} \leq \mathbb{P}_{1/2} \left\{ \exists t \in E : \max_{k \geq 1} B_k(t) = 0 \right\} \leq \frac{2}{\inf_{\rho \in \mathcal{P}(E)} I(\rho)}, \quad (5.21)$$

where $1/\inf \emptyset := 0$ and

$$I(\rho) := \iint \prod_{k=1}^{\infty} \left(1 + e^{-m(k)|t-s|}\right) \rho(ds) \rho(dt). \quad (5.22)$$

Inequalities (5.21) is valid for *every* increasing function m which satisfies $m(\mathbf{N}) \subset \mathbf{N}$. Now we concentrate on the functions m that satisfy the monotonicity requirements of Theorem 5.1, and prove that $I(\rho)$ and $J(\rho)$ converge and diverge together [for those functions m]. Our method is an adaptation of a reduction argument of [3], used to estimate Riesz-type products of the form $\prod_{k=1}^{\infty} (1 + \exp(-m(k)\lambda))$ for $\lambda > 0$ small.

First of all, note that for integers $n \geq 1$ and real numbers $\lambda > 0$,

$$\begin{aligned} \prod_{k=1}^n \left(1 + e^{-m(k)\lambda}\right) &= 1 + \sum_{\substack{S \subseteq \{1, \dots, n\} \\ S \neq \emptyset}} e^{-\lambda \sum_{k \in S} m(k)} \\ &\leq 1 + \sum_{\substack{S \subseteq \{1, \dots, n\} \\ S \neq \emptyset}} e^{-\lambda m(\max S)}. \end{aligned} \quad (5.23)$$

For every $j = 1, \dots, n$ there are 2^{j-1} subsets $S \subset \{1, \dots, n\}$ with $\max S = j$. Therefore,

$$\begin{aligned} \prod_{k=1}^n \left(1 + e^{-m(k)\lambda}\right) &\leq 1 + \sum_{j=1}^n 2^{j-1} e^{-\lambda m(j)} \leq 1 + \int_1^{\infty} e^{-\lambda m(\log_2 s)} ds \\ &= 1 + \int_{m(0)}^{\infty} e^{-\lambda t} dg(t) \leq 1 + (\mathcal{L}g)(\lambda). \end{aligned} \quad (5.24)$$

It follows that $I(\rho) \leq 1 + J(\rho)$.

For the complementary bound we begin by expressing our product as follows: First, we write $\sum_{k \in S} m(k)$ as $\max(S) \sum_{k \in S} m(k)/m(\max S)$. Then, we write $m(t) = 2^t f(2^t)$ where f is increasing, and find that $\prod_{k=1}^n (1 + e^{-m(k)\lambda})$ is bounded below by

$$1 + \sum_{\substack{S \subseteq \{1, \dots, n\} \\ S \neq \emptyset}} \exp\left(-\lambda m(\max S) \cdot \sum_{k \in S} 2^{k - \max S}\right). \quad (5.25)$$

Since $\sum_{k \in S} 2^{k - \max S} \leq \sum_{j=0}^{\infty} 2^{-j} = 2$, it follows that

$$\begin{aligned} \prod_{k=1}^n \left(1 + e^{-m(k)\lambda}\right) &\geq 1 + \sum_{\substack{S \subseteq \{1, \dots, n\} \\ S \neq \emptyset}} e^{-2\lambda m(\max S)} = 1 + \sum_{j=1}^n 2^{j-1} e^{-2\lambda m(j)} \\ &\geq 1 + \frac{1}{2} \int_2^{\infty} \exp(-2\lambda m(\log_2 t)) dt. \end{aligned} \quad (5.26)$$

Because $tf(t) = m(\log_2 t)$ for f increasing,

$$\prod_{k=1}^n \left(1 + e^{-m(k)\lambda}\right) \geq 1 + \frac{1}{4} \int_4^{\infty} e^{-\lambda t f(t/2)} dt \geq 1 + \frac{1}{4} \int_{m(2)}^{\infty} e^{-\lambda s} dg(s). \quad (5.27)$$

If $D := \sup\{|t - s| : s, t \in E\}$ then for all $\lambda \in (0, D)$,

$$\begin{aligned} \int_0^{m(2)} e^{-\lambda s} dg(s) &\leq g(m(2)) = 4, \\ \int_{m(2)}^{\infty} e^{-\lambda s} dg(s) &\geq \int_{m(2)}^{\infty} e^{-Ds} dg(s) := \frac{4}{C}. \end{aligned} \quad (5.28)$$

[We are *defining* C in this way.] Thus, $(\mathcal{L}g)(\lambda) \leq (1 + C) \int_{m(2)}^{\infty} e^{-\lambda s} dg(s)$ for all $\lambda \in (0, D)$, and therefore

$$\prod_{k=1}^n \left(1 + e^{-m(k)\lambda}\right) \geq 1 + \frac{(\mathcal{L}g)(\lambda)}{4(1 + C)} \geq \frac{1 + (\mathcal{L}g)(\lambda)}{4(1 + C)}. \quad (5.29)$$

This proves that $I(\rho) \geq (4 + 4C)^{-1}(1 + J(\rho))$, whence the theorem. \square

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