DYNAMICAL PERCOLATION ON GENERAL TREES

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ABSTRACT. Häggström, Peres, and Steif (1997) have introduced a dynamical version of percolation on a graph G. When G is a tree they derived a necessary and sufficient condition for percolation to exist at some time t. In the case that G is a spherically symmetric tree, Peres and Steif (1998) derived a necessary and sufficient condition for percolation to exist at some time t in a given target set D. The main result of the present paper is a necessary and sufficient condition for the existence of percolation, at some time $t \in D$, in the case that the underlying tree is not necessary spherically symmetric. This answers a question of Yuval Peres (personal communication). We present also calculations of the Hausdorff dimension of exceptional times of percolation.

1. Introduction

Let G be a locally finite graph with edge set E, and assign to each edge a weight of one or zero at random. All edge weights are independent and identically distributed. Choose and fix $p \in (0\,,1)$ to be the probability that a given edge weight is one, and define P_p to be the resulting product measure on the collection of random edge weights.

Recall that an edge is said to be "open" if its edge weight is one. Otherwise, the edge is deemed "closed." A fundamental problem of bond percolation is to decide when there can exist an infinite connected cluster of open edges in G (Grimmett, 1999). Choose and fix some vertex ρ in G, and consider the event $\{\rho \leftrightarrow \infty\}$ that percolation occurs through ρ. That is, let $\{\rho \leftrightarrow \infty\}$ to be the event that there exists an infinite connected cluster of open edges that emanate from the vertex ρ. Then the stated problem of percolation theory is, when is $\mathrm{P}_p\{\rho \leftrightarrow \infty\} > 0$?

There does not seem to be a general answer to this question, although much is known (Grimmett, 1999). For instance, there always exists a critical probability p_c such that

$$\mathrm{P}_p\left\{\rho \leftrightarrow \infty\right\} = \begin{cases} \text{positive}, & \text{if } p > p_c, \\ \text{zero}, & \text{if } p < p_c. \end{cases} \tag{1.1}$$

However, $\mathrm{P}_{p_c}\{\rho \leftrightarrow \infty\}$ can be zero for some graphs G, and positive for others.

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When G is a tree then much more is known. In this case Lyons (1992) has proved that $\mathrm{P}_p\{\rho \leftrightarrow \infty\} > 0$ if and only if there exists a probability measure μ on ∂G such that

$$\iint \frac{\mu(dv)\,\mu(dw)}{p^{|v\wedge w|}} < \infty. \tag{1.2}$$

In the language of population genetics, $v \wedge w$ denotes the "greatest common ancester" of v and w, and $|z|$ is the "age," or depth, of any vertex z.

Lyons's theorem improves on the earlier efforts of Lyons (1989; 1990) and aspects of the work of and Dubins and Freedman (1967) and Evans (1992). Benjamini et al. (1995) and Marchal (1998) contain two different optimal improvements on Lyons's theorem.

Häggström, Peres, and Steif (1997) added equilibrium dynamics to percolation problems. Next is a brief description. At time zero we construct all edge weights according to P_p. Then we update each edge weight, independently of all others, in a stationary-Markov fashion: If an edge weight is zero then it flips to a one at rate p; if an edge weight is one then it flips to a zero at rate $q := 1 - p$.

Let us write $\{\rho \overset{t}{\leftrightarrow} \infty\}$ for the event that we have percolation at time t. By stationarity, $\mathrm{P}_p\{\rho \overset{t}{\leftrightarrow} \infty\}$ does not depend on t; this is the probability of percolation in the context of Lyons (1992). In particular, if $p < p_c$ then $\mathrm{P}_p\{\rho \overset{t}{\leftrightarrow} \infty\} = 0$ for all $t \geq 0$. The results of Häggström et al. (1997) imply that there exists a tree G such that $\mathrm{P}_{p_c}(\cup_{t>0}\{\rho \overset{t}{\leftrightarrow} \infty\}) = 1$ although $\mathrm{P}_{p_c}\{\rho \overset{t}{\leftrightarrow} \infty\} = 0$ for all $t \geq 0$. We add that, in all cases, the event $\cup_{t>0}\{\rho \overset{t}{\leftrightarrow} \infty\}$ is indeed measurable, and thanks to ergodicity has probability zero or one.

Now let us specialize to the case that G is *spherically symmetric*. This means that all vertices of a given height have the same number of children. In this case, Häggström et al. (1997) studied dynamical percolation on G in greater depth, and proved that for all $p \in (0\,,1)$,

$$\mathrm{P}_p\left(\bigcup_{t\geq 0}\left\{\rho \overset{t}{\leftrightarrow} \infty\right\}\right) = 1 \quad \text{if and only if} \quad \sum_{l=1}^{\infty} \frac{p^{-l}}{l|G_l|} < \infty. \tag{1.3}$$

Here, G_n denotes the finite subtree of all vertices of height at most n, and $|G_n|$ denotes the number of vertices of G_n. This theorem has been extended further by Peres and Steif (1998). In order to describe their results, we follow Peres and Steif (1998) and consider only the non-trivial case where G is an infinite tree. In that case, Theorem 1.5 of Peres and Steif (1998) asserts that for all closed sets $D \subseteq [0\,,1]$ that are non-random, $\mathrm{P}_p(\cup_{t\in D}\{\rho \overset{t}{\leftrightarrow} \infty\}) > 0$ if and only if there exists a probability measure ν on D such that

$$\iint \sum_{l=1}^{\infty} \frac{1}{|G_{l+1}|}\left(1 + \frac{q}{p}e^{-|t-s|}\right)^l \nu(ds)\,\nu(dt) < \infty. \tag{1.4}$$

The principle aim of this paper is to study general trees G, i.e., not necessarily spherically symmetric ones, and describe when there is positive probability of percolation for some time t in a given "target set" D. Our description (Theorem 2.1) answers a question of Yuval Peres (personal communication), and confirms Conjecture 1 of Pemantle and Peres (1995) for a large family of concrete target percolations. In addition, when D is a singleton, our description recovers the characterization (1.2)—due to Lyons (1992)—of percolation on general trees.

As we mentioned earlier, it can happen that $p = p_c$ and yet percolation occurs at some time t. Let $S(G)$ denote the collection of all such exceptional times. When G is spherically symmetric, Häggström et al. (1997, Theorem 1.6) compute the Hausdorff dimension of $S(G)$. Here we do the same in the case that G is generic (Theorem 2.5). In order to do this we appeal to the theory of Lévy processes (Bertoin, 1996; Khoshnevisan, 2002; Sato, 1999); the resulting formula for dimension is more complicated when G is not regular. We apply our formula to present simple bounds for the Hausdorff dimension of $S(G) \cap D$ for a non-random target set D in the case that G is spherically symmetric (Proposition 5.3). When D is a regular fractal our upper and lower bounds agree, and we obtain an almost-sure identity for the Hausdorff dimension of $S(G) \cap D$.

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2. Main Results

Our work is in terms of various capacities for which we need some notation.

Let S be a topological space, and suppose $f : S \times S \to \mathbf{R}_+ \cup \{\infty\}$ is measurable and μ is a Borel probability measure on S. Then we define the f-energy of μ to be

$$I_f(\mu) := \iint f(x\,, y)\, \mu(dx)\, \mu(dy). \tag{2.1}$$

We define also the f-capacity of a Borel set $F \subseteq S$ as

$$\mathrm{Cap}_f(F) := \left[\inf_{\mu \in \mathscr{P}(F)} I_f(\mu) \right]^{-1}, \tag{2.2}$$

where $\inf \varnothing := \infty$, $1/\infty := 0$, and $\mathscr{P}(F)$ denotes the collection of all probability measures on F. Now we return to the problem at hand.

For $v, w \in \partial G$ and $s, t \geq 0$ define

$$h\left((v\,,s)\,;(w\,,t)\right) := \left(1 + \frac{q}{p}e^{-|s-t|}\right)^{|v\wedge w|}. \tag{2.3}$$

Peres and Steif (1998) have proved that if $\mathrm{P}_p\{\rho \leftrightarrow \infty\} = 0$, then for all closed sets $D \subset [0\,,1]$,

$$\mathrm{P}_p\left(\bigcup_{t\in D}\left\{\rho \overset{t}{\leftrightarrow} \infty\right\}\right) \geq \frac{1}{2}\mathrm{Cap}_h\left(\partial G \times D\right). \tag{2.4}$$

In addition, they prove that when G is spherically symmetric,

$$\mathrm{P}_p\left(\bigcup_{t\in D}\left\{\rho \overset{t}{\leftrightarrow} \infty\right\}\right) \leq 960e^3\mathrm{Cap}_h\left(\partial G \times D\right). \tag{2.5}$$

Their method is based on the fact that when G is spherically symmetric one can identify the form of the minimizing measure in the definition of $\mathrm{Cap}_h(\partial G \times D)$. In fact, the minimizing measure can be written as the uniform measure on ∂G—see (5.1)—times some probability measure on D. Whence follows also (1.4).

In general, G is not spherically symmetric, thus one does not know the form of the minimizing measure. We use other arguments that are based on random-field methods in order to obtain the following result. We note that the essence of our next theorem is in its upper bound because it holds without any exogenous conditions.

Theorem 2.1. *Suppose* $\mathrm{P}_p\{\rho \leftrightarrow \infty\} = 0$. *Then, for all compact sets* $D \subseteq \mathbf{R}_+$,

$$\frac{1}{2}\mathrm{Cap}_h\left(\partial G \times D\right) \leq \mathrm{P}_p\left(\bigcup_{t\in D}\left\{\rho \overset{t}{\leftrightarrow} \infty\right\}\right) \leq 512\mathrm{Cap}_h\left(\partial G \times D\right). \tag{2.6}$$

The condition that $\mathrm{P}_p\{\rho \leftrightarrow \infty\} = 0$ *is not needed in the upper bound.*

Thus, we can use the preceding theorem in conjunction with (2.4) to deduce the following.

Corollary 2.2. *Percolation occurs at some time* $t \in D$ *if and only if* $\partial G \times D$ *has positive* h*-capacity.*

We make two additional remarks.

Remark 2.3. Clearly, when $D = \{t\}$ is a singleton then $\mu \in \mathscr{P}(\partial G \times D)$ if and only if $\mu(A \times B) = \nu(A)\delta_t(B)$ for some $\nu \in \mathscr{P}(\partial G)$; also, $I_h(\nu \times \delta_t) = \iint p^{-|v\wedge w|}\,\nu(dv)\,\nu(dw)$. Therefore, Corollary 2.2 contains Lyons's theorem (Lyons, 1992), although our multiplicative constant is worse than that of Lyons.

Remark 2.4. It is not too hard to modify our methods and prove that when G is spherically symmetric,

$$\mathrm{P}_p\left(\bigcup_{t\in D}\left\{\rho \overset{t}{\leftrightarrow} \infty\right\}\right) \le \frac{512}{I_h(\mathfrak{m}_{\partial G}\times\nu)}, \tag{2.7}$$

where $\mathfrak{m}_{\partial G}$ is the uniform measure on ∂G. See Theorem 5.1 below. From this we readily recover (2.5) with the better constant 512 in place of $960e^3 \approx 19282.1$. This verifies the conjecture of Yuval Peres that $960e^3$ is improvable (personal communication), although it is unlikely that our 512 is optimal.

Next we follow the development of Häggström et al. (1997, Theorem 1.6), and consider the dimension of the set of times at which percolation occurs. This problem is non-trivial only when $p = p_c$.

Consider the random subset $S(G) := S(G)(\omega)$ of $[0\,,1]$ defined as

$$S(G) := \left\{t \in \mathbf{R}_+ :\ \rho \overset{t}{\leftrightarrow} \partial G\right\}. \tag{2.8}$$

Note in particular that, as events,

$$\{S(G)\cap D \neq \varnothing\} = \bigcup_{t\in D}\left\{\rho \overset{t}{\leftrightarrow} \infty\right\}. \tag{2.9}$$

Define, for our $\alpha \in (0\,,1)$, the function $\phi(\alpha) : (\partial G \times \mathbf{R}_+)^2 \to \mathbf{R}_+ \cup \{\infty\}$, as follows.

$$\phi(\alpha)\Big((v\,,t)\,;(w\,,s)\Big) := \frac{h((v\,,t)\,;(w\,,s))}{|t-s|^\alpha}. \tag{2.10}$$

Then we offer the following result on the fine structure of $S(G)$.

Theorem 2.5. *Let D be a non-random compact subset of $\mathbf{R}_+$. If $\mathrm{P}_p\{\rho \leftrightarrow \infty\} > 0$ then $S(G)\cap D$ has positive Lebesgue measure a.s. on $\{S(G)\cap D \neq \varnothing\}$. If $\mathrm{P}_p\{\rho \leftrightarrow \infty\} = 0$, then $S(G)\cap D$ has zero Lebesgue measure a.s., and a.s. on $\{S(G)\cap D \neq \varnothing\}$,*

$$\dim_{\mathrm{H}}\left(S(G)\cap D\right) = \sup\left\{0<\alpha<1 :\ \mathrm{Cap}_{\phi(\alpha)}(D) > 0\right\}, \tag{2.11}$$

where $\sup\varnothing := 0$.

For a stronger statement see Remark 4.4 below. When G is spherically symmetric this theorem often simplifies considerably; see Proposition 5.3 below.

3. Proof of Theorem 2.1

We prove only the upper bound; the lower bound (2.4) was proved much earlier in Peres and Steif (1998).

Without loss of generality, we may assume that G has no leaves [except the root]. Otherwise, we can replace G everywhere by G', where the latter is the maximal subtree of G that has no leaves [except the root]. This "leaflessness" assumption is in force throughout the proof. Also without loss of generality, we may assume that $\mathrm{P}_p\{\cup_{t\in D}\{\rho \overset{t}{\leftrightarrow} \infty\}) > 0$, for otherwise there is nothing left to prove.

As in Peres and Steif (1998), we first derive the theorem in the case that G is a finite tree. A little thought shows that we can still assume without loss of generality that G has no leaves. By this we now mean that there exists an integer k such that whenever a vertex v satisfies $|v| < k$, then v necessarily has a descendant w with $|w| = k$. The maximal such integer k is the *height* of G, and is denoted as n hereforth.

Define

$$\Xi := \partial G \times D. \tag{3.1}$$

Let $\mu \in \mathscr{P}(\Xi)$, and define

$$Z(\mu) := \frac{1}{p^n} \int_{(v,t)\in\Xi} \mathbf{1}_{\{\rho \overset{t}{\leftrightarrow} v\}}\, \mu(dv\, dt). \tag{3.2}$$

During the course of their derivation of (2.4), Peres and Steif (1998) have demonstrated that

$$\mathrm{E}_p[Z(\mu)] = 1 \quad \text{and} \quad \mathrm{E}_p\left[Z^2(\mu)\right] \leq 2I_h(\mu). \tag{3.3}$$

In fact, (2.4) follows immediately from this and the Paley–Zygmund inequality (1932): *For all non-negative $f \in L^2(\mathrm{P}_p)$,*

$$\mathrm{P}_p\{f > 0\} \geq \frac{(\mathrm{E}_p f)^2}{\mathrm{E}_p[f^2]}. \tag{3.4}$$

Now we prove Theorem 2.1. Because G is assumed to be finite, we can embed it in the plane. We continue to write G for the said embedding of G in $\mathbf{R}^2$; this should not cause too much confusion since we will not refer to the abstract tree G until the end of the proof.

Since G is assumed to be leafless, we can identify ∂G with the collection of vertices $\{v : |v| = n\}$ of maximal length. [Recall that n denotes the height of G.]

There are four natural partial orders on $\partial G \times \mathbf{R}_+$ which we describe next. Let $(v\,,t)$ and $(w\,,s)$ be two elements of $\partial G \times \mathbf{R}_+$:

(1) We say that $(v\,,t) <_{(-,-)} (w\,,s)$ if $t \leq s$ and v lies to the left of w in the planar embedding of G.
(2) If $t \geq s$ and v lies to the left of w [in the planar embedding of G], then we say that $(v\,,t) <_{(-,+)} (w\,,s)$.
(3) If $t \leq s$ and v lies to the right of w, then we say that $(v\,,t) <_{(+,-)} (w\,,s)$.
(4) If $t \geq s$ and v lies to the right of w, then we say that $(v\,,t) <_{(+,+)} (w\,,s)$.

[One really only needs two of these, but having four simplifies the ensuing presentations slightly.] The key feature of these partial orders is that, together, they totally order $\partial G\times\mathbf{R}_+$. By this we mean that

$$(v\,,t),(w\,,s)\in\partial G\times\mathbf{R}_+\ \Rightarrow\ {}^{\exists}\sigma,\tau\in\{-\,,+\}:\ (v\,,t)<_{(\sigma,\tau)}(w\,,s). \tag{3.5}$$

Define, for all $(v\,,t)\in\partial G\times\mathbf{R}_+$ and $\sigma,\tau\in\{-\,,+\}$,

$$\mathscr{F}_{(\sigma,\tau)}(v\,,t):=\text{sigma-algebra generated by }\left\{\mathbf{1}_{\{\rho\overset{s}{\leftrightarrow}w\}}\,;\,(w\,,s)<_{(\sigma,\tau)}(v\,,t)\right\}, \tag{3.6}$$

where the conditions that $s\ge 0$ and $w\in\partial G$ are implied tacitly. It is manifestly true that for every fixed $\sigma,\tau\in\{-\,,+\}$, the collection of sigma-algebras $\mathscr{F}_{(\sigma,\tau)}:=\{\mathscr{F}_{(\sigma,\tau)}(v\,,t)\}_{t\ge 0,v\in\partial G}$ is a filtration in the sense that

$$(w\,,s)<_{(\sigma,\tau)}(v\,,t)\implies\mathscr{F}_{(\sigma,\tau)}(w\,,s)\subseteq\mathscr{F}_{(\sigma,\tau)}(v\,,t). \tag{3.7}$$

Also, it follows fairly readily that each $\mathscr{F}_{(\sigma,\tau)}$ is commuting in the sense of Khoshnevisan (2002, pp. 35 and 233). When $(\sigma\,,\tau)=(\pm\,,+)$ this assertion is easy enough to check directly; when $(\sigma\,,\tau)=(\pm\,,-)$, it follows from the time-reversability of our dynamics together with the case $\tau=+$. Without causing too much confusion we can replace $\mathscr{F}_{(\sigma,\tau)}(v\,,t)$ by its completion. That is, we add to $\mathscr{F}_{(\sigma,\tau)}(v\,,t)$ subsets of all P_p-null sets for all $p\in(0\,,1)$. Also, we may—and will—replace the latter further by making it right-continuous in the partial order $<_{(\sigma,\tau)}$. As a consequence of this and Cairoli's maximal inequality (Khoshnevisan, 2002, Theorem 2.3.2, p. 235), for all twice-integrable random variables Y, and all $\sigma,\tau\in\{-\,,+\}$,

$$\mathrm{E}_p\left(\sup_{(v,t)\in\Xi}\left|\mathrm{E}_p\left[Y\,\middle|\,\mathscr{F}_{(\sigma,\tau)}(v\,,t)\right]\right|^2\right)\le 16\mathrm{E}_p\left[Y^2\right]. \tag{3.8}$$

Next we bound from below $\mathrm{E}_p[Z\,|\,\mathscr{F}_{(\sigma,\tau)}(w\,,s)]$, where $s\ge 0$ and $w\in\partial G$ are fixed:

$$\begin{aligned}&\mathrm{E}_p\left[Z(\mu)\,\middle|\,\mathscr{F}_{(\sigma,\tau)}(w\,,s)\right]\\&\qquad\ge\int_{\substack{(v,t)\in\Xi:\\(w,s)<_{(\sigma,\tau)}(v,t)}}p^{-n}\,\mathrm{P}_p\left(\rho\overset{t}{\leftrightarrow}v\,\middle|\,\mathscr{F}_{(\sigma,\tau)}(w\,,s)\right)\,\mu(dv\,dt)\cdot\mathbf{1}_{\{\rho\overset{s}{\leftrightarrow}w\}}.\end{aligned} \tag{3.9}$$

By the Markov property, P_p-a.s. on $\{\rho\overset{s}{\leftrightarrow}w\}$,

$$\mathrm{P}_p\left(\rho\overset{t}{\leftrightarrow}v\,\middle|\,\mathscr{F}_{(\sigma,\tau)}(w\,,s)\right)=p^{n-|v\wedge w|}\left(p+qe^{-(t-s)}\right)^{|v\wedge w|}. \tag{3.10}$$

See equation (6) of Häggström, Peres, and Steif (1997). It follows then that P_p a.s.,

$$\mathrm{E}_p\left[Z(\mu)\,\middle|\,\mathscr{F}_{(\sigma,\tau)}(w\,,s)\right]\ge\int_{\substack{(v,t)\in\Xi:\\(w,s)<_{(\sigma,\tau)}(v,t)}}h\left((w\,,s)\,;(v\,,t)\right)\,\mu(dv\,dt)\cdot\mathbf{1}_{\{\rho\overset{s}{\leftrightarrow}w\}}. \tag{3.11}$$

Thanks to the preceding, and (3.5), for all $s \geq 0$ and $w \in \partial G$ the following holds P_p a.s.:

$$\sum_{\sigma,\tau\in\{-,+\}} \mathrm{E}_p\left[Z(\mu)\;\middle|\;\mathscr{F}_{(\sigma,\tau)}(w\,,s)\right] \geq \int_\Xi h\left((w\,,s)\,;(v\,,t)\right)\,\mu(dv\,dt)\cdot \mathbf{1}_{\{\rho\overset{s}{\leftrightarrow}w\}}. \tag{3.12}$$

It is possible to check that the right-hand side is a right-continuous function of s. Because ∂G is finite, we can therefore combine all null sets and deduce that P_p almost surely, (3.12) holds simultaneously for all $s \geq 0$ and $w \in \partial G$.

Recall that we assumed, at the onset of the proof, that $\mathrm{P}_p(\cup_{t\in D}\{\rho \overset{t}{\leftrightarrow} \infty\}) > 0$. From this it follows easily that we can find random variables $\mathbf{s}$ and $\mathbf{w}$ such that:

(1) $\mathbf{s}(\omega) \in D \cup \{\infty\}$ for all ω, where ∞ is a point not in $\mathbf{R}_+$;
(2) $\mathbf{w}(\omega) \in \partial G \cup \{\delta\}$, where δ is an abstract "cemetery" point not in ∂G;
(3) $(\mathbf{w}(\omega)\,,\mathbf{s}(\omega)) \neq (\delta\,,\infty)$ if and only if there exists $t \in D$ and $v \in \partial G$ such that $\rho \overset{t}{\leftrightarrow} v$;
(4) $(\mathbf{w}(\omega)\,,\mathbf{s}(\omega)) \neq (\delta\,,\infty)$ if and only if $\rho \overset{\mathbf{s}(\omega)}{\leftrightarrow} \mathbf{w}(\omega)$.

Define a measure μ on Ξ by letting, for all Borel sets $A \times B \subseteq \Xi$,

$$\mu(A\times B) := \mathrm{P}_p\left((\mathbf{w}\,,\mathbf{s}) \in A\times B\,|\,(\mathbf{w}\,,\mathbf{s}) \neq (\delta\,,\infty)\right). \tag{3.13}$$

Note that $\mu \in \mathscr{P}(\Xi)$ because $\mathrm{P}_p(\cup_{t\in D}\{\rho \overset{t}{\leftrightarrow} \infty\}) = \mathrm{P}_p\{(\mathbf{w}\,,\mathbf{s}) \neq (\delta\,,\infty)\} > 0$.

We apply (3.12) with this particular $\mu \in \mathscr{P}(\Xi)$, and replace $(w\,,s)$ by $(\mathbf{w}\,,\mathbf{s})$, to find that a.s.,

$$\begin{aligned}\sum_{\sigma,\tau\in\{-,+\}} \sup_{(w,s)\in\Xi} &\mathrm{E}_p\left[Z(\mu)\;\middle|\;\mathscr{F}_{(\sigma,\tau)}(w\,,s)\right] \\ &\geq \int_\Xi h\left((\mathbf{w}\,,\mathbf{s})\,;(v\,,t)\right)\,\mu(dv\,dt)\cdot\mathbf{1}_{\cup_{t\in D}\{\rho\overset{t}{\leftrightarrow}\infty\}}.\end{aligned} \tag{3.14}$$

According to (3.8), and thanks to the inequality,

$$(a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2), \tag{3.15}$$

we can deduce that

$$\begin{aligned}&\mathrm{E}_p\left[\left(\sum_{\sigma,\tau\in\{-,+\}}\sup_{(w,s)\in\Xi}\mathrm{E}_p\left[Z(\mu)\;\middle|\;\mathscr{F}_{(\sigma,\tau)}(w\,,s)\right]\right)^2\right] \\ &\leq 4\sum_{\sigma,\tau\in\{-,+\}}\mathrm{E}_p\left[\sup_{(w,s)\in\Xi}\left|\mathrm{E}_p\left[Z(\mu)\;\middle|\;\mathscr{F}_{(\sigma,\tau)}(w\,,s)\right]\right|^2\right] \leq 256\mathrm{E}_p\left[Z^2(\mu)\right] \\ &\leq 512 I_h(\mu).\end{aligned} \tag{3.16}$$

See (3.3) for the final inequality. On the other hand, thanks to the definition of μ, and by the Cauchy–Schwarz inequality,

$$\begin{aligned}
&\mathrm{E}_p\left[\left(\int_\Xi h\left((\mathbf{w}\,,\mathbf{s})\,;(v\,,t)\right)\,\mu(dv\,dt)\right)^2 \,\middle|\, \bigcup_{t\in D}\left\{\rho \overset{t}{\leftrightarrow} \infty\right\}\right]\\
&\geq \left(\mathrm{E}_p\left[\int_\Xi h\left((\mathbf{w}\,,\mathbf{s})\,;(v\,,t)\right)\,\mu(dv\,dt) \,\middle|\, \bigcup_{t\in D}\left\{\rho \overset{t}{\leftrightarrow} \infty\right\}\right]\right)^2\\
&= \left[I_h(\mu)\right]^2.
\end{aligned} \tag{3.17}$$

Because G is finite, it follows that $0 < I_h(\mu) < \infty$. Therefore, (3.14), (3.16), and (3.17) together imply the theorem in the case that G is finite. The general case follows from the preceding by monotonicity.

4. Proof of Theorem 2.5

The assertions about the Lebesgue measure of $S(G) \cap D$ are mere consequences of the fact that $\mathrm{P}_p\{\rho \overset{t}{\leftrightarrow} \infty\}$ does not depend on t, used in conjunction with the Fubini–Tonelli theorem. Next we proceed with the remainder of the proof.

Choose and fix $\alpha \in (0\,,1)$. Let $Y_\alpha := \{Y_\alpha(t)\}_{t\geq 0}$ to be a symmetric stable Lévy process on $\mathbf{R}$ with index $(1-\alpha)$. We can normalize Y_α so that $\mathrm{E}[\exp\{i\xi Y(1)\}] = \exp(-|\xi|^{1-\alpha})$ for all $\xi \in \mathbf{R}$. We assume also that Y_α is totally independent of our dynamical percolation process. For more information on the process Y_α see the monographs of Bertoin (1996), Khoshnevisan (2002), and Sato (1999).

Recall the function $\phi(\alpha)$ from (2.10). Our immediate goal is to demonstrate the following.

Theorem 4.1. *Suppose* $\mathrm{P}_p\{\rho \leftrightarrow \infty\} = 0$. *Choose and fix* $M > 1$. *Then there exists a finite constant* $A = A(M) > 1$ *such that for all compact sets* $D \subseteq [-M\,,M]$,

$$\frac{1}{A}\mathrm{Cap}_{\phi(\alpha)}(D) \leq \mathrm{P}_p\left\{S(G) \cap D \cap \overline{Y_\alpha([1\,,2])} \neq \varnothing\right\} \leq A\mathrm{Cap}_{\phi(\alpha)}(D), \tag{4.1}$$

where $\overline{\mathrm{U}}$ *denotes the Euclidean closure of* U. *The condition that* $\mathrm{P}_p\{\rho \leftrightarrow \infty\} = 0$ *is not needed in the upper bound.*

Remark 4.2. We do not require the following strengthened form of Theorem 4.1, but it is simple enough to derive that we describe it here: *Theorem 4.1 continues to holds if* $\overline{Y_\alpha([1\,,2])}$ *were replaced by* $Y_\alpha([1\,,2])$. Indeed, a well-known theorem of Kanda (1978) implies that all semipolar sets for Y_α are polar; i.e., Y_α satisfies Hunt's (H) hypothesis (Hunt, 1957; 1958). This readily implies the assertion of this Remark.

From here on, until after the completion of the proof of Theorem 4.1, we will assume without loss of much generality that G is a finite tree of height n. The extension to the case where G is infinite is made by standard arguments.

Let D be as in Theorem 4.1. For all $\mu \in \mathscr{P}(\partial G \times D)$ and $\varepsilon \in (0,1)$, we define

$$Z_\varepsilon(\mu) := \frac{1}{(2\varepsilon)p^n} \int_\Xi \int_1^2 \mathbf{1}_{\{\rho \overset{t}{\leftrightarrow} v\} \cap \{|Y_\alpha(r) - t| \le \varepsilon\}} \, dr \, \mu(dv\, dt), \tag{4.2}$$

where $\Xi := \partial G \times D$, as before.

Next we collect some of the elementary properties of $Z_\varepsilon(\mu)$.

Lemma 4.3. *There exists $c > 1$ such that for all $\mu \in \mathscr{P}(\Xi)$ and $\varepsilon \in (0,1)$:*

1. $\mathrm{E}_p[Z_\varepsilon(\mu)] \ge 1/c$; *and*
2. $\mathrm{E}_p[Z_\varepsilon^2(\mu)] \le c I_{\phi(\alpha)}(\mu)$.

Proof. Define $p_r(a)$ to be the density of $Y_\alpha(r)$ at a. Then, subordination shows that: (i) $p_r(a) > 0$ for all $r > 0$ and $a \in \mathbf{R}$; and (ii) there exists $c_1 > 0$ such that $p_r(a) \ge c_1$ uniformly for all $r \in [1,2]$ and $a \in [-M-1, M+1]$. See Khoshnevisan (2002, pp. 377–384). This readily proves the first assertion of the lemma. Next we can note that by the Markov property of Y_α,

$$\begin{aligned} &\int_1^2 \int_1^2 \mathrm{P}\left\{|Y_\alpha(r) - t| \le \varepsilon\,, |Y_\alpha(R) - s| \le \varepsilon\right\} dr\, dR \\ &\quad \le \int_1^2 \int_r^2 \mathrm{P}\left\{|Y_\alpha(r) - t| \le \varepsilon\right\} \mathrm{P}\left\{|Y_\alpha(R-r) - (t-s)| \le 2\varepsilon\right\} dR\, dr \\ &\qquad + \int_1^2 \int_R^2 \mathrm{P}\left\{|Y_\alpha(R) - s| \le \varepsilon\right\} \mathrm{P}\left\{|Y_\alpha(r-R) - (s-t)| \le 2\varepsilon\right\} dr\, dR. \end{aligned} \tag{4.3}$$

We can appeal to subordination once again to find that there exists $c_2 > 0$ such that $p_r(a) \le c_2$ uniformly for all $r \in [1,2]$ and $a \in [-M-1, M+1]$. This, and symmetry, together imply that

$$\begin{aligned} &\int_1^2 \int_1^2 \mathrm{P}\left\{|Y_\alpha(r) - t| \le \varepsilon\,, |Y_\alpha(R) - s| \le \varepsilon\right\} dr\, dR \\ &\quad \le 4c_2\varepsilon \int_0^2 \mathrm{P}\left\{|Y_\alpha(r) - (t-s)| \le 2\varepsilon\right\} dr \\ &\quad \le 4e^2 c_2 \varepsilon \int_0^\infty \mathrm{P}\left\{|Y_\alpha(r) - (t-s)| \le 2\varepsilon\right\} e^{-r}\, dr. \end{aligned} \tag{4.4}$$

Let $u(a) := \int_0^\infty p_t(a) e^{-t}\, dt$ denote the one-potential density of Y_α, and note that

$$\int_1^2 \int_1^2 \mathrm{P}\left\{|Y_\alpha(r) - t| \le \varepsilon\,, |Y_\alpha(R) - s| \le \varepsilon\right\} dr\, dR \le 4e^2 c_2 \varepsilon \int_{|t-s|-2\varepsilon}^{|t-s|+2\varepsilon} u(z)\, dz. \tag{4.5}$$

It is well known that there exist $c_3 > 1$ such that

$$\frac{1}{c_3|z|^\alpha} \le u(z) \le \frac{c_3}{|z|^\alpha}, \qquad \text{for all } z \in [-2M-2\,,2M+2]; \tag{4.6}$$

see Khoshnevisan (2002, Lemma 3.4.1, p. 383), for instance. It follows that

$$\int_1^2\int_1^2 \mathrm{P}\left\{|Y_\alpha(r)-t|\le\varepsilon\,,|Y_\alpha(R)-s|\le\varepsilon\right\}\,dr\,dR \le 4e^2c_2c_3\varepsilon\int_{|t-s|-2\varepsilon}^{|t-s|+2\varepsilon}\frac{dz}{|z|^\alpha}. \tag{4.7}$$

If $|t-s|\ge 4\varepsilon$, then we use the bound $|z|^{-\alpha}\le(|t-s|/2)^{-\alpha}$. Else, we use the estimate $\int_{|t-s|-2\varepsilon}^{|t-s|+2\varepsilon}(\cdots)\le\int_{-6\varepsilon}^{6\varepsilon}(\cdots)$. This leads us to the existence of a constant $c_4=c_4(M)>0$ such that for all $s,t\in D$ and $\varepsilon\in(0\,,1)$,

$$\int_{|t-s|-2\varepsilon}^{|t-s|+2\varepsilon}\frac{dz}{|z|^\alpha}\le c_4\varepsilon\min\left(\frac{1}{|t-s|}\wedge\frac{1}{\varepsilon}\right)^\alpha\le c_4\frac{\varepsilon}{|t-s|^\alpha}. \tag{4.8}$$

Part two of the lemma follows from this and (3.10). □

Now we prove the first inequality in Theorem 4.1.

Proof of Theorem 4.1: First Half. We can choose some $\mu\in\mathscr{P}(\Xi)$, and deduce from Lemma 4.3 and the Paley–Zygmund inequality (3.4) that $\mathrm{P}_p\{Z_\varepsilon(\mu)>0\}\ge 1/(c^3I_{\phi(\alpha)}(\mu))$. Let Y^ε denote the closed ε-enlargement of $Y_\alpha([1\,,2])$.

Recall that $S(G)$ is closed because $\mathrm{P}_p\{\rho\leftrightarrow\infty\}=0$ (Häggström et al., 1997, Lemma 3.2). Also note that $\{Z_\varepsilon(\mu)>0\}\subseteq\{S(G)\cap D\cap Y^\varepsilon\neq\varnothing\}$. Let $\varepsilon\to0^+$ to obtain the first inequality of Theorem 4.1 after we optimize over $\mu\in\mathscr{P}(\Xi)$. □

The second half of Theorem 4.1 is more difficult to prove. We begin by altering the definition of $Z_\varepsilon(\mu)$ slightly as follows: For all $\varepsilon\in(0\,,1)$ and $\mu\in\mathscr{P}(\Xi)$ define

$$W_\varepsilon(\mu):=\frac{1}{(2\varepsilon)p^n}\int_\Xi\int_1^\infty\mathbf{1}_{\{\rho\overset{t}{\leftrightarrow}v\}\cap\{|Y_\alpha(r)-t|\le\varepsilon\}}e^{-r}\,dr\,\mu(dv\,dt). \tag{4.9}$$

[It might help to recall that n denotes the height of the finite tree G.] We can sharpen the second assertion of Lemma 4.3, and replace $Z_\varepsilon(\mu)$ by $W_\varepsilon(\mu)$, as follows: There exists a constant $c=c(M)>0$ such that

$$\mathrm{E}_p\left[W_\varepsilon^2(\mu)\right]\le cI_{\phi_\varepsilon(\alpha)}(\mu), \tag{4.10}$$

where

$$\phi_\varepsilon(\alpha)\left((v\,,t)\,;(w\,,s)\right):=h\left((v\,,t)\,;(w\,,s)\right)\cdot\left(\frac{1}{|t-s|}\wedge\frac{1}{\varepsilon}\right)^\alpha. \tag{4.11}$$

The aforementioned sharpening rests on (4.8) and not much more. So we omit the details.

Define $\mathscr{Y}(t)$ to be the sigma-algebra generated by $\{Y_\alpha(r)\}_{0\le r\le t}$. We can add to $\mathscr{Y}(t)$ all P-null sets, and even make it right-continuous [with respect to the usual total order on $\mathbf{R}$].

Let us denote the resulting sigma-algebra by $\mathscr{Y}(t)$ still, and the corresponding filtration by $\mathscr{Y}$.

Choose and fix $\sigma, \tau \in \{-\,,+\}$, and for all $v \in \partial G$ and $r, t \geq 0$ define

$$\mathscr{G}_{(\sigma,\tau)}(v\,,t\,,r) := \mathscr{F}_{(\sigma,\tau)}(v\,,t) \times \mathscr{Y}(r). \tag{4.12}$$

We say that $(v\,,t\,,r) \lll_{(\sigma,\tau)} (w\,,s\,,u)$ when $(v\,,t) <_{(\sigma,\tau)} (w\,,s)$ and $r \leq u$. Thus, each $\lll_{(\sigma,\tau)}$ defines a partial order on $\partial G \times \mathbf{R}_+ \times \mathbf{R}_+$.

Choose and fix $\sigma, \tau \in \{-\,,+\}$. Because $\mathscr{F}_{(\sigma,\tau)}$ is a two-parameter, commuting filtration [in the partial order $<_{(\sigma,\tau)}$, and since $\mathscr{Y}$ is the [one-parameter] independent filtration generated by a reversible Feller process, it follows readily that $\mathscr{G}_{(\sigma,\tau)}$ is a three-parameter, commuting filtration in the partial order $\lll_{(\sigma,\tau)}$. In particular, the following analogue of (3.8) is valid: For all $V \in L^2(\mathrm{P}_p)$,

$$\mathrm{E}_p\left(\sup_{(v,t,r)\in\Xi\times\mathbf{R}_+}\left|\mathrm{E}_p\left[V\;\middle|\;\mathscr{F}_{(\sigma,\tau)}(v\,,t\,,r)\right]\right|^2\right) \leq 64\mathrm{E}_p\left[V^2\right]. \tag{4.13}$$

(Khoshnevisan, 2002, Theorem 2.3.2, p. 235).

Next, we note that or all $(w\,,s\,,u) \in \partial G \times D \times [1\,,2]$, and all $\sigma, \tau \in \{-\,,+\}$, the following is valid P_p-almost surely:

$$\begin{aligned}&\mathrm{E}_p\left[W_\varepsilon(\mu)\;\middle|\;\mathscr{G}_{(\sigma,\tau)}(w\,,s\,,u)\right]\\ &\qquad\geq \frac{1}{(2\varepsilon)p^n}\int_{(v,t)<_{(\sigma,\tau)}(w,s)}\int_u^\infty \mathscr{H}e^{-r}\,dr\,\mu(dv\,dt)\cdot\mathbf{1}_{\{\rho\overset{s}{\leftrightarrow}w\}\cap\{|Y_\alpha(u)-s|\leq\varepsilon/2\}}.\end{aligned} \tag{4.14}$$

Here,

$$\begin{aligned}\mathscr{H} &:= \mathrm{P}_p\left(\rho\overset{t}{\leftrightarrow}v\,, |Y_\alpha(r)-t|\leq\varepsilon\;\middle|\;\mathscr{G}_{(\sigma,\tau)}(w\,,s\,,u)\right)\\ &= \mathrm{P}_p\left(\rho\overset{t}{\leftrightarrow}v\;\middle|\;\mathscr{F}_{(\sigma,\tau)}(w\,,s)\right)\times\mathrm{P}\left(|Y_\alpha(r)-t|\leq\varepsilon\mid\mathscr{Y}(u)\right).\end{aligned} \tag{4.15}$$

By the Markov property, P_p-almost surely on $\{\rho\overset{s}{\leftrightarrow}w\}$,

$$\mathrm{P}_p\left(\rho\overset{t}{\leftrightarrow}v\;\middle|\;\mathscr{F}_{(\sigma,\tau)}(w\,,s)\right) = p^n h\left((v\,,t)\,;(w\,,s)\right). \tag{4.16}$$

See (3.10). On the other hand, the Markov property of Y_α dictates that almost surely on $\{|Y_\alpha(u)-s|\leq\varepsilon/2\}$,

$$\mathrm{P}\left(|Y_\alpha(r)-t|\leq\varepsilon\mid\mathscr{Y}(u)\right) \geq \mathrm{P}\left\{|Y_\alpha(r-u)-(t-s)|\leq\frac{\varepsilon}{2}\right\} := \mathscr{A}. \tag{4.17}$$

Note that because $u \in [1\,,2]$,

$$\begin{aligned}\int_u^\infty \mathscr{A} e^{-r}\,dr &\ge \frac{1}{e^2}\int_0^\infty \mathrm{P}\left\{|Y_\alpha(r)-(t-s)|\le \frac{\varepsilon}{2}\right\} e^{-r}\,dr\\ &= \frac{1}{e^2}\int_{|t-s|-(\varepsilon/2)}^{|t-s|+(\varepsilon/2)} u(z)\,dz\\ &\ge c_5(2\varepsilon)\min\left(\frac{1}{|t-s|}\wedge\frac{1}{\varepsilon}\right)^\alpha,\end{aligned}\tag{4.18}$$

where c_5 does not depend on $(\varepsilon\,,\mu\,;t\,,s)$. The ultimate inequality follows from a similar argument that was used earlier to derive (4.8). So we omit the details.

Thus, we can plug the preceding bounds into (4.14) and deduce that P_p-a.s.,

$$\begin{aligned}&\mathrm{E}_p\left[W_\varepsilon(\mu)\,\middle|\,\mathscr{G}_{(\sigma,\tau)}(w\,,s\,,u)\right]\\ &\qquad\ge c_5\int_{(v,t)<_{(\sigma,\tau)}(w,s)} \phi_\varepsilon(\alpha)\Big((v\,,t)\,;(w\,,s)\Big)\,\mu(dv\,dt)\times \mathbf{1}_{\{\rho\overset{s}{\leftrightarrow} w\}\cap\{|Y_\alpha(u)-s|\le\varepsilon/2\}}.\end{aligned}\tag{4.19}$$

Moreover, it is possible to check that there exists one null set outside which the preceding holds for all $(w\,,s\,,u)\in\Xi\times[1\,,2]$. We are in a position to complete our proof of Theorem 4.1.

Proof of Theorem 4.1: Second Half. Without loss of generality, we may assume that $\mathrm{P}_p\{S(G)\cap D\cap\overline{Y_\alpha([1\,,2])}\}>0$, for otherwise there is nothing to prove.

Let us introduce two abstract cemetery states: $\delta\notin\partial G$ and $\infty\notin\mathbf{R}_+$. Then, there exists a map $(\mathbf{w}_\varepsilon\,,\mathbf{s}_\varepsilon\,,\mathbf{u}_\varepsilon):\Omega\mapsto(\partial G\cup\{\delta\})\times(D\cup\{\infty\})\times([1\,,2]\cup\{\infty\})$ with the properties that:

(1) $(\mathbf{w}_\varepsilon\,,\mathbf{s}_\varepsilon\,,\mathbf{u}_\varepsilon)(\omega)\neq(\delta\,,\infty\,,\infty)$ if and only if there exists $(w\,,s\,,u)(\omega)\in\Xi\times[1\,,2]$ such that $\rho\overset{s(\omega)}{\leftrightarrow}w(\omega)$ and $|Y_\alpha(u)-s|(\omega)\le\varepsilon/2$; and
(2) If $(\mathbf{w}_\varepsilon\,,\mathbf{s}_\varepsilon\,,\mathbf{u}_\varepsilon)(\omega)\neq(\delta\,,\infty\,,\infty)$, then (1) holds with $(\mathbf{w}_\varepsilon\,,\mathbf{s}_\varepsilon\,,\mathbf{u}_\varepsilon)(\omega)$ in place of $(w\,,s\,,u)(\omega)$.

Consider the event,

$$H(\varepsilon):=\{\omega:\ (\mathbf{w}_\varepsilon\,,\mathbf{s}_\varepsilon\,,\mathbf{u}_\varepsilon)(\omega)\neq(\delta\,,\infty\,,\infty)\}\,.\tag{4.20}$$

Thus, we can deduce that $\mu_\varepsilon\in\mathscr{P}(\Xi)$, where

$$\mu_\varepsilon(A\times B):=\mathrm{P}_p\left((\mathbf{w}_\varepsilon\,,\mathbf{s}_\varepsilon)\in A\times B\mid H(\varepsilon)\right),\tag{4.21}$$

valid for all measurable $A\times B\subseteq\Xi$.

Because of (3.5), we may apply (4.19) with μ_ε in place of μ and $(\mathbf{w}_\varepsilon\,, \mathbf{s}_\varepsilon\,, \mathbf{u}_\varepsilon)$ in place of $(w\,, s\,, u)$ to find that P_p-a.s.,

$$\begin{aligned}\sum_{\sigma,\tau\in\{-,+\}} \sup_{(w,s,u)\in\Xi\times[1,2]} \mathrm{E}_p\left[W_\varepsilon(\mu_\varepsilon)\;\middle|\;\mathscr{G}_{(\sigma,\tau)}(w\,,s\,,u)\right] \\ \geq c_5 \int_\Xi \phi_\varepsilon(\alpha)\Big((v\,,t)\,;(\mathbf{w}_\varepsilon\,,\mathbf{s}_\varepsilon)\Big)\,\mu_\varepsilon(dv\,dt)\cdot \mathbf{1}_{H(\varepsilon)}.\end{aligned} \tag{4.22}$$

We can square both ends of this inequality and take expectations $[\mathrm{P}_p]$. Owing to (3.15), the expectation of the square of the left-most term is at most

$$\begin{aligned}4\sum_{\sigma,\tau\in\{-,+\}} \mathrm{E}_p\left(\sup_{(w,s,u)\in\Xi\times[1,2]} \left|\mathrm{E}_p\left[W_\varepsilon(\mu_\varepsilon)\;\middle|\;\mathscr{G}_{(\sigma,\tau)}(w\,,s\,,u)\right]\right|^2\right) &\leq 1024\mathrm{E}_p\left[W_\varepsilon^2(\mu_\varepsilon)\right] \\ &\leq 1024 c I_{\phi_\varepsilon(\alpha)}(\mu_\varepsilon).\end{aligned} \tag{4.23}$$

See (4.13) and (4.10). We emphasize that the constant c is finite and positive, and does not depend on $(\varepsilon\,, \mu_\varepsilon)$.

On the other hand, by the Cauchy–Schwarz inequality, the expectation of the square of the right-most term in (4.22) is equal to

$$\begin{aligned}& c_5^2 \mathrm{E}_p\left[\left(\int_\Xi \phi_\varepsilon(\alpha)\Big((v\,,t)\,;(\mathbf{w}_\varepsilon\,,\mathbf{s}_\varepsilon)\Big)\,\mu_\varepsilon(dv\,dt)\right)^2\;\middle|\;H(\varepsilon)\right]\mathrm{P}_p(H(\varepsilon)) \\ &\geq c_5^2\left(\mathrm{E}_p\left[\int_\Xi \phi_\varepsilon(\alpha)\Big((v\,,t)\,;(\mathbf{w}_\varepsilon\,,\mathbf{s}_\varepsilon)\Big)\,\mu_\varepsilon(dv\,dt)\;\middle|\;H(\varepsilon)\right]\right)^2 \mathrm{P}_p(H(\varepsilon)) \\ &= c_5^2\left(I_{\phi_\varepsilon(\alpha)}(\mu_\varepsilon)\right)^2 \mathrm{P}_p(H(\varepsilon)).\end{aligned} \tag{4.24}$$

We can combine (4.23) with (4.24) to find that for all $N>0$,

$$\mathrm{P}_p(H(\varepsilon)) \leq \frac{1024c}{c_5^2}\left[I_{\phi_\varepsilon(\alpha)}(\mu_\varepsilon)\right]^{-1} \leq \frac{1024c}{c_5^2}\left[I_{N\wedge\phi_\varepsilon(\alpha)}(\mu_\varepsilon)\right]^{-1}. \tag{4.25}$$

Perhaps it is needless to say that $N\wedge\phi_\varepsilon(\alpha)$ is the function whose evaluation at $((v\,,t)\,,(w\,,s))\in\Xi\times\Xi$ is the minimum of the constant N and $\phi_\varepsilon(\alpha)((v\,,t)\,(w\,,s))$. Evidently, $N\wedge\phi_\varepsilon(\alpha)$ is bounded and lower semicontinuous on $\Xi\times\Xi$. By compactness we can find $\mu_0\in\mathscr{P}(\Xi)$ such that μ_ε converges weakly to μ_0. As $\varepsilon\downarrow 0$, the sets $H(\varepsilon)$ decrease set-theoretically, and their intersection includes $\{S(G)\cap D\cap\overline{Y_\alpha([1\,,2])}\neq\varnothing\}$. As a result we have

$$\mathrm{P}_p\left\{S(G)\cap D\cap\overline{Y_\alpha([1\,,2])}\neq\varnothing\right\} \leq \frac{1024c}{c_5^2}\left[I_{N\wedge\phi(\alpha)}(\mu_0)\right]^{-1}. \tag{4.26}$$

Let $N\uparrow\infty$ and appeal to the monotone convergence theorem to find that

$$\mathrm{P}_p\left\{S(G)\cap D\cap\overline{Y_\alpha([1\,,2])}\neq\varnothing\right\} \leq \frac{1024c}{c_5^2}\left[I_{\phi(\alpha)}(\mu_0)\right]^{-1} \leq \frac{1024c}{c_5^2}\mathrm{Cap}_{\phi(\alpha)}(D). \tag{4.27}$$

This concludes the proof of Theorem 4.1. $\square$

Proof of Theorem 2.5. Define $R_\alpha := \overline{Y_\alpha([1\,,2])}$, and recall the theorem of McKean (1955): *For all Borel sets $G \subseteq \mathbf{R}$,*

$$\mathrm{P}\left\{R_\alpha \cap G \neq \varnothing\right\} > 0 \quad \Longleftrightarrow \quad \mathrm{Cap}_\alpha(G) > 0. \tag{4.28}$$

Here, $\mathrm{Cap}_\alpha(G)$ denotes the α-dimensional (Bessel-) Riesz capacity of G (Khoshnevisan, 2002, Appendix C). That is,

$$\mathrm{Cap}_\alpha(G) := \left[\inf_{\mu \in \mathscr{P}(G)} I_\alpha(\mu)\right]^{-1}, \tag{4.29}$$

where

$$I_\alpha(\mu) := \iint \frac{\mu(dx)\,\mu(dy)}{|x-y|^\alpha}. \tag{4.30}$$

Now let $R_\alpha^1, R_\alpha^2, \ldots$ be i.i.d. copies of R_α, all independent of our dynamical percolation process as well. Then, by the Borel–Cantelli lemma,

$$\mathrm{P}\left\{\bigcup_{j=1}^\infty R_\alpha^j \cap G \neq \varnothing\right\} = \begin{cases} 1, & \text{if } \mathrm{Cap}_\alpha(G) > 0, \\ 0, & \text{if } \mathrm{Cap}_\alpha(G) = 0. \end{cases} \tag{4.31}$$

Set $G := S(G) \cap D$ and condition, once on G and once on $\cup_{j=1}^\infty R_\alpha^j$. Then, the preceding and Theorem 4.1 together imply that

$$\mathrm{P}_p\Big\{\mathrm{Cap}_\alpha\left(S(G) \cap D\right) > 0\Big\} = \begin{cases} 1, & \text{if } \quad \mathrm{Cap}_{\phi(\alpha)}(D) > 0, \\ 0, & \text{if } \quad \mathrm{Cap}_{\phi(\alpha)}(D) = 0. \end{cases} \tag{4.32}$$

The remainder of the theorem follows from Frostman's theorem (1935): *For all Borel sets $F \subset \mathbf{R}$, $\dim_{\mathrm{H}} F = \sup\{0 < \alpha < 1 : \ \mathrm{Cap}_\alpha(F) > 0\}$.* For a pedagogic account see Khoshnevisan (2002, Theorem 2.2.1, p. 521). $\square$

Remark 4.4. An inspection of our proof reveals the following stronger fact: *Outside a single P_p-null set,*

$$\mathrm{Cap}_\alpha\left(S(G) \cap D\right) > 0 \iff \mathrm{Cap}_{\phi(\alpha)}(D) > 0, \qquad \text{for all } \alpha \in (0\,,1). \tag{4.33}$$

Indeed, if "for all α" is replaced by "for all rational α," then this follows immediately from (4.32). The full assertion is a consequence of the mentioned one and the fact that $\alpha \mapsto \mathrm{Cap}_\alpha(S(G) \cap D)$ and $\alpha \mapsto \mathrm{Cap}_{\phi(\alpha)}(D)$ are both non-increasing functions.

5. On Spherically Symmetric Trees

Suppose G is spherically symmetric, and let $\mathfrak{m}_{\partial G}$ denote the uniform measure on ∂G. One way to define m is as follows: For all $f : \mathbf{Z}_+ \to \mathbf{R}_+$ and all $v \in \partial G$,

$$\int_{\partial G} f(|v \wedge w|)\, \mathfrak{m}_{\partial G}(dw) = \sum_{l=0}^{n-1} \frac{f(l)}{|G_{l+1}|}, \tag{5.1}$$

where $n \in \mathbf{Z}_+ \cup \{\infty\}$ denotes the height of G. In particular, we may note that if G is infinite then for all $\nu \in \mathscr{P}(\mathbf{R}_+)$,

$$I_h(\mathfrak{m}_{\partial G} \times \nu) = \iint \sum_{l=0}^{\infty} \frac{1}{|G_{l+1}|} \left(1 + \frac{q}{p} e^{-|t-s|}\right)^l \nu(ds)\, \nu(dt). \tag{5.2}$$

This is the integral in (1.4).

Yuval Peres asked us if the constant $960e^3 \approx 19282.1$ in (2.5) can be improved upon. The following answers this question by replacing $960e^3$ by 512. Although we do not know how to improve this constant further, it seems unlikely to be the optimal one.

Theorem 5.1. *Suppose G is an infinite, spherically symmetric tree, and $\mathrm{P}_p\{\rho \leftrightarrow \infty\} = 0$. Then, for all compact sets $D \subseteq [0\,,1]$,*

$$\frac{1}{2 \inf_{\nu \in \mathscr{P}(D)} I_h(\mathfrak{m}_{\partial G} \times \nu)} \le \mathrm{P}_p\left(\bigcup_{t \in D} \left\{\rho \overset{t}{\leftrightarrow} \infty\right\}\right) \le \frac{512}{\inf_{\nu \in \mathscr{P}(D)} I_h(\mathfrak{m}_{\partial G} \times \nu)}, \tag{5.3}$$

where $\inf \varnothing := \infty$ and $1/\infty := 0$. The condition that $\mathrm{P}_p\{\rho \leftrightarrow \infty\} = 0$ is not needed for the upper bound.

The proof follows that of Theorem 2.1 closely. Therefore, we sketch the highlights of the proof only.

Sketch of Proof. The lower bound follows immediately from (2.4), so we concentrate on the upper bound only. As we have done before, we may, and will, assume without loss of generality that G is a finite tree of height n.

For all $\nu \in \mathscr{P}(D)$ consider $Z(\mathfrak{m}_{\partial G} \times \nu)$ defined in (3.2). That is,

$$Z(\mathfrak{m}_{\partial G} \times \nu) = \frac{1}{p^n |G_n|} \int_D \sum_{v \in G_n} \mathbf{1}_{\{\rho \overset{t}{\leftrightarrow} v\}}\, \nu(dt). \tag{5.4}$$

It might help to point out that in the present setting, G_n is identified with ∂G. According to (3.3),

$$\mathrm{E}_p[Z(\mathfrak{m}_{\partial G} \times \nu)] = 1 \quad \text{and} \quad \mathrm{E}_p\left[Z^2(\mathfrak{m}_{\partial G} \times \nu)\right] \le 2 I_h(\mathfrak{m}_{\partial G} \times \nu). \tag{5.5}$$

In accord with (3.12), outside a single null set, the following holds for all $w \in \partial G$, $\sigma, \tau \in \{-\,,+\}$, and $s \geq 0$:

$$
\begin{aligned}
&\sum_{\sigma,\tau\in\{-,+\}} \mathrm{E}_p\left[Z(\mathfrak{m}_{\partial G}\times\nu) \,\middle|\, \mathscr{F}_{(\sigma,\tau)}(w\,,s)\right] \\
&\qquad \geq \int_{(v,t)\in\Xi} \left(1+\frac{q}{p}e^{-|t-s|}\right)^{|v\wedge w|} (\mathfrak{m}_{\partial G}\times\nu)(dv\,dt)\cdot \mathbf{1}_{\{\rho \overset{s}{\leftrightarrow} w\}} \\
&\qquad = \int_D \sum_{l=0}^{n-1} \frac{1}{|G_{l+1}|}\left(1+\frac{q}{p}e^{-|t-s|}\right)^{l} \nu(dt)\times \mathbf{1}_{\{\rho \overset{s}{\leftrightarrow} w\}} \\
&\qquad = I_h(\mathfrak{m}_{\partial G}\times\nu)\cdot \mathbf{1}_{\{\rho \overset{s}{\leftrightarrow} w\}}.
\end{aligned} \tag{5.6}
$$

See (5.1) for the penultimate line. Next we take the supremum of the left-most term over all w, and replace w by $\mathbf{w}$ in the right-most term; then square and take expectations, as we did in the course of the proof of Theorem 2.1. $\square$

Finally, let us return briefly to the Hausdorff dimension of $S(G)\cap D$ in the case that G is spherically symmetric. First we recall Theorem 1.6 of Häggström et al. (1997): *If* $\mathrm{P}_p(\cup_{t\in D}\{\rho \overset{t}{\leftrightarrow} \infty\})=1$ *then* P_p*-a.s.,*

$$
\dim_{\text{H}} S(G) = \sup\left\{\alpha>0:\ \sum_{l=1}^{\infty}\frac{p^{-l}l^{\alpha-1}}{|G_l|}<\infty\right\}. \tag{5.7}
$$

Next we announce the Hausdorff dimension of $S(G)\cap D$ in the case that G is spherically symmetric.

Theorem 5.2. *Suppose that G is an infinite, spherically symmetric tree, and* $\mathrm{P}_p(\cup_{t\in D}\{\rho \overset{t}{\leftrightarrow} \infty\})=1$*. Then, for all compact sets $D\subseteq[0\,,1]$,*

$$
\dim_{\text{H}}\left(S(G)\cap D\right) = \sup\left\{0<\alpha<1:\ \inf_{\nu\in\mathscr{P}(D)} I_{\phi(\alpha)}(\mathfrak{m}_{\partial G}\times\nu)<\infty\right\}, \tag{5.8}
$$

P_p*-almost surely.*

The strategy of the proof is exactly the same as that of the proof of Theorem 2.5, but we use $Z(\mathfrak{m}_{\partial G}\times\nu)$ in place of $Z(\mu)$. The minor differences in the proofs are omitted here.

For the purposes of comparison, we mention the following consequence of (5.1): For all $\nu\in\mathscr{P}(\mathbf{R}_+)$,

$$
\begin{aligned}
I_{\phi(\alpha)}(\mathfrak{m}_{\partial G}\times\nu) &= \iint \sum_{l=0}^{\infty}\frac{1}{|G_{l+1}|}\left(1+\frac{q}{p}e^{-|t-s|}\right)^{l}\frac{\nu(ds)\,\nu(dt)}{|t-s|^{\alpha}} \\
&= \iint \sum_{l=0}^{\infty}\frac{p^{-l}}{|G_{l+1}|}\left(1-q\left\{1-e^{-|t-s|}\right\}\right)^{l}\frac{\nu(ds)\,\nu(dt)}{|t-s|^{\alpha}}.
\end{aligned} \tag{5.9}
$$

It may appear that this expression is difficult to work with. To illustrate that this is not always the case we derive the following bound which may be of independent interest. Our next result computes the Hausdorff dimension of $S(G) \cap D$ in case that D is a nice fractal; i.e., one whose packing and Hausdorff dimensions agree. Throughout, $\dim_{\mathrm{P}}$ denotes packing dimension (Tricot, 1982; Sullivan, 1984).

Proposition 5.3. *Suppose G is an infinite spherically symmetric tree. Suppose also that $\mathrm{P}_p(\cup_{t\in D}\{\rho \overset{t}{\leftrightarrow} \infty\}) > 0$ for a certain non-random compact set $D \subseteq \mathbf{R}_+$. If $\delta := \dim_{\mathrm{H}} D$ and $\Delta := \dim_{\mathrm{P}} D$, then P_p-almost surely on $\cup_{t\in D}\{\rho \overset{t}{\leftrightarrow} \infty\}$,*

$$[\dim_{\mathrm{H}} S(G) - (1-\delta)]_+ \le \dim_{\mathrm{H}} (S(G)\cap D) \le [\dim_{\mathrm{H}} S(G) - (1-\Delta)]_+ . \tag{5.10}$$

Proof. Without loss of generality, we may assume that G has no leaves [except ρ].

The condition $\mathrm{P}_p(\cup_{t\in D}\{\rho \overset{t}{\leftrightarrow} \infty\}) > 0$ and ergodicity together prove that there a.s. $[\mathrm{P}_p]$ exists a time t of percolation. Therefore, (1.3) implies that

$$\sum_{i=1}^{\infty} \frac{p^{-l}}{l|G_l|} < \infty. \tag{5.11}$$

This is in place throughout. Next we proceed with the harder lower bound first. Without loss of generality, we may assume that $\dim_{\mathrm{H}} S(G) > 1-\delta$, for otherwise there is nothing left to prove.

According to Frostman's lemma, there exists $\nu \in \mathscr{P}(D)$ such that for all $\varepsilon > 0$ we can find a constant C_ε with the following property:

$$\sup_{x\in D} \nu\left([x-r\,,x+r]\right) \le C_\varepsilon r^{\delta-\varepsilon}, \qquad \text{for all} \quad r>0. \tag{5.12}$$

(Khoshnevisan, 2002, Theorem 2.1.1, p. 517.) We shall fix this ν throughout the derivation of the lower bound.

Choose and fix α that satisfies

$$0 < \alpha < \frac{\dim_{\mathrm{H}} S(G) - 1}{1-\varepsilon} + \delta - \varepsilon. \tag{5.13}$$

[Because we assumed that $\dim_{\mathrm{H}} S(G) > 1-\delta$ the preceding bound is valid for all $\varepsilon > 0$ sufficiently small. Fix such a ε as well.] For this particular $(\nu\,,\alpha\,,\varepsilon)$ we apply to (5.9) the elementary bound $1 - q\{1-e^{-x}\} \le \exp(-qx/2)$, valid for all $0 \le x \le 1$, and obtain

$$I_{\phi(\alpha)}(\mathfrak{m}_{\partial G} \times \nu) \le \sum_{l=0}^{\infty} \frac{p^{-l}}{|G_{l+1}|} \iint \exp\left(-\frac{ql|t-s|}{2}\right) \frac{\nu(ds)\,\nu(dt)}{|t-s|^{\alpha}}. \tag{5.14}$$

We split the integral in two parts, according to whether or not $|t-s|$ is small, and deduce that

$$I_{\phi(\alpha)}(\mathfrak{m}_{\partial G}\times\nu) \le \sum_{l=0}^{\infty}\frac{p^{-l}}{|G_{l+1}|}\iint_{|t-s|\le l^{-(1-\varepsilon)}}\frac{\nu(ds)\,\nu(dt)}{|t-s|^{\alpha}} + \sum_{l=0}^{\infty}\frac{p^{-l}l^{(1-\varepsilon)\alpha}e^{-(ql^{\varepsilon})/2}}{|G_{l+1}|}. \tag{5.15}$$

Thanks to (5.11) the last term is a finite number, which we call K_ε. Thus,

$$I_{\phi(\alpha)}(\mathfrak{m}_{\partial G}\times\nu) \le \sum_{l=0}^{\infty}\frac{p^{-l}}{|G_{l+1}|}\iint_{|t-s|\le l^{-(1-\varepsilon)}}\frac{\nu(ds)\,\nu(dt)}{|t-s|^{\alpha}} + K_\varepsilon. \tag{5.16}$$

Integration by parts shows that if $f:\mathbf{R}\to\mathbf{R}_+\cup\{\infty\}$ is even, as well as right-continuous and non-increasing on $(0\,,\infty)$, then for all $0<a<b$,

$$\iint_{a\le|t-s|\le b} f(s-t)\,\nu(ds)\,\nu(dt) = f(x)F_\nu(x)\Big|_a^b + \int_a^b F_\nu(x)\,d|f|(x), \tag{5.17}$$

where $F_\nu(x) := (\nu\times\nu)\{(s\,,t)\in\mathbf{R}_+^2:\ |s-t|\le x\}\le C_\varepsilon x^{\delta-\varepsilon}$ thanks to (5.12). We apply this bound with $a\downarrow 0$, $b:=l^{-(1-\varepsilon)}$, and $f(x):=|x|^{-\alpha}$ to deduce that

$$\iint_{|t-s|\le l^{-(1-\varepsilon)}}\frac{\nu(ds)\,\nu(dt)}{|t-s|^{\alpha}} \le A l^{(1-\varepsilon)(\alpha-\delta+\varepsilon)}, \tag{5.18}$$

since $\alpha<\delta-\varepsilon$ by (5.13). Here, $A:=C_\epsilon(\delta-\varepsilon)/(-\alpha+\delta-\varepsilon)$. Consequently,

$$I_{\phi(\alpha)}(\mathfrak{m}_{\partial G}\times\nu) \le A\sum_{l=0}^{\infty}\frac{p^{-l}l^{(1-\varepsilon)(\alpha-\delta+\varepsilon)}}{|G_{l+1}|} + K_\varepsilon, \tag{5.19}$$

which is finite thanks to (5.13) and (5.7). It follows from Theorem 5.2 that P_p-almost surely, $\dim_{\mathrm{H}}(S(G)\cap D)\ge\alpha$. Let $\varepsilon\downarrow 0$ and $\alpha\uparrow\dim_{\mathrm{H}} S(G)-1+\delta$ in (5.13) to obtain the desired lower bound.

Choose and fix $\beta>\Delta$ and $\alpha>\dim_{\mathrm{H}} S(G)+\beta-1$. We appeal to (5.9) and the following elementary bound: For all integers $l\ge 1$ and all $0\le x\le 1/l$, we have $(1-q\{1-e^{-x}\})^l\ge p$. It follows from this and (5.9) that for all $\nu\in\mathscr{P}(E)$,

$$\begin{aligned} I_{\phi(\alpha)}(\mathfrak{m}_{\partial G}\times\nu) &\ge p\sum_{l=0}^{\infty}\frac{p^{-l}}{|G_{l+1}|}\iint_{|t-s|\le l^{-1}}\frac{\nu(ds)\,\nu(dt)}{|t-s|^{\alpha}} \\ &\ge p\sum_{l=0}^{\infty}\frac{p^{-l}l^{\alpha}}{|G_{l+1}|}\int\nu\left(\left(t-\frac{1}{l}\ ,\ t+\frac{1}{l}\right)\right)\nu(dt). \end{aligned} \tag{5.20}$$

Because $\beta>\dim_{\mathrm{P}} D$, the density theorem of Taylor and Tricot (1985, Theorem 5.4) implies that

$$\liminf_{\varepsilon\to 0^+}\frac{1}{\varepsilon^{\beta}}\int\nu\left((t-\varepsilon\ ,\ t+\varepsilon)\right)\nu(dt) \ge \int\liminf_{\varepsilon\to 0^+}\frac{\nu\left((t-\varepsilon\ ,\ t+\varepsilon)\right)}{\varepsilon^{\beta}}\nu(dt)=\infty. \tag{5.21}$$

We have also applied Fatou's lemma. Thus, there exists $c > 0$ such that for all $\nu \in \mathscr{P}(E)$,

$$I_{\phi(\alpha)}(\mathfrak{m}_{\partial G} \times \nu) \geq c \sum_{l=0}^{\infty} \frac{p^{-l} l^{\alpha-\beta}}{|G_{l+1}|} = \infty; \tag{5.22}$$

see (5.7). It follows from Theorem 5.2 that P_p-almost surely, $\dim_{\mathrm{H}}(S(G) \cap D) \leq \alpha$. Let $\beta \downarrow \Delta$ and then $\alpha \downarrow \dim_{\mathrm{H}} S(G) + \Delta - 1$, in this order, to finish. $\square$

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