

DYNAMICAL PERCOLATION ON GENERAL TREES

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ABSTRACT. Häggström, Peres, and Steif (1997) have introduced a dynamical version of percolation on a graph G . When G is a tree they derived a necessary and sufficient condition for percolation to exist at some time t . In the case that G is a spherically symmetric tree, Peres and Steif (1998) derived a necessary and sufficient condition for percolation to exist at some time t in a given target set D . The main result of the present paper is a necessary and sufficient condition for the existence of percolation, at some time $t \in D$, in the case that the underlying tree is not necessarily spherically symmetric. This answers a question of Yuval Peres (personal communication). We present also a formula for the Hausdorff dimension of the set of exceptional times of percolation.

1. Introduction

Let G be a locally finite connected graph. Choose and fix $p \in (0, 1)$, and let each edge be “open” or “closed” with respective probabilities p and $q := 1 - p$; all edge assignments are made independently. Define P_p to be the resulting product measure on the collection of random edge assignments.

A fundamental problem of bond percolation is to decide when there can exist an infinite connected cluster of open edges in G (Grimmett, 1999). Choose and fix some vertex ρ in G , and consider the event $\{\rho \leftrightarrow \infty\}$ that percolation occurs through ρ . That is, let $\{\rho \leftrightarrow \infty\}$ denote the event that there exists an infinite connected cluster of open edges that emanate from the vertex ρ . Then the stated problem of percolation theory is, when is $P_p\{\rho \leftrightarrow \infty\} > 0$? We may note that the positivity of $P_p\{\rho \leftrightarrow \infty\}$ does not depend on our choice of ρ .

There does not seem to be a general answer to this question, although much is known (Grimmett, 1999). For instance, there always exists a critical probability p_c such that

$$(1.1) \quad P_p\{\rho \leftrightarrow \infty\} = \begin{cases} \text{positive,} & \text{if } p > p_c, \\ \text{zero,} & \text{if } p < p_c. \end{cases}$$

However, $P_{p_c}\{\rho \leftrightarrow \infty\}$ can be zero for some graphs G , and positive for others.

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When G is a tree, much more is known. In this case, Lyons (1992) has proved that $P_p\{\rho \leftrightarrow \infty\} > 0$ if and only if there exists a probability measure μ on ∂G such that

$$(1.2) \quad \iint \frac{\mu(dv)\mu(dw)}{p^{|v \wedge w|}} < \infty.$$

In the language of population genetics, $v \wedge w$ denotes the “greatest common ancestor” of v and w , and $|z|$ is the “age,” or height, of the vertex z . Also, ∂G denotes the boundary of G . This is the collection of all infinite rays, and is metrized with the hyperbolic metric $d(v, w) := \exp(-|v \wedge w|)$. It is not hard to check that $(\partial G, d)$ is a compact metric space. Furthermore, one can apply the celebrated theorem of Frostman (1935) in conjunction with Lyons’s theorem to find that $p_c = \exp(-\dim_{\text{H}} \partial G)$, where $\dim_{\text{H}} \partial G$ denotes the Hausdorff dimension of the metric space $(\partial G, d)$.

Lyons’s theorem improves on the earlier efforts of Lyons (1989; 1990) and aspects of the work of Dubins and Freedman (1967) and Evans (1992). Benjamini et al. (1995) and Marchal (1998) contain two different optimal improvements on Lyons’s theorem.

Häggström, Peres, and Steif (1997) added equilibrium dynamics to percolation problems. Next is a brief description. At time zero we construct all edge assignments according to P_p . Then we update each edge weight, independently of all others, in a stationary-Markov fashion: If an edge is closed then it flips to an open one at rate p ; if an edge weight is open then it flips to a closed edge at rate $q := 1 - p$.

Let us write $\{\rho \overset{t}{\leftrightarrow} \infty\}$ for the event that we have percolation at time t . By stationarity, $P_p\{\rho \overset{t}{\leftrightarrow} \infty\}$ does not depend on t . In particular, if $p < p_c$ then $P_p\{\rho \overset{t}{\leftrightarrow} \infty\} = 0$ for all $t \geq 0$. If G is a tree, then $P_p\{\rho \overset{t}{\leftrightarrow} \infty\}$ is the probability of percolation in the context of Lyons (1992). The results of Häggström et al. (1997) imply that there exists a tree G such that $P_{p_c}(\cup_{t>0}\{\rho \overset{t}{\leftrightarrow} \infty\}) = 1$ although $P_{p_c}\{\rho \overset{t}{\leftrightarrow} \infty\} = 0$ for all $t \geq 0$. We add that, in all cases, the event $\cup_{t>0}\{\rho \overset{t}{\leftrightarrow} \infty\}$ is indeed measurable, and thanks to ergodicity has probability zero or one.

Now let us specialize to the case that G is a *spherically symmetric* tree. This means that all vertices of a given height have the same number of children. In this case, Häggström et al. (1997) studied dynamical percolation on G in greater depth and proved that for all $p \in (0, 1)$,

$$(1.3) \quad P_p \left(\bigcup_{t \geq 0} \{\rho \overset{t}{\leftrightarrow} \infty\} \right) = 1 \quad \text{if and only if} \quad \sum_{l=1}^{\infty} \frac{p^{-l}}{l|G_l|} < \infty.$$

Here, G_n denotes the collection of all vertices of height n , and $|G_n|$ denotes its cardinality. This theorem has been extended further by Peres and Steif (1998). In order to describe their results we follow their lead and consider only the non-trivial case where G is an infinite tree.

In that case, Theorem 1.4 of Peres and Steif (1998) asserts that for all nonrandom closed sets $D \subseteq [0, 1]$, $P_p(\cup_{t \in D} \{\rho \xleftrightarrow{t} \infty\}) > 0$ if and only if there exists a probability measure ν on D such that

$$(1.4) \quad \iint \sum_{l=1}^{\infty} \frac{1}{|G_l|} \left(1 + \frac{q}{p} e^{-|t-s|}\right)^l \nu(ds) \nu(dt) < \infty.$$

The principle aim of this paper is to study general trees G —i.e., not necessarily spherically symmetric ones—and describe when there is positive probability of percolation for some time t in a given “target set” D . Our description (Theorem 2.1) answers a question of Yuval Peres (personal communication), and confirms Conjecture 1 of Pemantle and Peres (1995) for a large family of concrete target percolations. In addition, when D is a singleton, our description recovers the characterization (1.2)—due to Lyons (1992)—of percolation on general trees.

As was mentioned earlier, it can happen that $P_{p_c}\{\rho \leftrightarrow \infty\} = 0$, and yet percolation occurs at some time t $[P_{p_c}]$. Let $S(G)$ denote the collection of all such exceptional times. When G is spherically symmetric, Häggström et al. (1997, Theorem 1.6) compute the Hausdorff dimension of $S(G)$. Here we do the same in the case that G is a generic tree (Theorem 2.6). In order to do this we appeal to the theory of Lévy processes (Bertoin, 1996; Khoshnevisan, 2002; Sato, 1999); the resulting formula for dimension is more complicated when G is not spherically symmetric. We apply our formula to present simple bounds for the Hausdorff dimension of $S(G) \cap D$ for a non-random target set D in the case that G is spherically symmetric (Proposition 5.3). When D is a regular fractal our upper and lower bounds agree, and we obtain an almost-sure identity for the Hausdorff dimension of $S(G) \cap D$.

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2. Main Results

Our work is in terms of various capacities for which we need some notation.

Let S be a topological space, and suppose $f : S \times S \rightarrow \mathbf{R}_+ \cup \{\infty\}$ is measurable and μ is a Borel probability measure on S . Then we define the f -energy of μ to be

$$(2.1) \quad I_f(\mu) := \iint f(x, y) \mu(dx) \mu(dy).$$

We define also the f -capacity of a Borel set $F \subseteq S$ as

$$(2.2) \quad \text{Cap}_f(F) := \left[\inf_{\mu \in \mathcal{P}(F)} I_f(\mu) \right]^{-1},$$

where $\inf \emptyset := \infty$, $1/\infty := 0$, and $\mathcal{P}(F)$ denotes the collection of all probability measures on F . Now we return to the problem at hand.

For $v, w \in \partial G$ and $s, t \geq 0$ define

$$(2.3) \quad h((v, s); (w, t)) := \left(1 + \frac{q}{p} e^{-|s-t|} \right)^{|v \wedge w|}.$$

Peres and Steif (1998) have proved that if $P_p\{\rho \leftrightarrow \infty\} = 0$, then for all closed sets $D \subset [0, 1]$,

$$(2.4) \quad P_p \left(\bigcup_{t \in D} \left\{ \rho \overset{t}{\leftrightarrow} \infty \right\} \right) \geq \frac{1}{2} \text{Cap}_h(\partial G \times D).$$

In addition, they prove that when G is spherically symmetric,

$$(2.5) \quad P_p \left(\bigcup_{t \in D} \left\{ \rho \overset{t}{\leftrightarrow} \infty \right\} \right) \leq 960e^3 \text{Cap}_h(\partial G \times D).$$

Their method is based on the fact that when G is spherically symmetric one can identify the form of the minimizing measure in the definition of $\text{Cap}_h(\partial G \times D)$. In fact, the minimizing measure can be written as the uniform measure on ∂G —see (5.1)—times some probability measure on D . Whence follows also (1.4).

In general, G is not spherically symmetric, thus one does not know the form of the minimizing measure. We use other arguments that are based on random-field methods in order to obtain the following result. We note that the essence of our next theorem is in its upper bound because it holds without any exogenous conditions.

Theorem 2.1. *Suppose $P_p\{\rho \leftrightarrow \infty\} = 0$. Then, for all compact sets $D \subseteq \mathbf{R}_+$,*

$$(2.6) \quad \frac{1}{2} \text{Cap}_h(\partial G \times D) \leq P_p \left(\bigcup_{t \in D} \left\{ \rho \overset{t}{\leftrightarrow} \infty \right\} \right) \leq 512 \text{Cap}_h(\partial G \times D).$$

The condition that $P_p\{\rho \leftrightarrow \infty\} = 0$ is not needed in the upper bound.

Thus, we can use the preceding theorem in conjunction with (2.4) to deduce the following.

Corollary 2.2. *Percolation occurs at some time $t \in D$ if and only if $\partial G \times D$ has positive h -capacity.*

We make three additional remarks.

Remark 2.3. When $D = \{t\}$ is a singleton, $\mu \in \mathcal{P}(\partial G \times D)$ if and only if $\mu(A \times B) = \nu(A)\delta_t(B)$ for some $\nu \in \mathcal{P}(\partial G)$; also, $I_h(\nu \times \delta_t) = \iint p^{-|v \wedge w|} \nu(dv) \nu(dw)$. Therefore, Theorem 2.1 contains Lyons's theorem (1992), although our multiplicative constant is worse than that of Lyons.

Remark 2.4. It is not too hard to modify our methods and prove that when G is spherically symmetric,

$$(2.7) \quad P_p \left(\bigcup_{t \in D} \left\{ \rho \overset{t}{\leftrightarrow} \infty \right\} \right) \leq \frac{512}{I_h(\mathbf{m}_{\partial G} \times \nu)},$$

where $\mathbf{m}_{\partial G}$ is the uniform measure on ∂G . See Theorem 5.1 below. From this we readily recover (2.5) with the better constant 512 in place of $960e^3 \approx 19282.1$. This verifies the conjecture of Yuval Peres that $960e^3$ is improvable (personal communication), although it is unlikely that our 512 is optimal.

Remark 2.5. The abstract capacity condition of Corollary 2.2 can be simplified, for example when D is a “strong β -set.” [Strong β -sets are a little more regular than the s -sets of Besicovitch (1942).] We mention here only the following consequence of Theorem 6.2 and Example 6.1: When D is a strong β -set, $P_p(\cup_{t \in D} \{\rho \overset{t}{\leftrightarrow} \infty\}) > 0$ if and only if there exists $\mu \in \mathcal{P}(\partial G)$ such that

$$(2.8) \quad \iint \frac{\mu(dv) \mu(dw)}{|v \wedge w|^\beta p^{|v \wedge w|}} < \infty.$$

This implies both Lyons's theorem (1.2), and that of Häggström et al. (1.3). See remark 6.3.

Next we follow the development of Häggström et al. (1997, Theorem 1.6), and consider the Hausdorff dimension of the set of times at which percolation occurs. The matter is non-trivial only when $p = p_c$.

Consider the random subset $S(G) := S(G)(\omega)$ of $[0, 1]$ defined as

$$(2.9) \quad S(G) := \left\{ t \geq 0 : \rho \overset{t}{\leftrightarrow} \infty \right\}.$$

Note in particular that, as events,

$$(2.10) \quad \{S(G) \cap D \neq \emptyset\} = \bigcup_{t \in D} \left\{ \rho \overset{t}{\leftrightarrow} \infty \right\}.$$

Define, for all $\alpha \in (0, 1)$, the function $\phi(\alpha) : (\partial G \times \mathbf{R}_+)^2 \rightarrow \mathbf{R}_+ \cup \{\infty\}$, as follows:

$$(2.11) \quad \phi(\alpha) \left((v, t); (w, s) \right) := \frac{h((v, t); (w, s))}{|t - s|^\alpha}.$$

Then we offer the following result on the fine structure of $S(G)$.

Theorem 2.6. *Let D be a non-random compact subset of $[0, 1]$. If $P_p\{\rho \leftrightarrow \infty\} > 0$ then $S(G)$ has positive Lebesgue measure a.s. If $P_p\{\rho \leftrightarrow \infty\} = 0$, then $S(G)$ has zero Lebesgue measure a.s., and a.s. on $\{S(G) \cap D \neq \emptyset\}$,*

$$(2.12) \quad \dim_{\text{H}}(S(G) \cap D) = \sup \{0 < \alpha < 1 : \text{Cap}_{\phi(\alpha)}(\partial G \times D) > 0\},$$

where $\sup \emptyset := 0$.

When G is spherically symmetric this theorem can be simplified considerably; see Proposition 5.3 below. In the case that G is spherically symmetric and $D = [0, 1]$, Theorem 2.6 is a consequence of Theorem 1.5 of Peres and Steif (1998).

3. Proof of Theorem 2.1

We prove only the upper bound; the lower bound (2.4) was proved much earlier in Peres and Steif (1998).

Without loss of generality we may assume that G has no leaves. Otherwise we can replace G everywhere by G' , where the latter is the maximal subtree of G that has no leaves. This “leaflessness” assumption is in force throughout the proof. Also, without loss of generality, we may assume that $P_p(\cup_{t \in D} \{\rho \overset{t}{\leftrightarrow} \infty\}) > 0$, for there is nothing left to prove otherwise.

As in Peres and Steif (1998), we first derive the theorem in the case that G is a finite tree. Let n denote its *height*. That is, $n := \max |v|$ where the maximum is taken over all vertices v . We can—and will—assume without loss of generality that G has *no leaves*. That is, whenever a vertex v satisfies $|v| < n$, then v necessarily has a descendant w with $|w| = n$.

Define

$$(3.1) \quad \Xi := \partial G \times D.$$

Let $\mu \in \mathcal{P}(\Xi)$, and define

$$(3.2) \quad Z(\mu) := \frac{1}{p^n} \int_{(v,t) \in \Xi} \mathbf{1}_{\{\rho \overset{t}{\leftrightarrow} v\}} \mu(dv dt).$$

During the course of their derivation of (2.4), Peres and Steif (1998) have demonstrated that

$$(3.3) \quad E_p[Z(\mu)] = 1 \quad \text{and} \quad E_p[Z^2(\mu)] \leq 2I_h(\mu).$$

Equation (2.4) follows immediately from this and the Paley–Zygmund inequality (1932): *For all non-negative $f \in L^2(P_p)$,*

$$(3.4) \quad P_p\{f > 0\} \geq \frac{(E_p f)^2}{E_p[f^2]}.$$

Now we prove the second half of Theorem 2.1. Because G is assumed to be finite, we can embed it in the plane. We continue to write G for the said embedding of G in $\mathbf{R}^2$; this

should not cause too much confusion since we will not refer to the abstract tree G until the end of the proof.

Since G is assumed to be leafless, we can identify ∂G with the collection of vertices $\{v : |v| = n\}$ of maximal length. [Recall that n denotes the height of G .]

There are four natural partial orders on $\partial G \times \mathbf{R}_+$ which we describe next. Let (v, t) and (w, s) be two elements of $\partial G \times \mathbf{R}_+$:

- (1) We say that $(v, t) <_{(-,-)} (w, s)$ if $t \leq s$ and v lies to the left of w in the planar embedding of G .
- (2) If $t \geq s$ and v lies to the left of w [in the planar embedding of G], then we say that $(v, t) <_{(-,+)} (w, s)$.
- (3) If $t \leq s$ and v lies to the right of w , then we say that $(v, t) <_{(+,-)} (w, s)$.
- (4) If $t \geq s$ and v lies to the right of w , then we say that $(v, t) <_{(+,+)} (w, s)$.

One really only needs two of these, but having four simplifies the ensuing presentations slightly.

The key feature of these partial orders is that, together, they totally order $\partial G \times \mathbf{R}_+$. By this we mean that

$$(3.5) \quad (v, t), (w, s) \in \partial G \times \mathbf{R}_+ \Rightarrow \exists \sigma, \tau \in \{-, +\} : (v, t) <_{(\sigma, \tau)} (w, s).$$

Define, for all $(v, t) \in \partial G \times \mathbf{R}_+$ and $\sigma, \tau \in \{-, +\}$,

$$(3.6) \quad \mathcal{F}_{(\sigma, \tau)}(v, t) := \text{sigma-algebra generated by } \left\{ \mathbf{1}_{\{\rho \xrightarrow{s} w\}} ; (w, s) <_{(\sigma, \tau)} (v, t) \right\},$$

where the conditions that $s \geq 0$ and $w \in \partial G$ are implied tacitly. It is manifestly true that for every fixed $\sigma, \tau \in \{-, +\}$, the collection of sigma-algebras $\mathcal{F}_{(\sigma, \tau)} := \{\mathcal{F}_{(\sigma, \tau)}(v, t)\}_{t \geq 0, v \in \partial G}$ is a filtration in the sense that

$$(3.7) \quad (w, s) <_{(\sigma, \tau)} (v, t) \implies \mathcal{F}_{(\sigma, \tau)}(w, s) \subseteq \mathcal{F}_{(\sigma, \tau)}(v, t).$$

Also, it follows fairly readily that each $\mathcal{F}_{(\sigma, \tau)}$ is commuting in the sense of Khoshnevisan (2002, pp. 35 and 233). When $(\sigma, \tau) = (\pm, +)$ this assertion is easy enough to check directly; when $(\sigma, \tau) = (\pm, -)$, it follows from the time-reversability of our dynamics together with the case $\tau = +$. Without causing too much confusion we can replace $\mathcal{F}_{(\sigma, \tau)}(v, t)$ by its completion $[\mathcal{P}_p]$. Also, we may—and will—replace the latter further by making it right-continuous in the partial order $<_{(\sigma, \tau)}$. As a consequence of this and Cairoli's maximal inequality (Khoshnevisan, 2002, Theorem 2.3.2, p. 235), for all twice-integrable random variables Y , and all $\sigma, \tau \in \{-, +\}$,

$$(3.8) \quad \mathbf{E}_p \left(\sup_{(v, t) \in \Xi} \left| \mathbf{E}_p \left[Y \mid \mathcal{F}_{(\sigma, \tau)}(v, t) \right] \right|^2 \right) \leq 16 \mathbf{E}_p [Y^2].$$

In order to obtain this from Theorem 2.3.2 of Khoshnevisan (*loc. cit.*) set $N = p = 2$, identify the parameter t in that book by our (v, t) , and define the process M there—at time-point (v, t) —to be our $E_p[Y \mid \mathcal{F}_{(\sigma, \tau)}(v, t)]$.

Next we bound from below $E_p[Z(\mu) \mid \mathcal{F}_{(\sigma, \tau)}(w, s)]$, where $s \geq 0$ and $w \in \partial G$ are fixed:

$$(3.9) \quad \begin{aligned} E_p \left[Z(\mu) \mid \mathcal{F}_{(\sigma, \tau)}(w, s) \right] \\ \geq \int_{\substack{(v, t) \in \Xi: \\ (w, s) <_{(\sigma, \tau)} (v, t)}} p^{-n} P_p \left(\rho \overset{t}{\leftrightarrow} v \mid \mathcal{F}_{(\sigma, \tau)}(w, s) \right) \mu(dv dt) \cdot \mathbf{1}_{\{\rho \overset{s}{\leftrightarrow} w\}}. \end{aligned}$$

By the Markov property, P_p -a.s. on $\{\rho \overset{s}{\leftrightarrow} w\}$,

$$(3.10) \quad P_p \left(\rho \overset{t}{\leftrightarrow} v \mid \mathcal{F}_{(\sigma, \tau)}(w, s) \right) = p^{n-|v \wedge w|} (p + qe^{-|t-s|})^{|v \wedge w|}.$$

See equation (6) of Häggström, Peres, and Steif (1997). It follows then that P_p a.s.,

$$(3.11) \quad E_p \left[Z(\mu) \mid \mathcal{F}_{(\sigma, \tau)}(w, s) \right] \geq \int_{\substack{(v, t) \in \Xi: \\ (w, s) <_{(\sigma, \tau)} (v, t)}} h((w, s); (v, t)) \mu(dv dt) \cdot \mathbf{1}_{\{\rho \overset{s}{\leftrightarrow} w\}}.$$

Thanks to the preceding, and (3.5), for all $s \geq 0$ and $w \in \partial G$ the following holds P_p a.s.:

$$(3.12) \quad \sum_{\sigma, \tau \in \{-, +\}} E_p \left[Z(\mu) \mid \mathcal{F}_{(\sigma, \tau)}(w, s) \right] \geq \int_{\Xi} h((w, s); (v, t)) \mu(dv dt) \cdot \mathbf{1}_{\{\rho \overset{s}{\leftrightarrow} w\}}.$$

It is possible to check that the right-hand side is a right-continuous function of s . Because ∂G is finite, we can therefore combine all null sets and deduce that P_p almost surely, (3.12) holds simultaneously for all $s \geq 0$ and $w \in \partial G$.

Recall that we assumed, at the onset of the proof, that $P_p(\cup_{t \in D} \{\rho \overset{t}{\leftrightarrow} \infty\}) > 0$. From this it follows easily that we can find random variables $\mathbf{s}$ and $\mathbf{w}$ such that:

- (1) $\mathbf{s}(\omega) \in D \cup \{\infty\}$ for all ω , where ∞ is a point not in $\mathbf{R}_+$;
- (2) $\mathbf{w}(\omega) \in \partial G \cup \{\delta\}$, where δ is an abstract “cemetery” point not in ∂G ;
- (3) $(\mathbf{w}(\omega), \mathbf{s}(\omega)) \neq (\delta, \infty)$ if and only if there exists $t \in D$ and $v \in \partial G$ such that $\rho \overset{t}{\leftrightarrow} v$;
- (4) $(\mathbf{w}(\omega), \mathbf{s}(\omega)) \neq (\delta, \infty)$ if and only if $\rho \overset{\mathbf{s}(\omega)}{\leftrightarrow} \mathbf{w}(\omega)$.

Here, and henceforth, $\{\rho \overset{t}{\leftrightarrow} v\}$ denotes the event that at time t every edge that conjoins ρ and v is open.

There are many ways of constructing $\mathbf{s}$ and $\mathbf{w}$. We explain one for the sake of completeness. Define

$$(3.13) \quad \mathbf{s} := \inf \left\{ s \in D : \exists v \in \partial G \text{ such that } \rho \overset{s}{\leftrightarrow} v \right\},$$

where $\inf \emptyset := \infty$. If $\mathbf{s}(\omega) = \infty$ then we set $\mathbf{w}(\omega) := \delta$. Else, we define $\mathbf{w}(\omega)$ to be the left-most (say) ray in ∂G whose edges are all open at time $\mathbf{s}(\omega)$. It might help to recall that

G is assumed to be a finite tree, and we are identifying it with its planar embedding so that “left-most” can be interpreted unambiguously.

Define a measure μ on Ξ by letting, for all Borel sets $A \times B \subseteq \Xi$,

$$(3.14) \quad \mu(A \times B) := P_p((\mathbf{w}, \mathbf{s}) \in A \times B \mid (\mathbf{w}, \mathbf{s}) \neq (\delta, \infty)).$$

Note that $\mu \in \mathcal{P}(\Xi)$ because $P_p(\cup_{t \in D} \{\rho \xleftrightarrow{t} \infty\}) = P_p\{(\mathbf{w}, \mathbf{s}) \neq (\delta, \infty)\} > 0$.

We apply (3.12) with this particular $\mu \in \mathcal{P}(\Xi)$, and replace (w, s) by $(\mathbf{w}, \mathbf{s})$, to find that a.s.,

$$(3.15) \quad \sum_{\sigma, \tau \in \{-, +\}} \sup_{(w, s) \in \Xi} E_p \left[Z(\mu) \mid \mathcal{F}_{(\sigma, \tau)}(w, s) \right] \geq \int_{\Xi} h((\mathbf{w}, \mathbf{s}); (v, t)) \mu(dv dt) \cdot \mathbf{1}_{\cup_{t \in D} \{\rho \xleftrightarrow{t} \infty\}}.$$

According to (3.8), and thanks to the inequality,

$$(3.16) \quad (a + b + c + d)^2 \leq 4(a^2 + b^2 + c^2 + d^2),$$

we can deduce that

$$(3.17) \quad \begin{aligned} E_p \left[\left(\sum_{\sigma, \tau \in \{-, +\}} \sup_{(w, s) \in \Xi} E_p \left[Z(\mu) \mid \mathcal{F}_{(\sigma, \tau)}(w, s) \right] \right)^2 \right] \\ \leq 4 \sum_{\sigma, \tau \in \{-, +\}} E_p \left[\sup_{(w, s) \in \Xi} \left| E_p \left[Z(\mu) \mid \mathcal{F}_{(\sigma, \tau)}(w, s) \right] \right|^2 \right] \\ \leq 256 E_p [Z^2(\mu)] \\ \leq 512 I_h(\mu). \end{aligned}$$

See (3.3) for the final inequality. On the other hand, thanks to the definition of μ , and by the Cauchy–Schwarz inequality,

$$(3.18) \quad \begin{aligned} E_p \left[\left(\int_{\Xi} h((\mathbf{w}, \mathbf{s}); (v, t)) \mu(dv dt) \right)^2 \middle| \bigcup_{t \in D} \{\rho \xleftrightarrow{t} \infty\} \right] \\ \geq \left(E_p \left[\int_{\Xi} h((\mathbf{w}, \mathbf{s}); (v, t)) \mu(dv dt) \middle| \bigcup_{t \in D} \{\rho \xleftrightarrow{t} \infty\} \right] \right)^2 \\ = [I_h(\mu)]^2. \end{aligned}$$

Because G is finite, it follows that $0 < I_h(\mu) < \infty$. Therefore, (3.15), (3.17), and (3.18) together imply the theorem in the case that G is finite. The general case follows from the preceding by monotonicity.

4. Proof of Theorem 2.6

The assertions about the Lebesgue measure of $S(G)$ are mere consequences of the fact that $P_p\{\rho \xleftrightarrow{t} \infty\}$ does not depend on t , used in conjunction with the Fubini–Tonelli theorem. Indeed,

$$(4.1) \quad E_p[\text{meas } S(G)] = \int_0^\infty P_p\{\rho \xleftrightarrow{t} \infty\} dt.$$

Next we proceed with the remainder of the proof.

Choose and fix $\alpha \in (0, 1)$. Let $Y_\alpha := \{Y_\alpha(t)\}_{t \geq 0}$ to be a symmetric stable Lévy process on $\mathbf{R}$ with index $(1 - \alpha)$. We can normalize Y_α so that $E[\exp\{i\xi Y(1)\}] = \exp(-|\xi|^{1-\alpha})$ for all $\xi \in \mathbf{R}$. We assume also that Y_α is independent of our dynamical percolation process. For more information on the process Y_α see the monographs of Bertoin (1996), Khoshnevisan (2002), and Sato (1999).

Recall the function $\phi(\alpha)$ from (2.11). Our immediate goal is to demonstrate the following.

Theorem 4.1. *Suppose $P_p\{\rho \leftrightarrow \infty\} = 0$. Choose and fix $M > 1$. Then there exists a finite constant $A = A(M) > 1$ such that for all compact sets $D \subseteq [-M, M]$,*

$$(4.2) \quad \frac{1}{A} \text{Cap}_{\phi(\alpha)}(\partial G \times D) \leq P_p\left\{S(G) \cap D \cap \overline{Y_\alpha([1, 2])} \neq \emptyset\right\} \leq A \text{Cap}_{\phi(\alpha)}(\partial G \times D),$$

where $\overline{U}$ denotes the closure of U . The condition that $P_p\{\rho \leftrightarrow \infty\} = 0$ is not needed in the upper bound.

Remark 4.2. The time interval $[1, 2]$ could just as easily be replaced with an arbitrary, but fixed, closed interval $I \subset (0, \infty)$ that is bounded away from the origin.

Remark 4.3. We do not require the following strengthened form of Theorem 4.1, but it is simple enough to derive that we describe it here: *Theorem 4.1 continues to hold if $\overline{Y_\alpha([1, 2])}$ were replaced by $Y_\alpha([1, 2])$.* Indeed, a well-known theorem of Kanda (1978) implies that all semipolar sets for Y_α are polar; i.e., Y_α satisfies Hunt’s (H) hypothesis (Hunt, 1957; 1958). This readily implies the assertion of this Remark.

From here on, until after the completion of the proof of Theorem 4.1, we will assume without loss of much generality that G is a finite tree of height n . The extension to the case where G is infinite is made by standard arguments.

Let D be as in Theorem 4.1. For all $\mu \in \mathcal{P}(\partial G \times D)$ and $\varepsilon \in (0, 1)$ define

$$(4.3) \quad Z_\varepsilon(\mu) := \frac{1}{(2\varepsilon)p^n} \int_{\Xi} \int_1^2 \mathbf{1}_{\{\rho \xleftrightarrow{t} v\} \cap \{|Y_\alpha(r) - t| \leq \varepsilon\}} dr \mu(dv dt),$$

where $\Xi := \partial G \times D$, as before.

Next we collect some of the elementary properties of $Z_\varepsilon(\mu)$.

Lemma 4.4. *There exists $c > 1$ such that for all $\mu \in \mathcal{P}(\Xi)$ and $\varepsilon \in (0, 1)$:*

- (1) $E_p[Z_\varepsilon(\mu)] \geq 1/c$; and
- (2) $E_p[Z_\varepsilon^2(\mu)] \leq cI_{\phi(\alpha)}(\mu)$.

Proof. Define $p_r(a)$ to be the density of $Y_\alpha(r)$ at a . Bochner's *subordination* implies that: (i) $p_r(a) > 0$ for all $r > 0$ and $a \in \mathbf{R}$; and (ii) there exists $c_1 > 0$ such that $p_r(a) \geq c_1$ uniformly for all $r \in [1, 2]$ and $a \in [-M-1, M+1]$. We recall here that subordination is the assertion that $p_r(a)$ is a stable mixture of Gaussian densities (Bochner, 1949). For related results, see also Nelson (1958). Khoshnevisan (2002, pp. 377–384) contains a modern-day account that includes explicit proofs of assertions (i) and (ii) above. The first assertion of the lemma follows.

Next we can note that by the Markov property of Y_α ,

$$\begin{aligned}
 & \int_1^2 \int_1^2 \mathbf{P} \{ |Y_\alpha(r) - t| \leq \varepsilon, |Y_\alpha(R) - s| \leq \varepsilon \} \, dr \, dR \\
 (4.4) \quad & \leq \int_1^2 \int_r^2 \mathbf{P} \{ |Y_\alpha(r) - t| \leq \varepsilon \} \mathbf{P} \{ |Y_\alpha(R-r) - (s-t)| \leq 2\varepsilon \} \, dR \, dr \\
 & \quad + \int_1^2 \int_R^2 \mathbf{P} \{ |Y_\alpha(R) - s| \leq \varepsilon \} \mathbf{P} \{ |Y_\alpha(r-R) - (s-t)| \leq 2\varepsilon \} \, dr \, dR.
 \end{aligned}$$

We can appeal to subordination once again to find that there exists $c_2 > 0$ such that $p_r(a) \leq c_2$ uniformly for all $r \in [1, 2]$ and $a \in [-M-1, M+1]$. This, and symmetry, together imply that

$$\begin{aligned}
 & \int_1^2 \int_1^2 \mathbf{P} \{ |Y_\alpha(r) - t| \leq \varepsilon, |Y_\alpha(R) - s| \leq \varepsilon \} \, dr \, dR \\
 (4.5) \quad & \leq 4c_2\varepsilon \int_0^2 \mathbf{P} \{ |Y_\alpha(r) - (t-s)| \leq 2\varepsilon \} \, dr \\
 & \leq 4e^2c_2\varepsilon \int_0^\infty \mathbf{P} \{ |Y_\alpha(r) - (t-s)| \leq 2\varepsilon \} e^{-r} \, dr.
 \end{aligned}$$

Let $u(a) := \int_0^\infty p_t(a)e^{-t} \, dt$ denote the one-potential density of Y_α , and note that

$$(4.6) \quad \int_1^2 \int_1^2 \mathbf{P} \{ |Y_\alpha(r) - t| \leq \varepsilon, |Y_\alpha(R) - s| \leq \varepsilon \} \, dr \, dR \leq 4e^2c_2\varepsilon \int_{|t-s|-2\varepsilon}^{|t-s|+2\varepsilon} u(z) \, dz.$$

It is well known that there exist $c_3 > 1$ such that

$$(4.7) \quad \frac{1}{c_3|z|^\alpha} \leq u(z) \leq \frac{c_3}{|z|^\alpha}, \quad \text{for all } z \in [-2M-2, 2M+2];$$

see Khoshnevisan (2002, Lemma 3.4.1, p. 383), for instance. It follows that

$$(4.8) \quad \int_1^2 \int_1^2 \mathbb{P} \{ |Y_\alpha(r) - t| \leq \varepsilon, |Y_\alpha(R) - s| \leq \varepsilon \} \, dr \, dR \leq 4e^2 c_2 c_3 \varepsilon \int_{|t-s|-2\varepsilon}^{|t-s|+2\varepsilon} \frac{dz}{|z|^\alpha}.$$

If $|t - s| \geq 4\varepsilon$, then we use the bound $|z|^{-\alpha} \leq (|t - s|/2)^{-\alpha}$. Else, we use the estimate $\int_{|t-s|-2\varepsilon}^{|t-s|+2\varepsilon} (\dots) \leq \int_{-6\varepsilon}^{6\varepsilon} (\dots)$. This leads us to the existence of a constant $c_4 = c_4(M) > 0$ such that for all $s, t \in D$ and $\varepsilon \in (0, 1)$,

$$(4.9) \quad \begin{aligned} \int_{|t-s|-2\varepsilon}^{|t-s|+2\varepsilon} \frac{dz}{|z|^\alpha} &\leq c_4 \varepsilon \min \left(\frac{1}{|t-s|} \wedge \frac{1}{\varepsilon} \right)^\alpha \\ &\leq c_4 \frac{\varepsilon}{|t-s|^\alpha}. \end{aligned}$$

Part two of the lemma follows from this and (3.10). $\square$

Now we prove the first inequality in Theorem 4.1.

Proof of Theorem 4.1: First Half. We can choose some $\mu \in \mathcal{P}(\Xi)$, and deduce from Lemma 4.4 and the Paley–Zygmund inequality (3.4) that $\mathbb{P}_p\{Z_\varepsilon(\mu) > 0\} \geq 1/(c^3 I_{\phi(\alpha)}(\mu))$. Let Y^ε denote the closed ε -enlargement of $Y_\alpha([1, 2])$.

Recall that $S(G)$ is closed because $\mathbb{P}_p\{\rho \leftrightarrow \infty\} = 0$ (Häggström et al., 1997, Lemma 3.2). Also note that

$$(4.10) \quad \{Z_\varepsilon(\mu) > 0\} \subseteq \{S(G) \cap D \cap Y^\varepsilon \neq \emptyset\}.$$

Let $\varepsilon \rightarrow 0^+$ to obtain the first inequality of Theorem 4.1 after we optimize over $\mu \in \mathcal{P}(\Xi)$. $\square$

The second half of Theorem 4.1 is more difficult to prove. We begin by altering the definition of $Z_\varepsilon(\mu)$ slightly as follows: For all $\varepsilon \in (0, 1)$ and $\mu \in \mathcal{P}(\Xi)$ define

$$(4.11) \quad W_\varepsilon(\mu) := \frac{1}{(2\varepsilon)p^n} \int_\Xi \int_1^\infty \mathbf{1}_{\{\rho \overset{t}{\leftrightarrow} v\} \cap \{|Y_\alpha(r) - t| \leq \varepsilon\}} e^{-r} \, dr \, \mu(dv \, dt).$$

[It might help to recall that n denotes the height of the finite tree G .] We can sharpen the second assertion of Lemma 4.4, and replace $Z_\varepsilon(\mu)$ by $W_\varepsilon(\mu)$, as follows: There exists a constant $c = c(M) > 0$ such that

$$(4.12) \quad \mathbb{E}_p [W_\varepsilon^2(\mu)] \leq c I_{\phi_\varepsilon(\alpha)}(\mu),$$

where

$$(4.13) \quad \phi_\varepsilon(\alpha)((v, t); (w, s)) := h((v, t); (w, s)) \cdot \left(\frac{1}{|t-s|} \wedge \frac{1}{\varepsilon} \right)^\alpha.$$

The aforementioned sharpening rests on (4.9) and not much more. So we omit the details.

Define $\mathcal{Y}(t)$ to be the sigma-algebra generated by $\{Y_\alpha(r)\}_{0 \leq r \leq t}$. We can add to $\mathcal{Y}(t)$ all $\mathbb{P}$ -null sets, and even make it right-continuous [with respect to the usual total order on $\mathbf{R}$].

Let us denote the resulting sigma-algebra by $\mathcal{Y}(t)$ still, and the corresponding filtration by $\mathcal{Y}$.

Choose and fix $\sigma, \tau \in \{-, +\}$, and for all $v \in \partial G$ and $r, t \geq 0$ define

$$(4.14) \quad \mathcal{G}_{(\sigma, \tau)}(v, t, r) := \mathcal{F}_{(\sigma, \tau)}(v, t) \times \mathcal{Y}(r).$$

We say that $(v, t, r) \ll_{(\sigma, \tau)} (w, s, u)$ when $(v, t) <_{(\sigma, \tau)} (w, s)$ and $r \leq u$. Thus, each $\ll_{(\sigma, \tau)}$ defines a partial order on $\partial G \times \mathbf{R}_+ \times \mathbf{R}_+$.

Choose and fix $\sigma, \tau \in \{-, +\}$. Because $\mathcal{F}_{(\sigma, \tau)}$ is a two-parameter, commuting filtration in the partial order $<_{(\sigma, \tau)}$, and since $\mathcal{Y}$ is the [one-parameter] independent filtration generated by a reversible Feller process, it follows readily that $\mathcal{G}_{(\sigma, \tau)}$ is a three-parameter, commuting filtration in the partial order $\ll_{(\sigma, \tau)}$. In particular, the following analogue of (3.8) is valid: For all $V \in L^2(\mathbf{P}_p)$,

$$(4.15) \quad \mathbf{E}_p \left(\sup_{(v, t, r) \in \Xi \times \mathbf{R}_+} \left| \mathbf{E}_p [V \mid \mathcal{F}_{(\sigma, \tau)}(v, t, r)] \right|^2 \right) \leq 64 \mathbf{E}_p [V^2].$$

(Khoshnevisan, 2002, Theorem 2.3.2, p. 235).

Next, we note that for all $(w, s, u) \in \partial G \times D \times [1, 2]$, and all $\sigma, \tau \in \{-, +\}$, the following is valid $\mathbf{P}_p$ -almost surely:

$$(4.16) \quad \begin{aligned} & \mathbf{E}_p \left[W_\varepsilon(\mu) \mid \mathcal{G}_{(\sigma, \tau)}(w, s, u) \right] \\ & \geq \frac{1}{(2\varepsilon)p^n} \int_{(v, t) <_{(\sigma, \tau)} (w, s)} \int_u^\infty \mathcal{H} e^{-r} dr \mu(dv dt) \cdot \mathbf{1}_{\{\rho \overset{s}{\leftrightarrow} w\} \cap \{|Y_\alpha(u) - s| \leq \varepsilon/2\}}. \end{aligned}$$

Here,

$$(4.17) \quad \begin{aligned} \mathcal{H} &:= \mathbf{P}_p \left(\rho \overset{t}{\leftrightarrow} v, |Y_\alpha(r) - t| \leq \varepsilon \mid \mathcal{G}_{(\sigma, \tau)}(w, s, u) \right) \\ &= \mathbf{P}_p \left(\rho \overset{t}{\leftrightarrow} v \mid \mathcal{F}_{(\sigma, \tau)}(w, s) \right) \times \mathbf{P}(|Y_\alpha(r) - t| \leq \varepsilon \mid \mathcal{Y}(u)). \end{aligned}$$

By the Markov property, $\mathbf{P}_p$ -almost surely on $\{\rho \overset{s}{\leftrightarrow} w\}$,

$$(4.18) \quad \mathbf{P}_p \left(\rho \overset{t}{\leftrightarrow} v \mid \mathcal{F}_{(\sigma, \tau)}(w, s) \right) = p^n h((v, t); (w, s)).$$

See (3.10). On the other hand, the Markov property of Y_α dictates that almost surely on $\{|Y_\alpha(u) - s| \leq \varepsilon/2\}$,

$$(4.19) \quad \begin{aligned} \mathbf{P}(|Y_\alpha(r) - t| \leq \varepsilon \mid \mathcal{Y}(u)) &\geq \mathbf{P} \left\{ |Y_\alpha(r - u) - (t - s)| \leq \frac{\varepsilon}{2} \right\} \\ &:= \mathcal{A}. \end{aligned}$$

Note that because $u \in [1, 2]$,

$$\begin{aligned}
 \int_u^\infty \mathcal{A} e^{-r} dr &\geq \frac{1}{e^2} \int_0^\infty \mathbb{P} \left\{ |Y_\alpha(r) - (t-s)| \leq \frac{\varepsilon}{2} \right\} e^{-r} dr \\
 (4.20) \qquad &= \frac{1}{e^2} \int_{|t-s|-(\varepsilon/2)}^{|t-s|+(\varepsilon/2)} u(z) dz \\
 &\geq c_5(2\varepsilon) \min \left(\frac{1}{|t-s|} \wedge \frac{1}{\varepsilon} \right)^\alpha,
 \end{aligned}$$

where c_5 does not depend on $(\varepsilon, \mu; t, s)$. The ultimate inequality follows from a similar argument that was used earlier to derive (4.9). So we omit the details.

Thus, we can plug the preceding bounds into (4.16) and deduce that $\mathbb{P}_p$ -a.s.,

$$\begin{aligned}
 (4.21) \qquad &\mathbb{E}_p \left[W_\varepsilon(\mu) \mid \mathcal{G}_{(\sigma, \tau)}(w, s, u) \right] \\
 &\geq c_5 \int_{(v, t) <_{(\sigma, \tau)} (w, s)} \phi_\varepsilon(\alpha) \left((v, t); (w, s) \right) \mu(dv dt) \times \mathbf{1}_{\{\rho \xleftrightarrow{s} w\} \cap \{|Y_\alpha(u) - s| \leq \varepsilon/2\}}.
 \end{aligned}$$

Moreover, it is possible to check that there exists one null set outside which the preceding holds for all $(w, s, u) \in \Xi \times [1, 2]$. We are in a position to complete our proof of Theorem 4.1.

Proof of Theorem 4.1: Second Half. Without loss of very much generality, we may assume that

$$(4.22) \qquad \mathbb{P}_p\{S(G) \cap D \cap \overline{Y_\alpha([1, 2])}\} > 0,$$

for otherwise there is nothing to prove.

Let us introduce two abstract cemetery states: $\delta \notin \partial G$ and $\infty \notin \mathbf{R}_+$. Then, there exists a map $(\mathbf{w}_\varepsilon, \mathbf{s}_\varepsilon, \mathbf{u}_\varepsilon) : \Omega \mapsto (\partial G \cup \{\delta\}) \times (D \cup \{\infty\}) \times ([1, 2] \cup \{\infty\})$ with the properties that:

- (1) $(\mathbf{w}_\varepsilon, \mathbf{s}_\varepsilon, \mathbf{u}_\varepsilon)(\omega) \neq (\delta, \infty, \infty)$ if and only if there exists $(w, s, u)(\omega) \in \Xi \times [1, 2]$ such that $\rho \xleftrightarrow{s(\omega)} w(\omega)$ and $|Y_\alpha(u) - s|(\omega) \leq \varepsilon/2$; and
- (2) If $(\mathbf{w}_\varepsilon, \mathbf{s}_\varepsilon, \mathbf{u}_\varepsilon)(\omega) \neq (\delta, \infty, \infty)$, then (1) holds with $(\mathbf{w}_\varepsilon, \mathbf{s}_\varepsilon, \mathbf{u}_\varepsilon)(\omega)$ in place of $(w, s, u)(\omega)$.

See the proof of Theorem 2.1 for a closely-related construction.

Consider the event,

$$(4.23) \qquad H(\varepsilon) := \{\omega : (\mathbf{w}_\varepsilon, \mathbf{s}_\varepsilon, \mathbf{u}_\varepsilon)(\omega) \neq (\delta, \infty, \infty)\}.$$

Thus, we can deduce that $\mu_\varepsilon \in \mathcal{P}(\Xi)$, where

$$(4.24) \qquad \mu_\varepsilon(A \times B) := \mathbb{P}_p((\mathbf{w}_\varepsilon, \mathbf{s}_\varepsilon) \in A \times B \mid H(\varepsilon)),$$

valid for all measurable $A \times B \subseteq \Xi$.

Because of (3.5), we may apply (4.21) with μ_ε in place of μ and $(\mathbf{w}_\varepsilon, \mathbf{s}_\varepsilon, \mathbf{u}_\varepsilon)$ in place of (w, s, u) to find that P_p -a.s.,

$$(4.25) \quad \sum_{\sigma, \tau \in \{-, +\}} \sup_{(w, s, u) \in \Xi \times [1, 2]} E_p \left[W_\varepsilon(\mu_\varepsilon) \mid \mathcal{G}_{(\sigma, \tau)}(w, s, u) \right] \\ \geq c_5 \int_{\Xi} \phi_\varepsilon(\alpha) \left((v, t); (\mathbf{w}_\varepsilon, \mathbf{s}_\varepsilon) \right) \mu_\varepsilon(dv dt) \cdot \mathbf{1}_{H(\varepsilon)}.$$

We can square both ends of this inequality and take expectations $[P_p]$. Owing to (3.16), the expectation of the square of the left-most term is at most

$$(4.26) \quad 4 \sum_{\sigma, \tau \in \{-, +\}} E_p \left(\sup_{(w, s, u) \in \Xi \times [1, 2]} \left| E_p \left[W_\varepsilon(\mu_\varepsilon) \mid \mathcal{G}_{(\sigma, \tau)}(w, s, u) \right] \right|^2 \right) \leq 1024 E_p [W_\varepsilon^2(\mu_\varepsilon)] \\ \leq 1024 c I_{\phi_\varepsilon(\alpha)}(\mu_\varepsilon).$$

See (4.15) and (4.12). We emphasize that the constant c is finite and positive, and does not depend on $(\varepsilon, \mu_\varepsilon)$.

On the other hand, by the Cauchy–Schwarz inequality, the expectation of the square of the right-most term in (4.25) is equal to

$$(4.27) \quad c_5^2 E_p \left[\left(\int_{\Xi} \phi_\varepsilon(\alpha) \left((v, t); (\mathbf{w}_\varepsilon, \mathbf{s}_\varepsilon) \right) \mu_\varepsilon(dv dt) \right)^2 \mid H(\varepsilon) \right] P_p(H(\varepsilon)) \\ \geq c_5^2 \left(E_p \left[\int_{\Xi} \phi_\varepsilon(\alpha) \left((v, t); (\mathbf{w}_\varepsilon, \mathbf{s}_\varepsilon) \right) \mu_\varepsilon(dv dt) \mid H(\varepsilon) \right] \right)^2 P_p(H(\varepsilon)) \\ = c_5^2 (I_{\phi_\varepsilon(\alpha)}(\mu_\varepsilon))^2 P_p(H(\varepsilon)).$$

We can combine (4.26) with (4.27) to find that for all $N > 0$,

$$(4.28) \quad P_p(H(\varepsilon)) \leq \frac{1024c}{c_5^2} [I_{\phi_\varepsilon(\alpha)}(\mu_\varepsilon)]^{-1} \\ \leq \frac{1024c}{c_5^2} [I_{N \wedge \phi_\varepsilon(\alpha)}(\mu_\varepsilon)]^{-1}.$$

Perhaps it is needless to say that $N \wedge \phi_\varepsilon(\alpha)$ is the function whose evaluation at $((v, t), (w, s)) \in \Xi \times \Xi$ is the minimum of the constant N and $\phi_\varepsilon(\alpha)((v, t)(w, s))$. Evidently, $N \wedge \phi_\varepsilon(\alpha)$ is bounded and lower semicontinuous on $\Xi \times \Xi$. By compactness we can find $\mu_0 \in \mathcal{P}(\Xi)$ such that μ_ε converges weakly to μ_0 . As $\varepsilon \downarrow 0$, the sets $H(\varepsilon)$ decrease set-theoretically, and their intersection includes $\{S(G) \cap D \cap \overline{Y_\alpha([1, 2])} \neq \emptyset\}$. As a result we have

$$(4.29) \quad P_p \left\{ S(G) \cap D \cap \overline{Y_\alpha([1, 2])} \neq \emptyset \right\} \leq \frac{1024c}{c_5^2} [I_{N \wedge \phi(\alpha)}(\mu_0)]^{-1}.$$

Let $N \uparrow \infty$ and appeal to the monotone convergence theorem to find that

$$(4.30) \quad \begin{aligned} \mathbb{P}_p \left\{ S(G) \cap D \cap \overline{Y_\alpha([1, 2])} \neq \emptyset \right\} &\leq \frac{1024c}{c_5^2} [I_{\phi(\alpha)}(\mu_0)]^{-1} \\ &\leq \frac{1024c}{c_5^2} \text{Cap}_{\phi(\alpha)}(\partial G \times D). \end{aligned}$$

This concludes the proof of Theorem 4.1. $\square$

Proof of Theorem 2.6. Define $R_\alpha := \overline{Y_\alpha([1, 2])}$, and recall the theorem of McKean (1955): *For all Borel sets $B \subseteq \mathbf{R}$,*

$$(4.31) \quad \mathbb{P} \{ R_\alpha \cap B \neq \emptyset \} > 0 \iff \text{Cap}_\alpha(B) > 0.$$

Here, $\text{Cap}_\alpha(B)$ denotes the α -dimensional (Bessel-) Riesz capacity of B (Khoshnevisan, 2002, Appendix C). That is,

$$(4.32) \quad \text{Cap}_\alpha(B) := \left[\inf_{\mu \in \mathcal{P}(B)} I_\alpha(\mu) \right]^{-1},$$

where

$$(4.33) \quad I_\alpha(\mu) := \iint \frac{\mu(dx) \mu(dy)}{|x - y|^\alpha}.$$

Now let $R_\alpha^1, R_\alpha^2, \dots$ be i.i.d. copies of R_α , all independent of our dynamical percolation process as well. Then, by the Borel–Cantelli lemma,

$$(4.34) \quad \mathbb{P} \left\{ \bigcup_{j=1}^{\infty} R_\alpha^j \cap B \neq \emptyset \right\} = \begin{cases} 1, & \text{if } \text{Cap}_\alpha(B) > 0, \\ 0, & \text{if } \text{Cap}_\alpha(B) = 0. \end{cases}$$

Set $B := S(G) \cap D$ and condition, once on G and once on $\cup_{j=1}^{\infty} R_\alpha^j$. Then, the preceding and Theorem 4.1 together imply that

$$(4.35) \quad \mathbb{P}_p \left\{ \text{Cap}_\alpha(S(G) \cap D) > 0 \right\} = \begin{cases} 1, & \text{if } \text{Cap}_{\phi(\alpha)}(\partial G \times D) > 0, \\ 0, & \text{if } \text{Cap}_{\phi(\alpha)}(\partial G \times D) = 0. \end{cases}$$

The remainder of the theorem follows from Frostman’s theorem (1935): *For all Borel sets $F \subset \mathbf{R}$,*

$$(4.36) \quad \dim_{\text{H}} F = \sup \{ 0 < \alpha < 1 : \text{Cap}_\alpha(F) > 0 \},$$

which we apply with $F := S(G) \cap D$. For a pedagogic account of Frostman’s theorem see Khoshnevisan (2002, Theorem 2.2.1, p. 521). $\square$

5. On Spherically Symmetric Trees

Suppose G is spherically symmetric, and let $\mathbf{m}_{\partial G}$ denote the uniform measure on ∂G . One way to define $\mathbf{m}_{\partial G}$ is as follows: For all $f : \mathbf{Z}_+ \rightarrow \mathbf{R}_+$ and all $v \in \partial G$,

$$(5.1) \quad \int_{\partial G} f(|v \wedge w|) \mathbf{m}_{\partial G}(dw) = \sum_{l=0}^{n-1} \frac{f(l)}{|G_l|},$$

where $n \in \mathbf{Z}_+ \cup \{\infty\}$ denotes the height of G . In particular, we may note that if G is infinite then for all $\nu \in \mathcal{P}(\mathbf{R}_+)$,

$$(5.2) \quad I_h(\mathbf{m}_{\partial G} \times \nu) = \iint \sum_{l=0}^{\infty} \frac{1}{|G_l|} \left(1 + \frac{q}{p} e^{-|t-s|}\right)^l \nu(ds) \nu(dt).$$

This is the integral in (1.4).

Yuval Peres asked us if the constant $960e^3 \approx 19282.1$ in (2.5) can be improved upon. The following answers this question by replacing $960e^3$ by 512. Although we do not know how to improve this constant further, it seems unlikely to be the optimal one.

Theorem 5.1. *Suppose G is an infinite, spherically symmetric tree, and $P_p\{\rho \leftrightarrow \infty\} = 0$. Then, for all compact sets $D \subseteq [0, 1]$,*

$$(5.3) \quad \frac{1}{2 \inf_{\nu \in \mathcal{P}(D)} I_h(\mathbf{m}_{\partial G} \times \nu)} \leq P_p \left(\bigcup_{t \in D} \left\{ \rho \overset{t}{\leftrightarrow} \infty \right\} \right) \leq \frac{512}{\inf_{\nu \in \mathcal{P}(D)} I_h(\mathbf{m}_{\partial G} \times \nu)},$$

where $\inf \emptyset := \infty$ and $1/\infty := 0$. The condition that $P_p\{\rho \leftrightarrow \infty\} = 0$ is not needed for the upper bound.

The proof follows that of Theorem 2.1 closely. Therefore, we sketch the highlights of the proof only.

Sketch of Proof. The lower bound follows immediately from (2.4), so we concentrate on the upper bound only. As we have done before, we may, and will, assume without loss of generality that G is a finite tree of height n .

For all $\nu \in \mathcal{P}(D)$ consider $Z(\mathbf{m}_{\partial G} \times \nu)$ defined in (3.2). That is,

$$(5.4) \quad Z(\mathbf{m}_{\partial G} \times \nu) = \frac{1}{p^n |G_n|} \int_D \sum_{v \in G_n} \mathbf{1}_{\{\rho \overset{t}{\leftrightarrow} v\}} \nu(dt).$$

It might help to point out that in the present setting, G_n is identified with ∂G . According to (3.3),

$$(5.5) \quad \mathbb{E}_p[Z(\mathbf{m}_{\partial G} \times \nu)] = 1 \quad \text{and} \quad \mathbb{E}_p[Z^2(\mathbf{m}_{\partial G} \times \nu)] \leq 2I_h(\mathbf{m}_{\partial G} \times \nu).$$

In accord with (3.12), outside a single null set, the following holds for all $w \in \partial G$, $\sigma, \tau \in \{-, +\}$, and $s \geq 0$:

$$\begin{aligned}
 (5.6) \quad & \sum_{\sigma, \tau \in \{-, +\}} E_p \left[Z(\mathbf{m}_{\partial G} \times \nu) \mid \mathcal{F}_{(\sigma, \tau)}(w, s) \right] \\
 & \geq \int_{(v, t) \in \Xi} \left(1 + \frac{q}{p} e^{-|t-s|} \right)^{|v \wedge w|} (\mathbf{m}_{\partial G} \times \nu)(dv dt) \cdot \mathbf{1}_{\{\rho \overset{s}{\leftrightarrow} w\}} \\
 & = \int_D \sum_{l=0}^{n-1} \frac{1}{|G_l|} \left(1 + \frac{q}{p} e^{-|t-s|} \right)^l \nu(dt) \times \mathbf{1}_{\{\rho \overset{s}{\leftrightarrow} w\}} \\
 & = I_h(\mathbf{m}_{\partial G} \times \nu) \cdot \mathbf{1}_{\{\rho \overset{s}{\leftrightarrow} w\}}.
 \end{aligned}$$

See (5.1) for the penultimate line. Next we take the supremum of the left-most term over all w , and replace w by $\mathbf{w}$ in the right-most term; then square and take expectations, as we did in the course of the proof of Theorem 2.1. $\square$

Finally, let us return briefly to the Hausdorff dimension of $S(G) \cap D$ in the case that G is spherically symmetric. First we recall Theorem 1.6 of Häggström et al. (1997): *If $P_p(\cup_{t \in D} \{\rho \overset{t}{\leftrightarrow} \infty\}) = 1$ then P_p -a.s.,*

$$(5.7) \quad \dim_{\text{H}} S(G) = \sup \left\{ \alpha > 0 : \sum_{l=1}^{\infty} \frac{p^{-l} l^{\alpha-1}}{|G_l|} < \infty \right\}.$$

Next we announce the Hausdorff dimension of $S(G) \cap D$ in the case that G is spherically symmetric.

Theorem 5.2. *Suppose that the tree G is infinite and spherically symmetric. If, in addition, $P_p(\cup_{t \in D} \{\rho \overset{t}{\leftrightarrow} \infty\}) > 0$, then for all compact sets $D \subseteq [0, 1]$,*

$$(5.8) \quad \dim_{\text{H}} (S(G) \cap D) = \sup \left\{ 0 < \alpha < 1 : \inf_{\nu \in \mathcal{P}(D)} I_{\phi(\alpha)}(\mathbf{m}_{\partial G} \times \nu) < \infty \right\},$$

P_p -almost surely on $\{S(G) \cap D \neq \emptyset\}$.

The strategy of the proof is exactly the same as that of the proof of Theorem 2.6, but we use $Z(\mathbf{m}_{\partial G} \times \nu)$ in place of $Z(\mu)$. The minor differences in the proofs are omitted here.

For the purposes of comparison, we mention the following consequence of (5.1): For all $\nu \in \mathcal{P}(\mathbf{R}_+)$,

$$\begin{aligned}
 (5.9) \quad I_{\phi(\alpha)}(\mathbf{m}_{\partial G} \times \nu) &= \iint \sum_{l=0}^{\infty} \frac{1}{|G_l|} \left(1 + \frac{q}{p} e^{-|t-s|} \right)^l \frac{\nu(ds) \nu(dt)}{|t-s|^\alpha} \\
 &= \iint \sum_{l=0}^{\infty} \frac{p^{-l}}{|G_l|} (1 - q \{1 - e^{-|t-s|}\})^l \frac{\nu(ds) \nu(dt)}{|t-s|^\alpha}.
 \end{aligned}$$

It may appear that this expression is difficult to work with. To illustrate that this is not always the case we derive the following bound which may be of independent interest. Our next result computes the Hausdorff dimension of $S(G) \cap D$ in case that D is a nice fractal; i.e., one whose packing and Hausdorff dimensions agree. Throughout, $\dim_p$ denotes packing dimension (Tricot, 1982; Sullivan, 1984).

Proposition 5.3. *Suppose G is an infinite spherically symmetric tree. Suppose also that $P_p(\cup_{t \in D} \{\rho \xrightarrow{t} \infty\}) > 0$ for a certain non-random compact set $D \subseteq \mathbf{R}_+$. If $\delta := \dim_H D$ and $\Delta := \dim_p D$, then P_p -almost surely on $\cup_{t \in D} \{\rho \xrightarrow{t} \infty\}$,*

$$(5.10) \quad [\dim_H S(G) - (1 - \delta)]_+ \leq \dim_H (S(G) \cap D) \leq [\dim_H S(G) - (1 - \Delta)]_+.$$

Remark 5.4. An anonymous referee has pointed the following consequence: If the Hausdorff and packing dimensions of D are the same, then a.s. on $\{S(G) \cap D \neq \emptyset\}$,

$$(5.11) \quad 1 - \dim_H (S(G) \cap D) = (1 - \dim_H S(G)) + (1 - \dim_H D).$$

This agrees with the principle that “codimensions add.”

Proof. Without loss of generality, we may assume that G has no leaves [except ρ].

The condition $P_p(\cup_{t \in D} \{\rho \xrightarrow{t} \infty\}) > 0$ and ergodicity together prove that there a.s. $[P_p]$ exists a time t of percolation. Therefore, (1.3) implies that

$$(5.12) \quad \sum_{l=1}^{\infty} \frac{p^{-l}}{l|G_l|} < \infty.$$

This is in place throughout. Next we proceed with the harder lower bound first. Without loss of generality, we may assume that $\dim_H S(G) > 1 - \delta$, for otherwise there is nothing left to prove.

According to Frostman’s lemma, there exists $\nu \in \mathcal{P}(D)$ such that for all $\varepsilon > 0$ we can find a constant C_ε with the following property:

$$(5.13) \quad \sup_{x \in D} \nu([x - r, x + r]) \leq C_\varepsilon r^{\delta - \varepsilon}, \quad \text{for all } r > 0.$$

(Khoshnevisan, 2002, Theorem 2.1.1, p. 517.) We shall fix this ν throughout the derivation of the lower bound.

Choose and fix α that satisfies

$$(5.14) \quad 0 < \alpha < \frac{\dim_H S(G) - 1}{1 - \varepsilon} + \delta - \varepsilon.$$

[Because we assumed that $\dim_H S(G) > 1 - \delta$ the preceding bound is valid for all $\varepsilon > 0$ sufficiently small. Fix such a ε as well.] For this particular $(\nu, \alpha, \varepsilon)$ we apply to (5.9) the

elementary bound $1 - q\{1 - e^{-x}\} \leq \exp(-qx/2)$, valid for all $0 \leq x \leq 1$, and obtain

$$(5.15) \quad I_{\phi(\alpha)}(\mathbf{m}_{\partial G} \times \nu) \leq \sum_{l=0}^{\infty} \frac{p^{-l}}{|G_l|} \iint \exp\left(-\frac{ql|t-s|}{2}\right) \frac{\nu(ds)\nu(dt)}{|t-s|^\alpha}.$$

We split the integral in two parts, according to whether or not $|t-s|$ is small, and deduce that

$$(5.16) \quad I_{\phi(\alpha)}(\mathbf{m}_{\partial G} \times \nu) \leq \sum_{l=0}^{\infty} \frac{p^{-l}}{|G_l|} \iint_{|t-s| \leq l^{-(1-\varepsilon)}} \frac{\nu(ds)\nu(dt)}{|t-s|^\alpha} + \sum_{l=0}^{\infty} \frac{p^{-l} l^{(1-\varepsilon)\alpha} e^{-(ql^\varepsilon)/2}}{|G_l|}.$$

Thanks to (5.12) the last term is a finite number, which we call K_ε . Thus,

$$(5.17) \quad I_{\phi(\alpha)}(\mathbf{m}_{\partial G} \times \nu) \leq \sum_{l=0}^{\infty} \frac{p^{-l}}{|G_l|} \iint_{|t-s| \leq l^{-(1-\varepsilon)}} \frac{\nu(ds)\nu(dt)}{|t-s|^\alpha} + K_\varepsilon.$$

Integration by parts shows that if $f : \mathbf{R} \rightarrow \mathbf{R}_+ \cup \{\infty\}$ is even, as well as right-continuous and non-increasing on $(0, \infty)$, then for all $0 < a < b$,

$$(5.18) \quad \iint_{a \leq |t-s| \leq b} f(s-t) \nu(ds) \nu(dt) = f(x) F_\nu(x) \Big|_a^b + \int_a^b F_\nu(x) d|f|(x),$$

where $F_\nu(x) := (\nu \times \nu)\{(s, t) \in \mathbf{R}_+^2 : |s-t| \leq x\} \leq C_\varepsilon x^{\delta-\varepsilon}$ thanks to (5.13). We apply this bound with $a \downarrow 0$, $b := l^{-(1-\varepsilon)}$, and $f(x) := |x|^{-\alpha}$ to deduce that

$$(5.19) \quad \iint_{|t-s| \leq l^{-(1-\varepsilon)}} \frac{\nu(ds)\nu(dt)}{|t-s|^\alpha} \leq A l^{(1-\varepsilon)(\alpha-\delta+\varepsilon)},$$

since $\alpha < \delta - \varepsilon$ by (5.14). Here, $A := C_\varepsilon(\delta - \varepsilon)/(-\alpha + \delta - \varepsilon)$. Consequently,

$$(5.20) \quad I_{\phi(\alpha)}(\mathbf{m}_{\partial G} \times \nu) \leq A \sum_{l=0}^{\infty} \frac{p^{-l} l^{(1-\varepsilon)(\alpha-\delta+\varepsilon)}}{|G_l|} + K_\varepsilon,$$

which is finite thanks to (5.14) and (5.7). It follows from Theorem 5.2 that P_p -almost surely, $\dim_{\mathbf{H}}(S(G) \cap D) \geq \alpha$. Let $\varepsilon \downarrow 0$ and $\alpha \uparrow \dim_{\mathbf{H}} S(G) - 1 + \delta$ in (5.14) to obtain the desired lower bound.

Choose and fix $\beta > \Delta$ and $\alpha > \dim_{\mathbf{H}} S(G) + \beta - 1$. We appeal to (5.9) and the following elementary bound: For all integers $l \geq 1$ and all $0 \leq x \leq 1/l$, we have $(1 - q\{1 - e^{-x}\})^l \geq p$. It follows from this and (5.9) that for all $\nu \in \mathcal{P}(E)$,

$$(5.21) \quad \begin{aligned} I_{\phi(\alpha)}(\mathbf{m}_{\partial G} \times \nu) &\geq p \sum_{l=0}^{\infty} \frac{p^{-l}}{|G_l|} \iint_{|t-s| \leq l^{-1}} \frac{\nu(ds)\nu(dt)}{|t-s|^\alpha} \\ &\geq p \sum_{l=0}^{\infty} \frac{p^{-l} l^\alpha}{|G_l|} \int \nu\left(\left(t - \frac{1}{l}, t + \frac{1}{l}\right)\right) \nu(dt). \end{aligned}$$

Because $\beta > \dim_p D$, the density theorem of Taylor and Tricot (1985, Theorem 5.4) implies that

$$(5.22) \quad \liminf_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^\beta} \int \nu((t - \varepsilon, t + \varepsilon)) \nu(dt) \geq \int \liminf_{\varepsilon \rightarrow 0^+} \frac{\nu((t - \varepsilon, t + \varepsilon))}{\varepsilon^\beta} \nu(dt) = \infty.$$

We have also applied Fatou's lemma. Thus, there exists $c > 0$ such that for all $\nu \in \mathcal{P}(E)$,

$$(5.23) \quad I_{\phi(\alpha)}(\mathbf{m}_{\partial G} \times \nu) \geq c \sum_{l=0}^{\infty} \frac{p^{-l} l^{\alpha-\beta}}{|G_l|} = \infty;$$

see (5.7). It follows from Theorem 5.2 that P_p -almost surely, $\dim_H(S(G) \cap D) \leq \alpha$. Let $\beta \downarrow \Delta$ and then $\alpha \downarrow \dim_H S(G) + \Delta - 1$, in this order, to finish. $\square$

6. On Strong β -Sets

An anonymous referee has pointed out that the abstract capacity condition of Corollary 2.2 is in general difficult to check. And we agree. In the previous section we showed how such computations can be carried out when G is spherically symmetric but D is arbitrary. The goal of this section is to provide similar types of examples in the case that G is arbitrary but D is a “nice” set.

Henceforth we choose and fix some $\beta \in (0, 1]$, and assume that the target set $D \subset [0, 1]$ is a *strong β -set*; this is a stronger condition than the better known *s-set* condition of Besicovitch (1942). Specifically, we assume that there exists $\sigma \in \mathcal{P}(D)$ and constant $c_1, c_2 \in (0, \infty)$, such that

$$(6.1) \quad c_1 \varepsilon^\beta \leq \sigma([x - \varepsilon, x + \varepsilon]) \leq c_2 \varepsilon^\beta \quad \forall x \in D, \varepsilon \in (0, 1).$$

We can combine the density theorems of Frostman (1935) together with that of Taylor and Tricot (1985) to find that the packing and Hausdorff dimensions of D agree, and are both equal to β . Next we present an amusing example from symbolic dynamics; other examples abound.

Example 6.1 (Cantor Sets). Choose and fix an integer $b \geq 2$. We can write any $x \in [0, 1]$ as $x = \sum_{j=1}^{\infty} b^{-j} x_j$, where $x_j \in \{0, \dots, b-1\}$. In case of ambiguities we opt for the infinite expansion always. This defines the base- b digits $x_1, x_2, \dots$ of x uniquely. Let B denote a fixed subset of $\{0, \dots, b-1\}$, and define D to be the closure of

$$(6.2) \quad D_0 := \{x \in [0, 1] : x_j \in B \text{ for all } j \geq 1\}.$$

Then, D is a β -set with $\beta := \log_b |B|$, where $\log_b$ denotes the base- b logarithm and $|B|$ the cardinality of B . Indeed, let $X_1, X_2, \dots$ denote i.i.d. random variables with uniform distribution on B , and observe that for all integers $n \geq 1$ and all $x \in D$,

$$(6.3) \quad \mathbb{P}\{X_1 = x_1, \dots, X_n = x_n\} = |B|^{-n} = b^{-n\beta}.$$

If $y \in [0, 1]$ satisfies $y_1 = x_1, \dots, y_n = x_n$, then certainly $|x - y| \leq b \sum_{j=n+1}^{\infty} b^{-j} := Mb^{-n}$. Conversely, if $|x - y| \leq b^{-n}$ then it must be the case that $y_1 = x_1, \dots, y_n = x_n$. Let σ denote the distribution of $X = \sum_{j=1}^{\infty} b^{-j} X_j$, and we know a priori that $\sigma \in \mathcal{P}(D)$. In addition for all $x \in D$,

$$(6.4) \quad \sigma([x - b^{-n}, x + b^{-n}]) \leq b^{-n\beta} \leq \sigma([x - Mb^{-n}, x + Mb^{-n}]).$$

A direct monotonicity argument proves the assertion that D is a β -set with $\beta := \log_b |B|$. We note further that if $b := 3$ and $B := \{0, 2\}$ then D is nothing more than the usual ternary Cantor set in $[0, 1]$, σ is the standard Cantor–Lebesgue measure, and $\beta = \log_3 2$. For another noteworthy example set $B := \{0, \dots, b-1\}$ to find that $D = [0, 1]$, σ is the standard Lebesgue measure on $[0, 1]$, and $\beta := 1$.

Theorem 6.2. *Suppose G is an arbitrary locally finite tree and D is a strong β -set for some $\beta \in (0, 1]$. Then, $\mathbb{P}_p(\cup_{t \in D} \{\rho \xrightarrow{t} \infty\}) > 0$ iff D has positive g -capacity, where*

$$(6.5) \quad g(v, w) := \frac{1}{|v \wedge w|^\beta p^{|v \wedge w|}} \quad \forall v, w \in \partial G.$$

Remark 6.3. It is easy to see that $D := \{0\}$ is a strong 0-set with $\sigma := \delta_0$, and $D := [0, 1]$ is a strong 1-set with σ being the Lebesgue measure on $[0, 1]$; see the final sentence in Example 6.1. Therefore, it follows that Theorem 6.2 contains both Lyons's condition (1.2), as well as the condition (1.3) of Häggström et al. as special cases.

Sketch of proof of Theorem 6.2. We can employ a strategy similar to the one we used to prove Theorem 5.1, and establish first that $\mathbb{P}_p(\cup_{t \in D} \{\rho \xrightarrow{t} \infty\}) > 0$ if and only if $\text{Cap}_\psi(\partial G) > 0$, where $\psi(v, w) := R(|v \wedge w|)$ and

$$(6.6) \quad R(n) := \iint \left(1 + \frac{q}{p} e^{-|t-s|}\right)^n \sigma(ds) \sigma(dt) \quad \forall n \geq 1.$$

It might help to recall that σ was defined in (6.1). We integrate by parts as in (5.18) and then apply (6.1) to find that

$$(6.7) \quad a_1 n \int_0^1 x^\beta \left(1 + \frac{q}{p} e^{-x}\right)^{n-1} dx \leq R(n) \leq a_2 n \int_0^1 x^\beta \left(1 + \frac{q}{p} e^{-x}\right)^{n-1} dx,$$

where the a_i 's are positive and finite constants that do not depend on n . From here it is possible to check that $R(n)$ is bounded above and below by constant multiples of $n^{-\beta}p^{-n}$. Indeed, we use $\int_0^1(\cdots) \geq \int_{1/n}^1(\cdots)$ to obtain a lower bound. For an upper bound, we decompose $\int_0^1(\cdots)$ as $\int_0^{1/n}(\cdots) + \int_{1/n}^{1/2}(\cdots) + \int_{1/2}^1(\cdots)$, and verify that the first integral dominates the other two for all large values of n . This has the desired effect. $\square$

REFERENCES

- Benjamini, Itai, Robin Pemantle, and Yuval Peres. 1995. *Martin capacity for Markov chains*, Ann. Probab. **23**(3), 1332–1346.
- Bertoin, Jean. 1996. *Lévy Processes*, Cambridge University Press, Cambridge.
- Besicovitch, A. S. 1942. *A theorem on s -dimensional measure of sets of points*, Proc. Cambridge Philos. Soc. **38**, 24–27.
- Bochner, S. 1949. *Diffusion equation and stochastic processes*, Proc. Nat. Acad. Sci. U. S. A. **35**, 368–370.
- Dubins, Lester E. and David A. Freedman. 1967. *Random distribution functions*, Proc. Fifth Berkeley Sympos. Math. Statist. and Probability, 1965/1966, pp. Vol. II: Contributions to Probability Theory, Part 1, pp. 183–214.
- Evans, Steven N. 1992. *Polar and nonpolar sets for a tree indexed process*, Ann. Probab. **20**(2), 579–590.
- Frostman, Otto. 1935. *Potential d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions*, Medd. Lunds Univ. Mat. Sem. **3**, 1–118 (French).
- Grimmett, Geoffrey. 1999. *Percolation*, Second Edition, Springer-Verlag, Berlin.
- Häggström, Olle, Yuval Peres, and Jeffrey E. Steif. 1997. *Dynamical percolation*, Ann. Inst. H. Poincaré Probab. Statist. **33**(4), 497–528.
- Hunt, G. A. 1958. *Markoff processes and potentials. III*, Illinois J. Math. **2**, 151–213.
- . 1957. *Markoff processes and potentials. I, II*, Illinois J. Math. **1**, 44–93, 316–369.
- Kanda, Mamoru. 1978. *Characterization of semipolar sets for processes with stationary independent increments*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **42**(2), 141–154.
- Khoshnevisan, Davar. 2002. *Multiparameter Processes*, Springer-Verlag, New York.
- McKean, Henry P., Jr. 1955. *Sample functions of stable processes*, Ann. of Math. (2) **61**, 564–579.
- Lyons, Russell. 1989. *The Ising model and percolation on trees and tree-like graphs*, Comm. Math. Phys. **125**(2), 337–353.
- . 1990. *Random walks and percolation on trees*, Ann. Probab. **18**(3), 931–958.
- . 1992. *Random walks, capacity and percolation on trees*, Ann. Probab. **20**(4), 2043–2088.
- Marchal, Philippe. 1998. *The best bounds in a theorem of Russell Lyons*, Electron. Comm. Probab. **3**, 91–94 (electronic).
- Mattila, Pertti. 1995. *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, Cambridge.
- Nelson, Edward. 1958. *A functional calculus using singular Laplace integrals*, Trans. Amer. Math. Soc. **88**, 400–413.
- Paley, R. E. A. C. and A. Zygmund. 1932. *A note on analytic functions in the unit circle*, Proc. Cambridge Phil. Soc. **28**, 266–272.

- Pemantle, Robin and Yuval Peres. 1995. *Critical random walk in random environment on trees*, Ann. Probab. **23**(1), 105–140.
- Peres, Yuval and Jeffrey E. Steif. 1998. *The number of infinite clusters in dynamical percolation*, Probab. Theory Related Fields **111**(1), 141–165.
- Sato, Ken-iti. 1999. *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, Cambridge. Translated from the 1990 Japanese original; Revised by the author.
- Sullivan, Dennis. 1984. *Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups*, Acta Math. **153**(3-4), 259–277.
- Taylor, S. James and Claude Tricot. 1985. *Packing measure, and its evaluation for a Brownian path*, Trans. Amer. Math. Soc. **288**(2), 679–699.
- Tricot, Claude, Jr. 1982. *Two definitions of fractional dimension*, Math. Proc. Cambridge Philos. Soc. **91**(1), 57–74.

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