EXCEPTIONAL TIMES AND INVARIANCE FOR DYNAMICAL RANDOM WALKS

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ABSTRACT. Consider a sequence $\{X_i(0)\}_{i=1}^n$ of i.i.d. random variables. Associate to each $X_i(0)$ an independent mean-one Poisson clock. Every time a clock rings replace that X-variable by an independent copy and restart the clock. In this way, we obtain i.i.d. stationary processes $\{X_i(t)\}_{t\geq 0}$ $(i = 1, 2, \cdots)$ whose invariant distribution is the law ν of $X_1(0)$.

Benjamini et al. (2003) introduced the dynamical walk $S_n(t) = X_1(t) + \cdots + X_n(t)$, and proved among other things that the LIL holds for $n \mapsto S_n(t)$ for all t. In other words, the LIL is dynamically stable. Subsequently (2004b), we showed that in the case that the $X_i(0)$'s are standard normal, the classical integral test is not dynamically stable.

Presently, we study the set of times t when $n \mapsto S_n(t)$ exceeds a given envelope infinitely often. Our analysis is made possible thanks to a connection to the Kolmogorov ε-entropy. When used in conjunction with the invariance principle of this paper, this connection has other interesting by-products some of which we relate.

We prove also that the infinite-dimensional process $t \mapsto S_{\lfloor n\bullet \rfloor}(t)/\sqrt{n}$ converges weakly in $\mathscr{D}(\mathscr{D}([0,1]))$ to the Ornstein–Uhlenbeck process in $\mathscr{C}([0,1])$. For this we assume only that the increments have mean zero and variance one.

In addition, we extend a result of Benjamini et al. (2003) by proving that if the $X_i(0)$'s are lattice, mean-zero variance-one, and possess $2+\varepsilon$ finite absolute moments for some $\varepsilon > 0$, then the recurrence of the origin is dynamically stable. To prove this we derive a gambler's ruin estimate that is valid for all lattice random walks that have mean zero and finite variance. We believe the latter may be of independent interest.

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CONTENTS

1. INTRODUCTION AND MAIN RESULTS

Let $\{\xi_j^k\}_{j,k=0}^\infty$ denote a double-array of i.i.d. real-valued random variables with common distribution ν. Also let $\{\mathscr{C}_n\}_{n=1}^\infty$ denote a sequence of rate-one Poisson clocks that are totally independent from themselves as well as the ξ's. If the jump times of $\mathscr{C}_n$ are denoted by $0 = \tau_n(0) < \tau_n(1) < \tau_n(2) < \cdots$, then we define the discrete-time function-valued process $X = \{X_n(t);\, t \geq 0\}_{n=1}^\infty$ as follows: For all $n \geq 1$,

$$X_n(t) = \xi_n^k, \quad \text{if } \tau_n(k) \leq t < \tau_n(k+1). \tag{1.1}$$

For every $n \geq 1$, X_n is the random step function which starts, at time zero, at the value ξ_n^0. Then it proceeds iteratively by replacing its previous value by an independent copy every time the clock $\mathscr{C}_n$ rings. As a process indexed by t, $t \mapsto (X_1(t), X_2(t), \dots)$ is a stationary Markov process in $\mathbf{R}^\infty$, and its invariant measure is ν^∞.

The *dynamical walk* corresponding to the X's is the random field

$$S_n(t) = X_1(t) + \cdots + X_n(t), \qquad t \geq 0,\ n \geq 1. \tag{1.2}$$

One can think of the ensuing object in different ways. We take the following points of view interchangeably:

(1) For a given $t \geq 0$, $\{S_n(t)\}_{n=1}^\infty$ is a classical random walk with increment-distribution ν.
(2) For a given $n \geq 1$, $\{S_n(t)\}_{t\geq 0}$ is a right-continuous stationary Markov process in $\mathbf{R}$ whose invariant measure is $\nu * \cdots * \nu$ (n times).
(3) The process $\{S_n(\cdot)\}_{n=1}^\infty$ is a random walk with values in the Skorohod space $\mathscr{D}([0,1])$.

(4) The $\mathbf{R}^\infty$-valued process $t \mapsto (S_1(t), S_2(t), \ldots)$ is right-continuous, stationary, and Markov. Moreover, its invariant measure is the evolution law of a classical random walk with increment-distribution ν.

Dynamical walks were introduced recently by I. Benjamini, O. Häggström, Y. Peres, and J. Steif (2003) who posed the following question:

(1.3) Which a.s.-properties of the classical random walk $[\nu]$ hold simultaneously for all $t \in [0,1]$?

Random-walk properties that satisfy (1.3) are called *dynamically stable*; all others are called *dynamically sensitive*. This definition was introduced by Benjamini et al. (2003) who proved, among many other things, that if $\int_{-\infty}^{\infty} x^2\,\nu(dx)$ is finite then:

(1.4) The law of the iterated logarithm is dynamically stable.

In order to write this out properly, let us assume, without loss of generality, that $\int x\,\nu(dx) = 0$ and $\int x^2\,\nu(dx) = 1$. For any non-decreasing measurable function $H : \mathbf{R}_+ \to \mathbf{R}_+$ define

$$\Lambda_H = \left\{ t \in [0,1] :\ S_n(t) > H(n)\sqrt{n} \text{ infinitely often } [n] \right\}. \tag{1.5}$$

In words, Λ_H denotes the set of times $t \in [0,1]$ when $H(n)\sqrt{n}$ fails to be in the upper class [in the sense of P. Lévy] of the process $\{S_n(t)\}_{n=1}^{\infty}$. According to the Hewitt–Savage zero-one law, the event $\{\Lambda_H \neq \varnothing\}$ has probability zero or one.

Now set $H(n) = \sqrt{2c \ln\ln n}$. Then, dynamical stability of the LIL (1.4) is equivalent to the statement that $\Lambda_H = \varnothing$ a.s. if $c > 1$, whereas $\Lambda_H = [0,1]$ a.s. if $c < 1$. After ignoring a null set, we can write this in the following more conventional form:

$$\limsup_{n\to\infty} \frac{S_n(t)}{\sqrt{2n \ln\ln n}} = 1, \qquad {}^\forall t \in [0,1]. \tag{1.6}$$

Despite this, in the case that ν is standard normal, we have:

(1.7) The characterization of the upper class of a Gaussian random walk is dynamically sensitive.

Let Φ denote the standard normal distribution function and define $\bar\Phi = 1 - \Phi$. Recall that $H(n)\sqrt{n}$ is in the upper class of $S_n(0)$ if and only if $\int_1^\infty H^2(t)\bar\Phi(H(t))\,dt/t < \infty$ (Erdős, 1942). Then, (1.7) is a consequence of Erdős's theorem, used in conjunction with the following recent result (Khoshnevisan et al., 2004b, Theorem 1.5):

$$\Lambda_H \neq \varnothing \iff \int_1^\infty H^4(t)\bar\Phi(H(t))\,\frac{dt}{t} = \infty. \tag{1.8}$$

This leaves open the following natural question: Given a non-decreasing function H, how large is the collection of all times $t \in [0,1]$ at which H fails to be in the upper class of $\{S_n(t)\}_{n=1}^{\infty}$? In other words, we ask, "*How large is* Λ_H"? Define

$$\delta(H) = \sup\left\{ \zeta > 0 : \int_1^\infty H^\zeta(t)\bar\Phi(H(t))\,\frac{dt}{t} < \infty \right\}, \tag{1.9}$$

where $\sup\varnothing = 0$. The following describes the size of Λ_H in terms of its Hausdorff–Besicovitch dimension $\dim_{\mathscr{H}} \Lambda_H$.

Theorem 1.1. *Suppose ν is standard normal and $H : \mathbf{R}_+ \to \mathbf{R}_+$ is non-random and non-decreasing. Then with probability one,*

$$\dim_{\mathscr{H}} \Lambda_H = \min\left(1\,,\; \frac{4-\delta(H)}{2}\right), \tag{1.10}$$

where $\dim_{\mathscr{H}} A < 0$ means that A is empty.

In order to prove this we develop a series of technical results of independent interest. We describe one of them next.

First define, for any $\varepsilon > 0$, $k = \mathrm{K}_E(\varepsilon)$ to be the maximal number of points $x_1, \ldots, x_k \in E$ such that whenever $i \neq j$, $|x_i - x_j| \geq \varepsilon$. The function K_E is known as the *Kolmogorov ε-entropy* of E (Tihomirov, 1963), as well as the *packing number (or function)* of E (Mattila, 1995). Now suppose $\{z_j\}_{j=1}^\infty$ is any sequence of real numbers that satisfies

$$\inf_n z_n \geq 1, \; \lim_{n\to\infty} z_n = \infty, \;\text{ and }\; \lim_{n\to\infty} \frac{z_n}{n^{1/4}} = 0. \tag{1.11}$$

Then we have the following estimate; it expresses how the geometry of E affects probabilities of moderate deviations.

Theorem 1.2. *Let ν be standard normal. In addition, choose and fix a sequence $\{z_j\}_{j=1}^\infty$ that satisfies* (1.11). *Then there exists a finite constant $A_{1.12} > 1$ such that for all $n \geq 1$ and all non-empty non-random measurable sets $E \subseteq [0,1]$,*

$$A_{1.12}^{-1}\mathrm{K}_E\left(\frac{1}{z_n^2}\right)\bar{\Phi}(z_n) \leq \mathrm{P}\left\{\sup_{t\in E} S_n(t) \geq z_n\sqrt{n}\right\} \leq A_{1.12}\mathrm{K}_E\left(\frac{1}{z_n^2}\right)\bar{\Phi}(z_n). \tag{1.12}$$

Theorem 2.5 below appeals to Theorem 1.2 to characterize all non-random Borel sets $E \subseteq [0,1]$ that intersect Λ_H. Our characterization is not so easy to describe here in the Introduction. For now, suffice it to say that it readily yields Theorem 1.1. The following is another consequence of the said characterization: If ν is standard normal, then

$$\sup_{t\in E} \limsup_{n\to\infty} \frac{(S_n(t))^2 - 2n\ln\ln n}{n\ln\ln\ln n} = 3 + 2\dim_{\mathscr{P}} E. \tag{1.13}$$

Here, $\dim_{\mathscr{P}}$ denotes packing dimension (Mattila, 1995). The preceding display follows from (2.22) below.

On one hand, if we set E to be the entire interval $[0,1]$, then the right-hand side of (1.13) is equal to 5, and we obtain an earlier result of ours (2004b, Eq. 1.15). On the other hand, if we set E to be a singleton, then the right-hand side of (1.13) is equal to 3, and we obtain the second-term correction to the classical law of the iterated logarithm (Kolmogorov, 1929; Erdős, 1942).

Somewhat unexpectedly, the next result follows also from Theorem 1.2. To the best of our knowledge it is new.

Corollary 1.3. *Let $\{Z_t\}_{t\geq 0}$ denote the Ornstein–Uhlenbeck (OU) process on the real line that satisfies the s.d.e. $dZ = -Z\,dt + \sqrt{2}\,dW$ for a Brownian motion W. Then for every non-empty non-random closed set $E \subseteq [0,1]$, and all $z > 1$,*

$$A_{1.12}^{-1}\mathrm{K}_E\left(\frac{1}{z^2}\right)\bar{\Phi}(z) \leq \mathrm{P}\left\{\sup_{t\in E} Z_t \geq z\right\} \leq A_{1.12}\mathrm{K}_E\left(\frac{1}{z^2}\right)\bar{\Phi}(z). \tag{1.14}$$

Section 8 below contains further remarks along these lines.

For a proof of Corollary 1.3 consider the two-parameter processes,

$$\mathscr{S}_n(u,t) = \frac{X_1(t) + \cdots + X_{\lfloor un \rfloor}(t)}{\sqrt{n}} \quad (0 \le u, t \le 1),\ ^{\forall}n = 1, 2, \ldots . \tag{1.15}$$

Our recent work (2004b, Theorem 1.1) implies that if ν is standard normal, then $\mathscr{S}_n \Rightarrow \mathscr{U}$ in the sense of $\mathscr{D}([0,1]^2)$ (Bickel and Wichura, 1971), and $\mathscr{U}$ is the continuous centered Gaussian process with correlation function

$$\mathrm{E}\left[\mathscr{U}(u,t)\mathscr{U}(v,s)\right] = e^{-|t-s|}\min(u,v), \quad 0 \le u, v, s, t \le 1. \tag{1.16}$$

In particular, $\sup_{t\in E} S_n(t)/\sqrt{n}$ converges in distribution to $\sup_{t\in E}\mathscr{U}(1,t)$. Corollary 1.3 follows from this and the fact that $\sup_{t\in E}\mathscr{U}(1,t)$ has the same distribution as $\sup_{t\in E} Z_t$.

In this paper we apply stochastic calculus to strengthen our earlier central limit theorem (2004b, Theorem 1.1). Indeed we offer the following invariance principle.

Theorem 1.4. *If $\int_{-\infty}^{\infty} x\,\nu(dx) = 0$ and $\int_{-\infty}^{\infty} x^2\,\nu(dx) = 1$, then $\mathscr{S}_n \Rightarrow \mathscr{U}$ in the sense of $\mathscr{D}([0,1]^2)$.*

We close the introduction by presenting the following dynamic stability result.

Theorem 1.5. *Suppose ν is a distribution on $\mathbf{Z}$ which has mean zero and variance one. If there exists $\varepsilon > 0$ such that $\int_{-\infty}^{\infty} |x|^{2+\varepsilon}\,\nu(dx) < \infty$, then*

$$\mathrm{P}\left\{ \sum_{n=1}^{\infty} \mathbf{1}_{\{S_n(t)=0\}} = \infty \ \textit{for all}\ t \ge 0 \right\} = 1. \tag{1.17}$$

In words, under the conditions of Theorem 1.5, the recurrence of the origin is dynamically stable. When ν is supported by a finite subset of **Z** this was proved by Benjamini et al. (2003, Theorem 1.11). In order to generalize to the present setting, we first develop the following quantitative form of the classical gambler's ruin theorem. We state it next, since it may be of independent interest.

Consider i.i.d. integer-valued random variables $\{\xi_n\}_{n=1}^{\infty}$ such that $\mathrm{E}[\xi_1] = 0$ and $\sigma^2 = \mathrm{E}[\xi_1^2] < \infty$. Define $s_n = \xi_1 + \cdots + \xi_n$ to be the corresponding random walk, and let $T(z)$ denote the first-passage time to z; i.e.,

$$T(z) = \inf\{n \ge 1 : s_n = z\} \qquad ^{\forall}z \in \mathbf{Z} \quad (\inf\varnothing = \infty). \tag{1.18}$$

Theorem 1.6 (Gambler's Ruin)**.** *If G denotes the additive subgroup of $\mathbf{Z}$ generated by the possible values of $\{s_n\}_{n=1}^{\infty}$, then there exists a constant $A_{1.19} = A_{1.19}(\sigma, G) > 1$ such that*

$$\frac{A_{1.19}^{-1}}{1+|z|} \le \mathrm{P}\left\{T(z) \le T(0)\right\} \le \frac{A_{1.19}}{1+|z|} \qquad ^{\forall}z \in G. \tag{1.19}$$

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2. On the Kolmogorov ε-Entropy

2.1. **Λ_H-Polar Sets.** Let $H : \mathbf{R}_+ \to \mathbf{R}_+$ be non-decreasing and measurable, and recall the random set Λ_H from (1.5).

We say that a measurable set $E \subset [0,1]$ is Λ_H*-polar* if $\mathrm{P}\{\Lambda_H \cap E \neq \varnothing\} = 0$. If E is not Λ_H-polar, then the Hewitt–Savage law insures that $\mathrm{P}\{\Lambda_H \cap E \neq \varnothing\} = 1$. Our characterization of Λ_H-polar sets is described in terms of the function

$$\psi_H(E) = \int_1^\infty H^2(t)\mathrm{K}_E\left(\frac{1}{H^2(t)}\right)\bar{\Phi}(H(t))\,\frac{dt}{t}, \qquad \forall E \subseteq [0,1]. \tag{2.1}$$

Although ψ_H is subadditive, it is not a measure; e.g., ψ_H assigns equal mass $\int_1^\infty H^2(t)\bar{\Phi}(H(t))\,\frac{dt}{t}$ to all singletons. We will show that the function ψ_E determines the growth-rate of $\sup_{t\in E} S_n(t)$ in the following sense.

Theorem 2.1. *Suppose $E \subseteq [0,1]$ is Borel-measurable and $H : \mathbf{R}_+ \to \mathbf{R}_+$ is non-decreasing. Then,*

$$\limsup_{n\to\infty}\left[\sup_{t\in E} S_n(t) - H(n)\sqrt{n}\right] > 0 \text{ if and only if } \psi_H(E) = \infty. \tag{2.2}$$

Remark 2.2. In fact, we will prove that:

$$\begin{aligned}\psi_H(E) = \infty &\Longrightarrow \limsup_{n\to\infty}\left[\sup_{t\in E} S_n(t) - H(n)\sqrt{n}\right] = \infty;\\ \psi_H(E) < \infty &\Longrightarrow \limsup_{n\to\infty}\left[\sup_{t\in E} S_n(t) - H(n)\sqrt{n}\right] = -\infty.\end{aligned} \tag{2.3}$$

Definition 2.3. We write $\Psi_H(E) < \infty$ if we can decompose E as $E = \cup_{n=1}^\infty E_n$—where $E_1, E_2, \ldots,$ are closed—such that for all $n \geq 1$, $\psi_H(E_n) < \infty$. Else, we say that $\Psi_H(E) = \infty$.

Remark 2.4. One can have $\Psi_H(E) < \infty$ although $\psi_H(E) = \infty$. See Example 2.10 below.

The following then characterizes all polar sets of Λ_H; it will be shown to be a ready consequence of Theorem 2.1.

Theorem 2.5. *Suppose $E \subset [0,1]$ is a fixed compact set, and $H : \mathbf{R}_+ \to \mathbf{R}_+$ is non-decreasing. Then, E is Λ_H-polar if and only if $\Psi_H(E) = \infty$.*

Remark 2.6. The following variation of Remark 2.2 is valid:

$$\begin{aligned}\Psi_H(E) = \infty &\Longrightarrow \sup_{t\in E}\limsup_{n\to\infty}\left[S_n(t) - H(n)\sqrt{n}\right] = \infty;\\ \Psi_H(E) < \infty &\Longrightarrow \sup_{t\in E}\limsup_{n\to\infty}\left[S_n(t) - H(n)\sqrt{n}\right] = -\infty.\end{aligned} \tag{2.4}$$

2.2. **Relation to Minkowski Contents.** In the remainder of this section we say a few words about the function K_E. To begin with, let us note that the defining *maximal Kolmogorov sequence* $\{x_j\}_{j=1}^k$ has the property that any

$$w \in E \text{ satisfies } |w - x_j| \leq \varepsilon \text{ for some } j = 1, \ldots, k. \tag{2.5}$$

The Kolmogorov ε-entropy is related to the *Minkowski content* of $E \subseteq \mathbf{R}$. The latter can be defined as follows:

$$\mathrm{M}_n(E) = \sum_{i=-\infty}^{\infty} a_{i,n}(E), \text{ where } a_{i,n}(E) = \begin{cases} 1, & \text{if } \left[\frac{i}{n}, \frac{i+1}{n}\right) \cap E \neq \varnothing,\\ 0, & \text{otherwise.}\end{cases} \tag{2.6}$$

Here is the relation. See Dudley (1973, Theorem 6.0.1) and Mattila (1995, p. 78, eq. 5.8) for a related inequality.

Proposition 2.7. *For all non-empty sets $E \subseteq [0,1]$ and all integers $n \geq 1$,*

$$\mathrm{K}_E(1/n) \leq \mathrm{M}_n(E) \leq 3\mathrm{K}_E(1/n). \tag{2.7}$$

Remark 2.8. It is not difficult to see that both bounds can be attained.

Proof. Let $k = \mathrm{K}_E(1/n)$ and choose maximal (Kolmogorov) points $x_1 < x_2 \ldots < x_k$ such that any distinct pair (x_i, x_j) are distance at least $1/n$ apart. Define $\mathscr{E}$ to be the collection of all intervals $[\frac{i}{n}, \frac{i+1}{n})$, $0 \leq i < n$, such that any $I \in \mathscr{E}$ intersects E. Let $\mathscr{G}$ denote the collection of all $I \in \mathscr{E}$ such that some x_j is in I. These are the "good" intervals. Let $\mathscr{B} = \mathscr{E} \setminus \mathscr{G}$ denote the "bad" ones. Good intervals contain exactly one of the maximal Kolmogorov points, whereas bad ones contain none. Therefore, $\mathrm{K}_E(1/n) = |\mathscr{G}| \leq |\mathscr{E}| = \mathrm{M}_E(1/n)$, where $|\cdots|$ denotes cardinality. To complete our derivation we prove that $|\mathscr{B}| \leq 2\mathrm{K}_E(1/n)$.

We observe that any bad interval is necessarily adjacent to a good one. Therefore, we can write $\mathscr{B} = \mathscr{B}_L \cup \mathscr{B}_R$ where $\mathscr{B}_L$ [resp. $\mathscr{B}_R$] denotes the collection of all bad intervals I such that there exists a good interval adjacent to the left [resp. right] of I. By virtue of their definition, both $\mathscr{B}_L$ and $\mathscr{B}_R$ each have no more than $\mathrm{K}_E(1/n)$ elements. This completes the proof. □

An immediate consequence of this result is that if $\varepsilon \in [2^{-n-1}, 2^{-n}]$ then

$$\mathrm{K}_E(\varepsilon) \leq \mathrm{K}_E\left(2^{-n-1}\right) \leq M_{2^{n+1}}(E) \leq 2M_{2^n}(E) \leq 6\mathrm{K}_E\left(2^{-n}\right). \tag{2.8}$$

2.3. **Relation to Minkowski and Packing Dimensions.** There are well-known connections between ε-entropy and the (upper) Minkowski dimension, some of which we have already seen; many more of which one can find, in fine pedagogic form, in Mattila (1995, Ch. 5). We now present a relation that is particularly suited for our needs. Let H_ρ be any locally-bounded non-decreasing function such that

$$H_\rho(t) = \sqrt{2\ln\ln t + 2\rho \ln\ln\ln t}, \qquad {}^\forall t > e^{10000}. \tag{2.9}$$

One or two lines of calculations then reveal that

$$\psi_{H_\rho}(E) < \infty \text{ if and only if } \int_1^\infty \mathrm{K}_E(1/s) s^{\frac{1}{2}-\rho}\, ds < \infty. \tag{2.10}$$

Proposition 2.9. *For all compact linear sets E,*

$$\begin{aligned} \overline{\dim}_{\mathscr{M}} E &= \inf\left\{\rho > 0 : \ \psi_{H_\rho}(E) < \infty\right\} - \frac{3}{2}, \text{ and} \\ \dim_{\mathscr{P}} E &= \inf\left\{\rho > 0 : \ \Psi_{H_\rho}(E) < \infty\right\} - \frac{3}{2}. \end{aligned} \tag{2.11}$$

There are well-known examples of sets E whose packing and upper Minkowski dimension differ. Therefore, Proposition 2.9 provides us with an example of functions H (namely an appropriate H_ρ) and sets E such that $\psi_H(E)$ is infinite although $\Psi_H(E)$ is finite. This is good enough to address the issue raised in Remark 2.4. In fact, one can do more at little extra cost.

Example 2.10. Define

$$\mathscr{J}_\zeta(H) = \int_1^\infty H^\zeta(t)\bar{\Phi}(H(t))\,\frac{dt}{t} \qquad {}^\forall \zeta > 0. \tag{2.12}$$

Now consider any measurable non-decreasing function $H : \mathbf{R}_+ \to \mathbf{R}_+$ such that $\mathscr{J}_2(H) < \infty$ but $\mathscr{J}_{2+\varepsilon}(H) = \infty$ for some $\varepsilon > 0$. Then there are compact sets $E \subseteq [0,1]$ such that $\psi_H(E) = \infty$ although $\Psi_H(E) < \infty$. Our construction of such an E is based on a well-known example (Mattila, 1995, Exercise 1, p. 88).

Without loss of generality, we may assume that $\varepsilon \in (0,1)$. Bearing this in mind, define $r_0 = 1$ and $r_k = 1 - \sum_{j=1}^k j^{-1/\varepsilon}$ $(k = 1, 2, \ldots)$. Now consider

$$E = \{0\} \cup \bigcup_{k=0}^{\infty} \{r_k\}. \tag{2.13}$$

Then it is possible to prove that there is a constant $A > 1$ such that for all $\delta \in (0,1)$, $A^{-1}\delta^{\varepsilon} \le \mathrm{K}_E(\delta) \le A\delta^{\varepsilon}$. In particular, $\psi_H(E)$ is comparable to $\mathscr{J}_{2+\varepsilon}(H) = \infty$. On the other hand, because E is countable and $\mathscr{J}_2(H) < \infty$, we readily have $\Psi_H(E) < \infty$.

Our proof of Proposition 2.9 requires the following little lemma from geometric measure theory.

Lemma 2.11. *Suppose $H : \mathbf{R}_+ \to \mathbf{R}_+$ is non-decreasing and measurable, and $E \subseteq [0,1]$ is Borel and satisfies $\Psi_H(E) = \infty$. Then, there exists a compact set $G \subseteq E$ such that $\psi_H(I \cap G) = \infty$ for all rational intervals $I \subseteq [0,1]$ that intersect G.*

Proof. Let $\mathscr{R}$ denote the collection of all open rational intervals in $[0,1]$, and define

$$E_* = \bigcup_{I \in \mathscr{R}:\, \psi_H(E \cap I) < \infty} I. \tag{2.14}$$

A little thought makes it manifest that E_* is an open set in $[0,1]$, and $G = E \setminus E_*$ has the desired properties. □

Proof of Proposition 2.9. We will prove the assertion about $\overline{\dim}_{\mathscr{M}}$; the formula for $\dim_{\mathscr{P}}$ follows from the one for $\overline{\dim}_{\mathscr{M}}$, Lemma 2.11, and regularization (Mattila, 1995, p. 81).

Throughout the proof, we let $d = \overline{\dim}_{\mathscr{M}}(E)$ denote the Minkowski dimension of E (Mattila, 1995, p. 79). By its very definition, and thanks to Proposition 2.7, d can be written as

$$d = \overline{\dim}_{\mathscr{M}} E = \limsup_{s \to \infty} \frac{\log \mathrm{K}_E(1/s)}{\log s}. \tag{2.15}$$

Now

$$\begin{aligned} \int_1^{\infty} \mathrm{K}_E(1/s) s^{\frac12 - \rho}\, ds &= \sum_{n=0}^{\infty} \int_{2^n}^{2^{n+1}} \mathrm{K}_E(1/s) s^{\frac12-\rho}\, ds \\ &\ge 2^{-\rho} \sum_{n=0}^{\infty} \mathrm{K}_E\left(2^{-n}\right) 2^{-(\rho - \frac32)n} \\ &\ge 2^{-\rho} \limsup_{n \to \infty} \mathrm{K}_E\left(2^{-n}\right) 2^{-(\rho-\frac32)n}. \end{aligned} \tag{2.16}$$

Thus, if $2^n \le s \le 2^{n+1}$ and $\rho > 2$, then for all sufficiently large n,

$$s^{-(\rho - \frac32)} \mathrm{K}_E(1/s) \le 6 \cdot 2^{-n(\rho-\frac32)} \mathrm{K}_E\left(2^{-n}\right). \tag{2.17}$$

See (2.8). This development shows that

$$\int_1^\infty \mathrm{K}_E(1/s)s^{\frac{1}{2}-\rho}\,ds \geq \frac{1}{6\cdot 2^\rho}\limsup_{s\to\infty}\frac{\mathrm{K}_E(1/s)}{s^{(\rho-\frac{3}{2})}}. \tag{2.18}$$

Therefore, whenever $\rho - \frac{3}{2} < d$, the integral on the left-hand side is infinite. Thanks to (2.10), this means that

$$\inf\left\{\rho > 0:\ \psi_{H_\rho}(E) < \infty\right\} \leq \frac{3}{2} + d = \frac{3}{2} + \overline{\dim}_{\mathscr{M}} E. \tag{2.19}$$

This is half of the result for the Minkowski dimension. To prove the converse half, we argue similarly, and appeal to (2.8), to deduce that

$$\int_1^\infty \mathrm{K}_E(1/s)s^{\frac{1}{2}-\rho}\,ds \leq 6\sum_{n=0}^\infty \mathrm{K}_E\left(2^{-n}\right)2^{-n(\rho-\frac{3}{2})} \leq 6\sum_{n=0}^\infty 2^{n(d-\rho+\frac{3}{2})+o(n)}. \tag{2.20}$$

In particular, if $\rho > d+\frac{3}{2}$, then the left-hand side is finite. This and (2.10) together verify the asserted identity for $\overline{\dim}_{\mathscr{M}}$. □

Remark 2.12. In conjunction, Theorem 2.5 and Proposition 2.9 show that for any non-random Borel set $E \subseteq [0,1]$,

$$\begin{aligned} \rho > \frac{3}{2} + \dim_{\mathscr{P}} E &\implies \Lambda_{H_\rho} \cap E = \varnothing \\ \rho < \frac{3}{2} + \dim_{\mathscr{P}} E &\implies \Lambda_{H_\rho} \cap E \neq \varnothing. \end{aligned} \tag{2.21}$$

Moreover, the intersection argument of Khoshnevisan et al. (2000, Theorem 3.2) goes through unhindered to imply that if $\rho < \frac{3}{2} + \dim_{\mathscr{P}} E$, then $\dim_{\mathscr{P}}(\Lambda_H \cap E) = \dim_{\mathscr{P}} E$. In particular, we can apply this with $E = [0,1]$, and recall (1.8), to deduce the following:

$$\begin{aligned} \rho < \frac{5}{2} &\implies \dim_{\mathscr{P}} \Lambda_{H_\rho} = 1, \\ \rho > \frac{5}{2} &\implies \Lambda_{H_\rho} = \varnothing. \end{aligned} \tag{2.22}$$

Equation (1.13) is an immediate consequence of this. One could alternatively use the limsup-random-fractal theories of Khoshnevisan et al. (2000) and Dembo et al. (2000) to derive (2.22).

2.4. **An Application to Stable Processes.** Let $\{Y_\alpha(t)\}_{t\geq 0}$ denote a symmetric stable process with index $\alpha \in (0,1)$, and let us consider the random set $\mathscr{R}_\alpha = \mathrm{cl}(Y_\alpha([1,2]))$ denote the closed range of $\{Y_\alpha(t)\}_{t\in[1,2]}$.

Proposition 2.13. *Consider a given* $\alpha, \beta \in (0,1)$*. Then, for all* $M > 0$ *and* $p \geq 1$*, there exists a finite constant* $A_{2.23} = A_{2.23}(\alpha,\beta,p,M) > 1$ *such that for all intervals* $I \subset [-M, M]$ *with length* $\geq \beta$*, and all* $\varepsilon \in (0,1)$*,*

$$A_{2.23}^{-1}\varepsilon^{-\alpha p} \leq \mathrm{E}\left[\mathrm{K}^p_{\mathscr{R}_\alpha\cap I}(\varepsilon)\right] \leq A_{2.23}\varepsilon^{-\alpha p}. \tag{2.23}$$

Proof. Thanks to Proposition 2.7, it suffices to show that we can find $A_{2.24} > 1$ [depending only on α, M, p] such that for all $n \geq 1$,

$$A_{2.24}^{-1}n^{\alpha p} \leq \mathrm{E}\left[\mathrm{M}^p_n\left(\mathscr{R}_\alpha \cap I\right)\right] \leq A_{2.24}n^{\alpha p}. \tag{2.24}$$

This follows from connections to potential-theoretic notions, for which we need to introduce some notation.

Let $p_t(x,y)$ denote the transition densities of the process Y_α. As usual, P_x denotes the law of $x + Y_\alpha(\bullet)$ on path-space. Define $r(x,y)$ to be the 1-potential density of Y_α; i.e.,

$$r(x,y) = \int_0^\infty e^{-s} p_s(x,y)\, ds. \tag{2.25}$$

Finally, let $T(z,\varepsilon) = \inf\{s > 0 : |Y_\alpha(s) - z| \le \varepsilon\}$ designate the entrance time of the interval $[z-\varepsilon, z+\varepsilon]$; as usual, $\inf \varnothing = \infty$.

It is well known that for any $M > 0$, there exists a constant $A = A(M,\alpha) > 1$ such that

$$\begin{aligned} A^{-1}\varepsilon^{1-\alpha} &\le \inf_{x\in[-M,M]} \mathrm{P}\left\{\mathscr{R}_\alpha \cap [x-\varepsilon, x+\varepsilon] \neq \varnothing\right\} \\ &\le \sup_{x\in\mathbf{R}} \mathrm{P}\left\{\mathscr{R}_\alpha \cap [x-\varepsilon, x+\varepsilon] \neq \varnothing\right\} \le A\varepsilon^{1-\alpha}; \end{aligned} \tag{2.26}$$

see, for example Khoshnevisan (2002, Proposition 1.4.1, p. 351). In the case $p = 1$, this proves Equation (2.24). Because $L^p(\mathrm{P})$-norms are increasing in p, the lower bound in (2.24) follows, in fact, for all $p \ge 1$. Thus, it remains to prove the corresponding upper bound for $p > 1$.

Modern variants of classical probabilistic potential theory tell us that for all $x \notin [y-\varepsilon, y+\varepsilon]$,

$$\begin{aligned} &\int_0^\infty e^{-s} P_x\left\{T(y,\varepsilon) \le s\right\}\, ds \\ &\quad\le \mathbf{S}\left[\inf_{\mu\in\mathscr{P}([y-\varepsilon,y+\varepsilon])} \iint r(u,v)\,\mu(du)\,\mu(dv)\right]^{-1}. \end{aligned} \tag{2.27}$$

See Khoshnevisan (2002, Theorem 2.3.1, P. 368). Here, $\mathbf{S} = \sup_{z\in[y-\varepsilon,y+\varepsilon]} r(x,z)$, In the preceding, E is a linear Borel set, and $\mathscr{P}(E)$ denotes the collection of all probability measures on the Borel set E.

On the other hand, there exists a finite constant $A_{2.28} > 1$ such that whenever x, y are both in $[-2M, 2M]$,

$$A_{2.28}^{-1}|x-y|^{-1+\alpha} \le r(x,y) \le A_{2.28}|x-y|^{-1+\alpha}. \tag{2.28}$$

See, for example, Khoshnevisan (2002, Lemma 3.4.1, p. 383). Now as soon as we have $|x-y| \ge 2\varepsilon$ and $|z-y| \le \varepsilon$, it follows that $|x-z| \ge \frac{1}{2}|x-y|$. Therefore, the inequality $\int_0^\infty(\cdots) \ge \int_0^1(\cdots)$ leads us to the following:

$$\begin{aligned} &P_x\left\{T(y,\varepsilon) \le 1\right\} \\ &\quad\le 2^{1-\alpha} e A_{2.28}^2 |x-y|^{-1+\alpha}\left[\inf_{\mu\in\mathscr{P}([-\varepsilon,+\varepsilon])} \iint |u-v|^{-1+\alpha}\,\mu(du)\,\mu(dv)\right]^{-1}. \end{aligned} \tag{2.29}$$

The term $[\cdots]^{-1}$ is the $(1-\alpha)$-dimensional Riesz capacity of $[-\varepsilon,\varepsilon]$. It is a classical fact that the said capacity is, up to multiplicative constants, of exact order $\varepsilon^{1-\alpha}$. Therefore, there exists $A_{2.30} > 1$ such that for all $\varepsilon \in (0,1)$ and all $x, y \in [-2M, 2M]$ that satisfy $|x-y| \ge 2\varepsilon$,

$$P_x\left\{T(y,\varepsilon) \le 1\right\} \le A_{2.30}|x-y|^{-1+\alpha}\varepsilon^{1-\alpha}. \tag{2.30}$$

We now prove the upper bound in (2.24) for the case $p = 2$ and hence all $p \in [1,2]$. By the strong Markov property and time reversal, whenever $x, y \in [-2M, 2M]$

satisfy $|x-y| \geq 4\varepsilon$,

$$\begin{aligned}
&\mathrm{P}\left\{\mathscr{R}_\alpha \cap [x-\varepsilon, x+\varepsilon] \neq \varnothing\ ,\ \mathscr{R}_\alpha \cap [y-\varepsilon, y+\varepsilon] \neq \varnothing\right\} \\
&\quad \leq 2\mathrm{P}\left\{\mathscr{R}_\alpha \cap [x-\varepsilon, x+\varepsilon] \neq \varnothing\right\} \sup_{v \in [x-\varepsilon, x+\varepsilon]} P_v\left\{T(y,\varepsilon) \leq 1\right\} \\
&\quad \leq 2A_{2.30}|x-y|^{-1+\alpha}\varepsilon^{2(1-\alpha)}.
\end{aligned} \tag{2.31}$$

Equation (2.23) readily follows from this in the case that $p=2$. To derive the result for an arbitrary positive integer p, simply iterate this argument $p-1$ times. □

3. Proof of Theorem 1.2

This proof rests on half of the following preliminary technical result. Throughout this section $\{z_n\}_{n=1}^\infty$ is a fixed sequence that satisfies (1.11), and $E \subseteq [0,1]$ is a fixed non-random compact set.

Proposition 3.1. *Let $\{\delta_n\}_{n=1}^\infty$ be a fixed sequence of numbers in $[0,1]$ that satisfy*

$$\liminf_{n\to\infty} \delta_n z_n^2 > 0. \tag{3.1}$$

Then there exists a finite constant $A_{3.2} > 1$ such that for all $n \geq 1$,

$$A_{3.2}^{-1}\delta_n z_n^2 \bar\Phi(z_n) \leq \mathrm{P}\left\{\sup_{t\in[0,\delta_n]} S_n(t) \geq z_n\sqrt{n}\right\} \leq A_{3.2}\delta_n z_n^2 \bar\Phi(z_n). \tag{3.2}$$

Proof. We will need some of the notation, as well as results, of Khoshnevisan et al. (2004b). Therefore, we first recall the things that we need.

Let $\mathrm{P}_{\mathscr{N}}$ (resp. $\mathrm{E}_{\mathscr{N}}$) denote the 'quenched' measure $\mathrm{P}(\cdots\,|\,\mathscr{N})$ (resp. expectation operator $\mathrm{E}[\cdots\,|\,\mathscr{N}]$), where $\mathscr{N}$ denotes the σ-algebra generated by all of the clocks, and define $\mathscr{F}_t^n$ to be the σ-algebra generated by $\{S_j(s);\ 0\leq s\leq t\}_{j=1}^n$.

Define

$$\begin{aligned}
L_n(t) &= \int_0^t \mathbf{1}_{B_n(u)}\,du, \text{ where} \\
B_n(t) &= \left\{\omega\in\Omega:\ S_n(t) \geq z_n\sqrt{n}\right\}.
\end{aligned} \tag{3.3}$$

We replace the variable J_n of Khoshnevisan et al. (2004b, eq. 5.3) by our $L_n(2\delta_n)$, and go through the proof of Khoshnevisan et al. (2004b, Lemma 5.2) to see that there exists an $\mathscr{N}$-measurable event $A_{n,\frac12}$ such that for any $u\in[0,\delta_n]$, the following holds P-almost surely:

$$\begin{aligned}
\mathrm{E}_{\mathscr{N}}\left[L_n(2\delta_n)\,|\,\mathscr{F}_u^n\right] &\geq \frac{2}{3z_n^2}\int_0^{\frac32(2\delta_n-u)z_n^2} \bar\Phi\left(\sqrt{t}\right)\,dt\cdot \mathbf{1}_{A_{n,1/2}\cap B_n(u)} \\
&\geq \frac{2}{3z_n^2}\int_0^{\frac32\delta_n z_n^2} \bar\Phi\left(\sqrt{t}\right)\,dt\cdot \mathbf{1}_{A_{n,1/2}\cap B_n(u)} \\
&\geq \frac{A_{3.4}}{z_n^2}\cdot \mathbf{1}_{A_{n,1/2}\cap B_n(u)},
\end{aligned} \tag{3.4}$$

where $A_{3.4}$ is an absolute constant that is bounded below. Moreover, thanks to Khoshnevisan et al. (2004b, Theorem 2.1) and (3.1), there exists a finite constant $A_{3.5}\in(0,1)$ such that for all $n\geq 1$,

$$\mathrm{P}\left(A_{n,\frac12}^{\complement}\right) \leq z_n^2\delta_n\, e^{-A_{3.5}n/z_n^2}. \tag{3.5}$$

Now, $u \mapsto \mathrm{E}_{\mathscr{N}}[L_n(2\delta_n) \mid \mathscr{F}_u^n]$ is a non-negative and bounded $\mathrm{P}_{\mathscr{N}}$-martingale. Therefore, P-almost surely,

$$\begin{aligned}\mathbf{1}_{A_{n,1/2}} \mathrm{P}_{\mathscr{N}} \left\{ \sup_{u \in [0,\delta_n]} S_n(t) \geq z_n\sqrt{n} \right\} &= \mathrm{P}_{\mathscr{N}} \left\{ \sup_{u \in [0,\delta_n] \cap \mathbf{Q}} \mathbf{1}_{A_{n,1/2} \cap B_n(u)} \geq 1 \right\} \\ &\leq \mathrm{P}_{\mathscr{N}} \left\{ \sup_{u \in [0,\delta_n] \cap \mathbf{Q}} \mathrm{E}_{\mathscr{N}} \left[L_n(2\delta_n) \,\middle|\, \mathscr{F}_u^n \right] \geq \frac{A_{3.4}}{z_n^2} \right\} \qquad (3.6) \\ &\leq \frac{z_n^2}{A_{3.4}} \mathrm{E}_{\mathscr{N}} \left[L_n(2\delta_n) \right] = \frac{2}{A_{3.4}} \delta_n z_n^2 \bar{\Phi}(z_n).\end{aligned}$$

The ultimate inequality follows from Doob's maximal inequality for martingales, and the last equality from the stationarity of $t \mapsto S_n(t)$. Taking expectations and applying (3.5) yields

$$\mathrm{P} \left\{ \sup_{t \in [0,\delta_n]} S_n(t) \geq z_n\sqrt{n} \right\} \leq \frac{2}{A_{3.4}} \delta_n z_n^2 \left[\bar{\Phi}(z_n) + e^{-A_{3.5} n / z_n^2}. \right] \qquad (3.7)$$

Equation (1.11) shows that the first term on the right-hand side dominates the second one for all n sufficiently large. This yields the probability upper bound of the proposition. Now we work toward the lower bound.

By adapting the argument of Khoshnevisan et al. (2004b, eq. 6.12), we can conclude that P-almost surely there exists an $\mathscr{N}$-measurable P-a.s. finite random variable γ such that for all $n \geq \gamma$,

$$\mathrm{E}_{\mathscr{N}} \left[(L_n(\delta_n))^2 \right] \leq A_{3.8} \delta_n z_n^{-2} \bar{\Phi}(z_n). \qquad (3.8)$$

where $A_{3.8} > 1$ is a non-random and finite constant. [Replace J_n by $L_n(\delta_n)$ and proceed to revise equation (6.12) of Khoshnevisan et al. (2004b).] Since, by stationarity, $\mathrm{E}_{\mathscr{N}}[L_n(\delta_n)] = \delta_n \bar{\Phi}(z_n)$, the Paley–Zygmund inequality shows that P-almost surely for all $n \geq \gamma$,

$$\mathrm{P}_{\mathscr{N}} \left\{ L_n(\delta_n) > 0 \right\} \geq \frac{\left(\mathrm{E}_{\mathscr{N}}[L_n(\delta_n)] \right)^2}{\mathrm{E}_{\mathscr{N}} \left[(L_n(\delta_n))^2 \right]} \geq \frac{1}{A_{3.8}} \delta_n z_n^2 \bar{\Phi}(z_n). \qquad (3.9)$$

On the other hand,

$$\mathrm{P} \left\{ \sup_{t \in [0,\delta_n]} S_n(t) \geq z_n\sqrt{n} \right\} \geq \mathrm{P} \left\{ L_n(\delta_n) > 0 \right\} \geq \mathrm{P} \left\{ L_n(\delta_n) > 0 \,,\, n \geq \gamma \right\}. \qquad (3.10)$$

This is at least $A_{3.8}^{-1} \delta_n z_n^2 \bar{\Phi}(z_n) \mathrm{P}\{n \geq \gamma\}$. Therefore, the proposition follows for all n large, and hence all n by adjusting the constants. □

Proof of Theorem 1.2: Upper Bound. Let $k = \lfloor z_n^2 \rfloor + 1$, and recall the intervals $I_{j,k} = [j/k, (j+1)/k)$ for $0 \leq j \leq k$. Then,

$$\begin{aligned}\mathrm{P} \left\{ \sup_{t \in E} S_n(t) \geq z_n\sqrt{n} \right\} &\leq \sum_{\substack{0 \leq j \leq k: \\ I_{j,k} \cap E \neq \varnothing}} \mathrm{P} \left\{ \sup_{t \in I_{j,k}} S_n(t) \geq z_n\sqrt{n} \right\} \qquad (3.11) \\ &= \mathrm{M}_k(E) \mathrm{P} \left\{ \sup_{t \in [0,1/k]} S_n(t) \geq z_n\sqrt{n} \right\}.\end{aligned}$$

The last line follows from stationarity. Because $\liminf_{n\to\infty} k^{-1}z_n^2 = 1 > 0$, Proposition 3.1 applies, and we obtain the following:

$$\mathrm{P}\left\{\sup_{t\in E} S_n(t) \geq z_n\sqrt{n}\right\} \leq A_{3.1}\frac{z_n^2}{k}\mathrm{M}_k(E)\bar{\Phi}(z_n). \tag{3.12}$$

As $n \to \infty$, $z_n^2 = O(k)$, and $\mathrm{M}_k(E) \leq 3\mathrm{K}_E(1/k) \leq 18\mathrm{K}_E(z_n^{-2})$; cf. Proposition 2.7, as well as equation (2.8). The probability upper bound of Theorem 1.2 follows from this discussion. □

Proof of Theorem 1.2: Lower Bound. It is likely that one can use Proposition 3.1 for this bound as well, but we favor a more direct approach. Let $k = \mathrm{K}_E((16z_n)^{-2})$, and based on this find and fix maximal Kolmogorov points $x_1, \ldots, x_k$ in E such that whenever $i \neq j$, $|x_i - x_j| \geq (16z_n)^{-2}$. Without loss of generality, we may assume that $x_1 < x_2 < \cdots < x_k$. In terms of these maximal Kolmogorov points, we define

$$V_n = \sum_{j=1}^{k} \mathbf{1}_{\{S_n(x_j)\geq z_n\sqrt{n}\}}. \tag{3.13}$$

Evidently, P-almost surely,

$$\mathrm{E}_{\mathscr{N}}[V_n] = k\bar{\Phi}(z_n) \geq \mathrm{K}_E\left(z_n^{-2}\right)\bar{\Phi}(z_n). \tag{3.14}$$

Now we estimate the quenched second moment of V_n: There exists an $\mathscr{N}$-measurable P-almost surely finite random variable σ such that for all $n \geq \sigma$,

$$\begin{aligned}\mathrm{E}_{\mathscr{N}}\left[V_n^2\right] &\leq 2\sum\sum_{1\leq i\leq j\leq k} \mathrm{P}_{\mathscr{N}}\left\{S_n(x_i) \geq z_n\sqrt{n}\ ,\ S_n(x_j) \geq z_n\sqrt{n}\right\}\\ &\leq 4\sum\sum_{1\leq i\leq j\leq k} \exp\left(-\frac{1}{8}z_n^2(x_j - x_i)\right)\bar{\Phi}(z_n).\end{aligned} \tag{3.15}$$

See Khoshnevisan et al. (2004b, Lemma 6.2) for the requisite joint-probability estimate. Whenever $j > i$, we have $x_j - x_i = \sum_{l=i}^{j-1}(x_{l+1} - x_l) \geq \frac{1}{16}(j-i)z_n^{-2}$. Therefore, for all $n \geq \sigma$,

$$\begin{aligned}\mathrm{E}_{\mathscr{N}}\left[V_n^2\right] &\leq 4\sum\sum_{1\leq i\leq j\leq k} \exp\left(-\frac{1}{128}(j-i)\right)\bar{\Phi}(z_n)\\ &\leq \frac{4}{1-e^{-1/128}}k\bar{\Phi}(z_n) = A_{3.16}\mathrm{K}_E\left(\frac{1}{16z_n^2}\right)\bar{\Phi}(z_n)\\ &\leq 6^4A_{3.16}\mathrm{K}_E\left(\frac{1}{z_n^2}\right)\bar{\Phi}(z_n).\end{aligned} \tag{3.16}$$

The last line relies on four successive applications of (2.8), and is valid if n is at least $r = \inf\{k :\ z_k^2 \geq 4\}$. We combine (3.14), (3.16), and the Paley–Zygmund inequality to deduce that for all $n \geq \sigma \vee r$,

$$\mathrm{P}_{\mathscr{N}}\left\{V_n > 0\right\} \geq \frac{\left(\mathrm{E}_{\mathscr{N}}V_n\right)^2}{\mathrm{E}_{\mathscr{N}}\left[V_n^2\right]} \geq \frac{1}{6^4A_{3.16}}\mathrm{K}_E\left(z_n^{-2}\right)\bar{\Phi}(z_n), \tag{3.17}$$

P-almost surely. But for all $n \geq r$,

$$\begin{aligned} \mathrm{P}\left\{\sup_{t\in E} S_n(t) \geq z_n\sqrt{n}\right\} &\geq \mathrm{P}\left\{V_n > 0\right\} \geq \mathrm{P}\left\{V_n > 0\ ,\ n \geq \sigma\right\} \\ &\geq \frac{1}{6^4 A_{3.16}} \mathrm{K}_E\left(z_n^{-2}\right)\bar{\Phi}(z_n)\mathrm{P}\{n \geq \sigma\}. \end{aligned} \tag{3.18}$$

Because σ is finite P-almost surely, the lower bound in Theorem 1.2 follows for all sufficiently large n, and hence for all n after adjusting the constants. □

4. Proofs of Theorems 2.1, 2.5, and 1.1, and Remarks 2.2 and 2.6

The critical result is Theorem 2.1, and has a long and laborious proof. Fortunately, most of this argument appears, in a simplified setting, in Khoshnevisan et al. (2004b) from which we borrow liberally.

Throughout the following derivation, $\mathfrak{e}_n = \mathfrak{e}(n) = \lfloor e^{n/\ln_+(n)}\rfloor$, which is the so-called *Erdős sequence*.

Proof of Theorem 2.1. Without loss of generality, we can assume that

$$\sqrt{\ln_+\ln_+ t} \leq H(t) \leq 2\sqrt{\ln_+\ln_+ t} \qquad {}^\forall t > 0. \tag{4.1}$$

For the argument, follows Erdős (1942, eq.'s (1.2) and (3.4)).

We first dispose of the simple case $\psi_H(E) < \infty$.

By the reflection principle and by Theorem 1.2,

$$\begin{aligned} \mathrm{P}\left\{\max_{1\leq k\leq \mathfrak{e}(n+1)} \sup_{t\in E} S_k(t) \geq H(\mathfrak{e}_n)\sqrt{\mathfrak{e}_n}\right\} &\leq 2\mathrm{P}\left\{\sup_{t\in E} S_{\mathfrak{e}(n+1)}(t) \geq H(\mathfrak{e}_n)\sqrt{\mathfrak{e}_n}\right\} \\ &\leq 2A_{1.12}\mathrm{K}_E\left(\frac{1}{H^2(\mathfrak{e}_n)}\right)\bar{\Phi}(H(\mathfrak{e}_n)). \end{aligned} \tag{4.2}$$

Under (4.1), $\psi_H(E)$ is finite if and only if $\sum_n \mathrm{K}_E(1/H^2(\mathfrak{e}_n))\bar{\Phi}(H(\mathfrak{e}_n)) < \infty$. Hence, the case $\psi_H(E) < \infty$ follows from a monotonicity argument.

In the case $\psi_H(E) = \infty$, define for a fixed $\vartheta > 0$,

$$\begin{aligned} &S_n^* = \sup_{t\in E} S_{\mathfrak{e}(n)}(t), && H_n = H(\mathfrak{e}_n), \\ &\mathscr{I}_n = \left(H_n\sqrt{\mathfrak{e}_n}, \left(H_n + \frac{\vartheta}{H_n}\right)\sqrt{\mathfrak{e}_n}\right], && L_n = \sum_{j=1}^n \mathbf{1}_{\{S_j^*\in\mathscr{I}_j\}}, \\ &f(z) = \mathrm{K}_E(1/z^2)\bar{\Phi}(z). \end{aligned} \tag{4.3}$$

These are the present article's replacement of Khoshnevisan et al. (2004b, eq. 8.10). We can choose ϑ large enough (though independent of n) such that there exists $\eta \in (0,1)$ with the property that for all $n \geq 1$,

$$\eta \leq \frac{\mathrm{P}\left\{S_n^* \in \mathscr{I}_n\right\}}{\mathrm{P}\left\{S_n^* \geq H_n\sqrt{\mathfrak{e}_n}\right\}} \leq \eta^{-1}. \tag{4.4}$$

To see why this holds, we mimic the proof of Khoshnevisan et al. (2004b, Lemma 8.3), but in place of their Theorem 1.4, we use Theorem 1.2 of the present paper.

Now in light of (4.4) and condition $\psi_H(E) = \infty$, $\lim_{n\to\infty} \mathrm{E}[L_n] = \infty$. Therefore, by the Borel–Cantelli lemma, it suffices to show that

$$\limsup_{n\to\infty} \frac{\mathrm{E}\left[L_n^2\right]}{\left(\mathrm{E}[L_n]\right)^2} < \infty. \tag{4.5}$$

Everything comes down to estimating the following joint probability:

$$\mathscr{P}_{i,j} = \mathrm{P}\left\{S_i^* \in \mathscr{I}_i\ ,\ S_j^* \in \mathscr{I}_j\right\}, \qquad {}^\forall j > i \geq 1. \tag{4.6}$$

This painful task is performed by considering $\mathscr{P}_{i,j}$ on three different scales: (a) $j \geq i + \ln_+^{10}(i)$; (b) $j \in [i + \ln_+(i), i + \ln_+^{10}(i))$; and (c) $j \in (i, i + \ln_+(i))$. Fortunately, Lemmas 8.4–8.7 of Khoshnevisan et al. (2004b) do this for us at no cost. However, we note that they hold only after we replace their S_i^* with ours and all multiplicative constants are adjusted. Moreover, everywhere in their proofs, replace " $\sup_{t\in[0,1]}$" by " $\sup_{t\in E}$." Equation (4.5) follows from these estimates. □

Proof of Theorem 2.5. First, let us suppose that $\Psi_H(E) < \infty$. Then, we can write $E = \cup_{m=1}^\infty E_m$, with E_m's closed, such that for all m, $\psi_H(E_m) < \infty$. Theorem 2.1 proves, then, that for all m,

$$\sup_{t\in E_m} \limsup_{n\to\infty} \left[S_n(t) - H(n)\sqrt{n}\right] \leq 0, \text{ a.s.} \tag{4.7}$$

Maximize over $m = 1, 2, \ldots$ to prove half of Theorem 2.5.

To prove the second half of the theorem, we assume that $\Psi_H(E) = \infty$. By Lemma 2.11, we can find a compact set $G \subseteq E$ such that whenever I is a rational interval that intersects G, $\psi_H(I \cap G)$ is infinite. Now consider the random sets

$$\Lambda_H^n = \left\{t \in [0,1] : \sup_{\varepsilon>0} \inf_{t-\varepsilon<s<t+\varepsilon} \left[S_n(s) - H(n)\sqrt{n}\right] > 0\right\}. \tag{4.8}$$

By the regularity of the paths of S_n, Λ_H^n is open for every n.

By Theorem 2.5, for any rational interval I that intersects G, $\Lambda_H^n \cap (I \cap G)$ is non-empty infinitely often. In particular, $\cup_{i=n}^\infty \Lambda_H^i$ intersects $I \cap G$ infinitely often. Therefore, we have shown that $\cup_{i=n}^\infty \Lambda_H^i \cap G$ is an everywhere-dense relatively-open subset of the complete compact separable metric space G. By the Baire category theorem, $\cap_{n=1}^\infty \cup_{i=n}^\infty \Lambda_H^i \cap G$ is non-empty. In particular, there exist uncountably-many times $t \in G \subseteq E$ such that $t \in \limsup_n \Lambda_H^n = \Lambda_H$, whence the theorem. □

Proof of Theorem 1.1. We use a codimension argument. Let Y_α be the stable process of §2.4 which is chosen to be independent of the entire dynamical Gaussian walk, and let $\mathscr{R}_\alpha$ denote its (closed) range.

By Theorem 2.1 the following are equivalent for any dyadic interval I:

$$\begin{aligned} &\limsup_{n\to\infty} \left[\sup_{t\in\mathscr{R}_\alpha\cap I} S_n(t) - H(n)\sqrt{n}\right] > 0 \iff \psi_H\left(\mathscr{R}_\alpha \cap I\right) = \infty \\ &\limsup_{n\to\infty} \left[\sup_{t\in\mathscr{R}_\alpha\cap I} S_n(t) - H(n)\sqrt{n}\right] \leq 0 \iff \psi_H\left(\mathscr{R}_\alpha \cap I\right) < \infty. \end{aligned} \tag{4.9}$$

Recall (2.12). Thanks to (2.23),

$$\mathrm{E}\left[\psi_H\left(\mathscr{R}_\alpha \cap I\right)\right] \asymp \int_1^\infty H^{2(1+\alpha)}(t)\bar{\Phi}(H(t))\,\frac{dt}{t} = \mathscr{J}_{2(1+\alpha)}(H). \tag{4.10}$$

where '$\alpha \asymp \beta$' stands for 'α is finite if and only if β is'. Therefore, by (2.24) and the Paley–Zygmund inequality, $\psi_H\left(\mathscr{R}_\alpha \cap I\right)$ is infinite with positive probability if and only if its expectation is infinite. In particular,

$$\mathrm{P}\left\{\limsup_{n\to\infty} \sup_{t\in\mathscr{R}_\alpha\cap I} \left[S_n(t) - H(n)\sqrt{n}\right] > 0\right\} > 0 \iff \mathscr{J}_{2(1+\alpha)}(H) = \infty. \tag{4.11}$$

Because the condition on $\mathscr{J}_{2(1+\alpha)}$ does not involve the dyadic interval I, and since there are countably-many dyadic intervals, it follows from the category portion of the proof of Theorem 2.5 that

$$\mathrm{P}\left\{\sup_{t\in\mathscr{R}_\alpha}\limsup_{n\to\infty}\left[S_n(t)-H(n)\sqrt{n}\right]>0\right\}>0 \iff \mathscr{J}_{2(1+\alpha)}(H)=\infty. \tag{4.12}$$

That is, Λ_H intersects $\mathscr{R}_\alpha$ with positive probability if and only if $\mathscr{J}_{2(1+\alpha)}(H)=\infty$. But it is known that $\mathscr{R}_\alpha$ can hit a set E if and only if E has positive $(1-\alpha)$-dimensional Riesz capacity $\mathrm{Cap}_{1-\alpha}(E)$ (Khoshnevisan, 2002, Theorem 3.4.1, p. 384). Thus, by the Fubini–Tonneli theorem,

$$\mathrm{E}\left[\mathrm{Cap}_{1-\alpha}\left(\Lambda_H\right)\right]>0 \iff \mathscr{J}_{2(1+\alpha)}(H)=\infty. \tag{4.13}$$

Because $\alpha\in(0,1)$ is arbitrary, we have shown that for any $\zeta\in(0,1)$,

$$\mathrm{E}\left[\mathrm{Cap}_{2-(\zeta/2)}\left(\Lambda_H\right)\right]>0 \iff \mathscr{J}_\zeta(H)=\infty. \tag{4.14}$$

Frostman's theorem (Khoshnevisan, 2002, Theorem 2.2.1, p. 521) then implies the result. □

Proof of Remark 2.2. Because $\bar\Phi(x)\sim(2\pi)^{-1/2}x^{-1}\exp(-x^2/2)$ as $x\to\infty$,

$$\psi_H(E)<\infty \iff \int_1^\infty H(t)\mathrm{K}_E\left(\frac{1}{H^2(t)}\right)e^{-\frac{1}{2}H^2(t)}\,dt<\infty. \tag{4.15}$$

Therefore, we can appeal to (2.8) to see, after one or two lines of calculations, that

$$\psi_H(E)<\infty \iff {}^\forall c\in\mathbf{R}:\ \psi_{H+(c/H)}(E)<\infty. \tag{4.16}$$

Now we can prove the remark.

If $\psi_H(E)<\infty$, then the preceding remarks and Theorem 2.1 together prove that for any $c<0$,

$$\limsup_{n\to\infty}\left[\sup_{t\in E}S_n(t)-\sqrt{n}\left(H(n)+\frac{c}{H(n)}\right)\right]\le 0, \qquad \text{a.s.} \tag{4.17}$$

Thanks to (4.1), $H(n)=o(\sqrt{n})$ as $n\to\infty$. Thus, let $c\to-\infty$ to see that

$$\limsup_{n\to\infty}\left[\sup_{t\in E}S_n(t)-H(n)\sqrt{n}\right]=-\infty, \qquad \text{a.s.} \tag{4.18}$$

If $\psi_H(E)=\infty$, then we argue as above, but, this time, we let c tend to ∞. □

Proof of Remark 2.6. We follow the proof of Remark 2.2 verbatim, but apply Theorem 2.5 in place of Theorem 2.1 everywhere. □

5. PROOF OF THEOREM 1.4

A key idea of our proof of Theorem 1.4 is to appeal to martingale problems via the semi-martingale weak-convergence theory of Jacod and Shiryaev (2002). To elaborate on this connection a bit further let us note that $\{X_k\}_{k=1}^\infty$ are i.i.d. copies of a pure-jump Feller process with generator

$$Af(x)=\int_{-\infty}^\infty f(z)\,\nu(dz)-f(x) \qquad {}^\forall f\in\mathscr{C}_0(\mathbf{R}). \tag{5.1}$$

Before citing the result of Jacod and Shiryaev (2002) we need to introduce some more notation. This will be done in the first subsection. Let us note in advance that ours differs slightly from the notation of Jacod and Shiryaev (2002). In particular, our B corresponds to their B' and our C corresponds to their $\tilde{C}'$.

Throughout, we use the following particular construction of the process $\mathscr{U}$: Let $\{\beta(s,t)\}_{s,t\geq 0}$ denote the Brownian sheet, and define

$$\mathscr{U}(s,t) = \frac{\beta\left(s, e^{2t}\right)}{e^t} \qquad \forall s,t \geq 0. \tag{5.2}$$

The reader can check that $\mathscr{U}$ is indeed a continuous centered Gaussian process whose correlation function is given by (1.16).

We aim to prove the following:

Proposition 5.1. *Assume, in addition, that there exists* $\varepsilon > 0$ *such that*

$$\int_{-\infty}^{\infty} |x|^{2+\varepsilon}\,\nu(dx) < \infty. \tag{5.3}$$

Then, for each fixed $u \geq 0$*,* $\mathscr{S}_n(u,\cdot) \Rightarrow \mathscr{U}(u,\cdot)$ *in the sense of* $\mathscr{D}([0,1])$.

O. Rusakov (1995, Theorem 3.1) has demonstrated that a similar result holds for a closely-related model.

Because $u \mapsto \mathscr{S}_n(u,\bullet)$ is an infinite-dimensional Lévy process on $\mathscr{D}([0,1])$, a standard argument then yields the following. [See Lemma 2.4 of Eisenbaum and Khoshnevisan (2002), but replace $\mathscr{D}_T(\mathscr{C}(K))$ there by $\mathscr{D}(\mathscr{D}([0,1]))$.]

Proposition 5.2. *Under the additional constraint* (5.3)*, the finite-dimensional distributions of* $\mathscr{S}_n$ *converge to those of* $\mathscr{U}$.

In light of this, Proposition 5.1 and "tightness" together would yield Theorem 1.4 under (5.3). A truncation argument then removes (5.3). Our proof of Proposition uses the machinery of Jacod and Shiryaev (2002). Then we follow the general outline of Khoshnevisan et al. (2004b, §4) to establish tightness.

5.1. **Background on Semi-Martingales.** Let $\{X_t\}_{t\geq 0}$ be a cadlag semimartingale. We assume that X is defined on the canonical sample space $\mathscr{D}(\mathbf{R}_+)$.

Given a measurable function g, $\{v_t(g)\}_{t\geq 0}$ denotes the compensator of the process $t \mapsto \sum_{s\leq t,\, \Delta X_s \neq 0} g(\Delta X_s)$, where $\Delta X_t = X_t - X_{t-}$ designates the size of the jump of X at time t. We specialize our discussion further by considering the subclass of processes X that satisfy:

1. $X = M + B$, where B is continuous and adapted, and M is a local martingale.
2. $v_t(x^2) < \infty$ for all t. Of course, $v_t(x^2)$ stands for $v_t(g)$ where $g(x) = x^2$.

For such a process X, write

$$C_t = \langle M^c \rangle_t + v_t(x^2) \qquad \forall t \geq 0, \tag{5.4}$$

where M^c is the continuous part of M, and $\langle\cdot\rangle$ denotes quadratic variation.

Let $\mathscr{C}_{bz}$ denote the class of functions which are bounded and vanish near 0. Define

$$\tau_a = \inf\{t > 0 : \, |X_t| \vee |X_{t-}| \geq a\} \qquad \forall a > 0. \tag{5.5}$$

Now let $\{X^n\}_{n=1}^{\infty}$ denote a sequence of such semimartingales; B^n, C^n, τ_a^n, and $v^n(g)$ denote the corresponding characteristics for the process X^n.

Theorem 5.3 ((Jacod and Shiryaev, 2002, Theorem IX.3.48))**.** *If the following hold for a dense subset* D *of* $\mathbf{R}_+$*, then* $X_n \Rightarrow X$ *in the sense of* $\mathscr{D}(\mathbf{R}_+)$*:*

(1) *For each* $a > 0$ *there is an increasing and continuous non-random function* F^a *so that* $F^a(t) - V_{\tau_a \wedge t}(B)$, $F^a(t) - \langle M^c \rangle_{\tau_a \wedge t}$, *and* $F^a(t) - v_{\tau_a \wedge t}(x^2)$ *are increasing functions of* t*, where* $V_t(B)$ *denotes the total variation* B *on* $[0, t]$.

(2) *For all* $a > 0$ *and* $t > 0$,

$$\lim_{b \uparrow \infty} \sup_{\omega \in \mathscr{D}(\mathbf{R}_+)} v_{\tau_a \wedge t}\left(x^2 \mathbf{1}_{\{|x|>b\}}\right)(\omega) = 0. \tag{5.6}$$

(3) *The martingale problem for* X *has local uniqueness in the sense of Jacod and Shiryaev (2002).*

(4) *For all* $t \in D$ *and* $g \in \mathscr{C}_{bz}$*, the function* $\omega \mapsto (B_t(\omega), C_t(\omega), v_t(g)(\omega))$ *is Skorohod continuous.*

(5) X_0^n *converges in distribution to* X_0.

(6) *For all* $g \in \mathscr{C}_{bz}$, $v^n_{t \wedge \tau^n_a}(g) - v_{t \wedge \tau_a}(g) \xrightarrow{\mathrm{P}} 0$.

(7) *For all* $a, t > 0$, $\sup_{s \le t} |B^n_{s \wedge \tau^n_a} - B_{s \wedge \tau_a}(X^n)| \xrightarrow{\mathrm{P}} 0$.

(8) *For all* $t \in D$ *and* $a > 0$, $C^n_{t \wedge \tau^n_a} - C_{t \wedge \tau_a}(X^n) \xrightarrow{\mathrm{P}} 0$.

(9) *For all* $a, t, \varepsilon > 0$,

$$\lim_{b \uparrow \infty} \limsup_{n \to \infty} \mathrm{P}\left\{ v^n_{\tau^n_a \wedge t}\left(x^2 \mathbf{1}_{\{|x|>b\}}\right) > \varepsilon \right\} = 0. \tag{5.7}$$

5.2. **Proof of Proposition 5.1.** Write $O^u_t = \mathscr{U}(u, t)$. We then begin by noting the semi-martingale characteristics of the process $\{O^u_t\}_{t \ge 0}$. First, O^u solves the s.d.e.,

$$dX_t = -X_t\, dt + \sqrt{2u}\, d\beta^u_t, \tag{5.8}$$

where $\{\beta^u_t\}_{t \ge 0}$ is the Brownian motion $\{\beta(u, t)\}_{t \ge 0}$. It follows that $O^u_t = B_t(O^u) +$ a martingale, where $B_t : \mathscr{D}(\mathbf{R}_+) \to \mathbf{R}$ is defined by $B_t(\omega) = -\int_0^t \omega(s)\, ds$. Also note that $\langle O^u \rangle_t = 2ut$. Since $\{O^u_t\}_{t \ge 0}$ is path-continuous, $v_t(g) \equiv 0$.

Proof of Proposition 5.1. We will verify the conditions of Theorem 5.3 as they apply to $\{S_n(u, \cdot)\}_{n=1}^{\infty}$ and O^u.

The total variation of $r \mapsto B_r(\omega) = -\int_0^r \omega(s)\, ds$ on $[0, t]$ is $V_t(B(\omega)) = \int_0^t |\omega(s)|\, ds$. Therefore,

$$V_{\tau_a(\omega) \wedge t}(B(\omega)) \le a(\tau_a(\omega) \wedge t). \tag{5.9}$$

Since $\langle M^c \rangle_t = 2ut$ and $v \equiv 0$, $F^a(t) = [(2u \vee a) + 1]t$ satisfies condition (1).

Condition (2) is met automatically because $v_t(g) \equiv 0$.

O^u is a Feller diffusion with infinitesimal drift $a(x) = -x$ and infinitesimal variance $\sigma^2(x) = 2u$. In particular, a is Lipshitz-continuous and σ^2 is bounded. Hence, by Theorems III.2.32, III.2.33, and III.2.40 of Jacod and Shiryaev (2002), condition (3) is satisfied.

Because $\langle O^u \rangle_t = 2ut$, it follows that $C_t = 2ut$; cf. (5.4). In particular, $\mathscr{D}(\mathbf{R}_+) \ni \omega \mapsto C(\omega)$ is constant. Because $v_t(g) = 0$ also, this establishes the continuity condition (4) for both C and v. Since $\omega \mapsto \int_0^t \omega(s)\, ds$ is Skorohod-continuous condition (4) is satisfied.

Condition (5) follows from Donsker's Theorem; see, for example, (Billingsley, 1968, Theorem 10.1).

Fix a non-negative $g \in \mathscr{C}_{bz}$, define $L = \sup_x g(x)$, and suppose that g vanishes on $[-\delta, \delta]$. Then, we have, in differential notation,

$$\begin{aligned} dv_t^n(g) &= \mathrm{E}\left[d_t g\left(\Delta_t \mathscr{S}_n(u,t)\right) \mid \mathscr{F}_t^n \right] = \sum_{k=1}^{\lfloor un \rfloor} \int_{-\infty}^{\infty} g\left(\frac{x - X_k(t)}{\sqrt{n}} \right) \nu(dx)\, dt \\ &\leq L \sum_{k=1}^{\lfloor un \rfloor} \nu\{|x - X_k(t)| \geq \sqrt{n}\delta\}\, dt \\ &\leq L \sum_{k=1}^{\lfloor un \rfloor} \frac{1}{\delta^{2+\varepsilon} n^{1+(\varepsilon/2)}} \int_{-\infty}^{\infty} |x - X_k(t)|^{2+\varepsilon}\, \nu(dx)\, dt \\ &\leq \frac{2^{2+\varepsilon} L}{\delta^{2+\varepsilon} n^{1+(\varepsilon/2)}} \sum_{k=1}^{\lfloor un \rfloor} \left(\nu_{2+\varepsilon} + |X_k(t)|^{2+\varepsilon}\right) dt. \end{aligned} \tag{5.10}$$

Here, ν_α denote the αth absolute moment of the measure ν. This implies condition (6).

Next, let $\mathscr{F}_t = \vee_{n=1}^{\infty} \mathscr{F}_t^n$ denote the σ-algebra generated by $\{S_n(u); 0 \leq u \leq t\}_{n=1}^{\infty}$, and note that

$$\mathrm{E}\left[dX_k(t) \mid \mathscr{F}_t \right] = \left(\int_{-\infty}^{\infty} x\, \nu(dx) - X_k(t) \right) dt = -X_k(t)\, dt. \tag{5.11}$$

Summing over k gives

$$\mathrm{E}\left[d_t \mathscr{S}_n(u,t) \mid \mathscr{F}_t \right] = -\mathscr{S}_n(u,t)\, dt. \tag{5.12}$$

Consequently, $\mathscr{S}_n(u,t)$ has the following semi-martingale decomposition:

$$\mathscr{S}_n(u,t) = -\int_0^t \mathscr{S}_n(u,s)\, ds + \text{ a local } \mathscr{F}\text{-martingale} \qquad \forall t \geq 0. \tag{5.13}$$

Likewise, from (5.8) we conclude that

$$O_t^u = -\int_0^t O_s^u\, ds + \text{ a local } \mathscr{F}\text{-martingale} \qquad \forall t \geq 0. \tag{5.14}$$

Together (5.13) and (5.14) verify condition (7).

Because ν has mean zero and variance one,

$$\begin{aligned} d_t v_t^n(x^2) &= \mathrm{E}\left[d_t \mathscr{S}_n^2(u,t) \,\middle|\, \mathscr{F}_t \right] \\ &= \sum_{k=1}^{\lfloor un \rfloor} \int_{-\infty}^{\infty} \left(\frac{x - X_k(t)}{\sqrt{n}} \right)^2 \nu(dx)\, dt \\ &= \left(\frac{\lfloor un \rfloor}{n} + \frac{1}{n} \sum_{k=1}^{\lfloor un \rfloor} (X_k(t))^2 \right) dt. \end{aligned} \tag{5.15}$$

The pure-jump character of $\mathscr{S}_n(u, \cdot)$ implies that the quadratic variation of the continuous part of the local martingale in (5.13) is zero, whence $C_t^n = v_t^n(x^2)$. By the computation above and the law of large numbers, $C_t^n \xrightarrow{\mathrm{P}} 2ut = C_t = \langle O^u \rangle_t$. Therefore, condition (8) is satisfied.

Finally, after recalling that ν_α is the $\alpha^{\underline{\text{th}}}$ absolute moment of ν, we have

$$\begin{aligned} v_t^n\left(x^2\mathbf{1}_{\{|x|>b\}}\right) &= \int_0^t \mathrm{E}\left[d_s\mathscr{S}_n^2(u,s)\mathbf{1}_{\{|d_s\mathscr{S}_n(u,s)|>b\}}\;\middle|\;\mathscr{F}_s\right] \\ &= \frac{1}{n}\int_0^t\sum_{k=1}^{\lfloor un\rfloor}\int_{-\infty}^{\infty}(x-X_k(s))^2\,\mathbf{1}_{\{|x-X_k(s)|>b\}}\,\nu(dx)\,ds \\ &\le \frac{1}{n}\int_0^t\sum_{k=1}^{\lfloor un\rfloor}\left[\int_{-\infty}^{\infty}(x-X_k(s))^{2+\varepsilon}\,\nu(dx)\right]^{2/(2+\varepsilon)} \\ &\qquad\times\left[\nu\{|x-X_k(s)|>b\}\right]^{\varepsilon/(2+\varepsilon)}\,ds \\ &\le \frac{2^{2+\varepsilon}}{nb^{\varepsilon/(2+\varepsilon)}}\int_0^t\sum_{k=1}^{\lfloor un\rfloor}\left(\nu_{2+\varepsilon}+|X_k(s)|^{2+\varepsilon}\right)^{2/(2+\varepsilon)}\left(\nu_1+|X_k(s)|\right)^{\varepsilon/(2+\varepsilon)}\,ds. \end{aligned} \tag{5.16}$$

By the stationarity of X,

$$\begin{aligned} &\mathrm{E}\left[v_t^n\left(x^2\mathbf{1}_{\{|x|>b\}}\right)\right] \\ &\le \frac{2^{2+\varepsilon}t}{b^{\varepsilon/(2+\varepsilon)}}\mathrm{E}\left[\left(\nu_{2+\varepsilon}+|X_1(0)|^{2+\varepsilon}\right)^{2/(2+\varepsilon)}\left(\nu_1+|X_1(0)|\right)^{\varepsilon/(2+\varepsilon)}\right]. \end{aligned} \tag{5.17}$$

Also, since $t\mapsto v_t^n(x^2\mathbf{1}_{\{|x|>b\}})$ is non-decreasing we have

$$\mathrm{E}\left[v^n_{\tau_a^n\wedge t}\left(x^2\mathbf{1}_{\{|x|>b\}}\right)\right]\le\frac{At}{b^{\varepsilon/(2+\varepsilon)}}. \tag{5.18}$$

Therefore, by Markov's inequality, condition (9) holds. □

5.3. **Tightness.** This portion contains a variation on the argument in Khoshnevisan et al. (2004b, §4). We appeal to a criterion for tightness in $\mathscr{D}([0,1]^2)$ due to Bickel and Wichura (1971). [Because $\mathscr{D}([0,1]^2)\simeq\mathscr{D}(\mathscr{D}([0,1]))$, we will not make a distinction between the two spaces.]

A *block* is a two-dimensional half-open rectangle whose sides are parallel to the axes; i.e., I is a block if and only if it has the form $(s,t]\times(u,v]\subseteq(0,1]^2$. Two blocks I and I' are *neighboring* if either: (i) $I=(s,t]\times(u,v]$ and $I'=(s',t']\times(u,v]$ (horizontal neighboring); or (ii) $I=(s,t]\times(u,v]$ and $I'=(s,t]\times(u',v']$ (vertical neighboring).

Given any two-parameter stochastic process $Y=\{Y(s,t);\ s,t\in[0,1]\}$, and any block $I=(s,t]\times(u,v]$, the *increment of Y over I* [written as $\Delta Y(I)$] is defined as

$$\Delta Y(I)=Y(t,v)-Y(t,u)-Y(s,v)+Y(s,u). \tag{5.19}$$

Lemma 5.4 (Refinement to Bickel and Wichura (1971, Theorem 3))**.** *Let $\{Y_n\}_{n=1}^\infty$ denote a sequence of random fields in $\mathscr{D}([0,1]^2)$ such that for all $n\ge1$, $Y_n(s,t)=0$ if $st=0$. Suppose that there exist constants $A_{5.20}>1$, $\theta_1,\theta_2,\gamma_1,\gamma_2>0$ such that they are all independent of n, and whenever $I=(s,t]\times(u,v]$ and $J=(s',t']\times(u',v']$ are neighboring blocks, and if $s,t,s',t'\in n^{-1}\mathbf{Z}\cap[0,1]$, then*

$$\mathrm{E}\left[\,|\Delta Y_n(I)|^{\theta_1}\,|\Delta Y_n(J)|^{\theta_2}\,\right]\le A_{5.20}\,|I|^{\gamma_1}\,|J|^{\gamma_2}\,, \tag{5.20}$$

where $|I|$ and $|J|$ denote respectively the planar Lebesgue measures of I and J. If, in addition, $\gamma_1+\gamma_2>1$, then $\{Y_n\}_{n=1}^\infty$ is a tight sequence.

Additionally, we need the following *a priori* estimate.

Lemma 5.5. *In Theorem 1.4,*

$$\mathrm{E}\left[\max_{k\in\{1,\dots,n\}}\sup_{u\in[0,1]}|S_k(u)|^2\right]\le 64n \qquad \forall n\ge 1. \tag{5.21}$$

Proof. We choose and fix an integer $n\ge 1$. Also, we write $\mathrm{E}_{\mathscr{N}}$ for the conditional-expectation operator $\mathrm{E}[\cdots\mid\mathscr{N}]$, where $\mathscr{N}$ denotes the σ-algebra generated by the clocks.

We can collect the jump-times of the process $\{S_i(u)\}_{u\in[0,1]}$ for all $i=1,\dots,n$. These times occur at the jump-times of a homogeneous, mean-n Poisson process by time one. Define $T_0=0$ and enumerate the said jumps to obtain $0=T_0<T_1<T_2<\dots<T_{N(n)}$. The variable $N(n)$ has the Poisson distribution with mean n.

If $u\in[T_j,T_{j+1})$, then $S_n(u)=S_n(T_j)=\sum_{\ell=0}^{j-1}\{S_n(T_{\ell+1})-S_n(T_\ell)\}$. This proves that

$$\sup_{u\in[0,1]}|S_n(u)|=\max_{1\le j\le N(n)}\left|\sum_{\ell=0}^{j-1}\zeta_\ell\right|. \tag{5.22}$$

Here, the ζ's are independent of $\mathscr{N}$, and have the same distribution as $\nu\star\nu^-$ where $\nu^-(G)=\nu(-G)$. Moreover, the ζ_{2i}'s [resp. ζ_{2i+1}'s] form an independent collection. In accord with Doob's maximal $(2,2)$-inequality,

$$\begin{aligned}
&\mathrm{E}_{\mathscr{N}}\left[\sup_{u\in[0,1]}|S_n(u)|^2\right]\\
&\le 2\left\{\mathrm{E}_{\mathscr{N}}\left[\max_{1\le j\le N(n)}\left|\sum_{\ell<j:\text{ odd}}\zeta_\ell\right|^2\right]+\mathrm{E}_{\mathscr{N}}\left[\max_{1\le j\le N(n)}\left|\sum_{\ell<j:\text{ even}}\zeta_\ell\right|^2\right]\right\}\\
&\le 8\mathrm{E}_{\mathscr{N}}\left[\sum_{\ell=0}^{N(n)-1}\zeta_\ell^2\right]=16N(n)\qquad\text{a.s.}
\end{aligned} \tag{5.23}$$

[We have used also the inequality $(x+y)^2\le 2\{x^2+y^2\}$.] Take expectations to obtain

$$\mathrm{E}\left[\sup_{u\in[0,1]}|S_n(u)|^2\right]\le 16n. \tag{5.24}$$

It is easy to see that $n\to\sup_{u\in[0,1]}|S_n(u)|$ is a submartingale. Thus, Doob's strong $(2,2)$-inequality and (5.24) together imply the lemma. □

5.4. **Proof of Theorem 1.4.** We proceed in two steps.

Step 1. The $L^4(\mathrm{P})$ *Case.* First we derive the theorem when $\mathrm{E}\{|X_0(u)|^4\}$ is finite. In this case, (5.3) holds and so it remains to derive tightness. We do so by appealing to Lemma 5.4.

Consider first the vertical neighboring case. By the stationarity of the increments of random walks we need only consider the case where $I=(0,s]\times(0,u]$

and $J=(0,s]\times(u,v]$, where $s\in n^{-1}\mathbf{Z}$. Clearly,

$$\Delta\mathscr{S}_n(I)=\mathscr{S}_n(s,u)-\mathscr{S}_n(s,0)=\frac{S_{\lfloor sn\rfloor}(u)-S_{\lfloor sn\rfloor}(0)}{\sqrt{n}},$$
(5.25)
$$\Delta\mathscr{S}_n(J)=\mathscr{S}_n(s,v)-\mathscr{S}_n(s,u)=\frac{S_{\lfloor sn\rfloor}(v)-S_{\lfloor sn\rfloor}(u)}{\sqrt{n}}.$$

By the Cauchy-Schwarz inequality, $\|\Delta\mathscr{S}_n(I)\Delta\mathscr{S}_n(J)\|_2^2\le\|\Delta\mathscr{S}_n(I)\|_4^4\,\|\Delta\mathscr{S}_n(J)\|_4^4$. Note that the distribution of $\Delta\mathscr{S}_n(I)$ [resp. $\Delta\mathscr{S}_n(J)$] is the same as $\sum_{i=1}^{N_n(u)}(\xi_i-\xi_i')$ [resp. $\sum_{i=1}^{N_n(v-u)}(\xi_i-\xi_i')$], where: (i) $\{\xi_i\}_{i=1}^\infty$ is an i.i.d. sequence, each distributed according to ν; (ii) $\{\xi_i'\}_{i=1}^\infty$ is an independent copy of $\{\xi_i\}_{i=1}^\infty$; and (iii) $N_n(r)$ is a Poisson random variable, with mean $\lfloor nr\rfloor$, that is independent of all of the ξ's. These remarks, together with a direct computation, show that there exists a finite constant K such that $\|\Delta\mathscr{S}_n(I)\Delta\mathscr{S}_n(J)\|_2\le K|I|\,|J|$. A similar inequality is valid for the horizontal neighboring case. That is simpler to derive than the preceding, and so we omit the details. This and Lemma 5.4 together prove tightness in the case that the $X_k(0)$'s are in $L^4(\mathrm{P})$. According to Proposition 5.2, Theorem 1.4 follows suit in the case that $X_1(0)\in L^4(\mathrm{P})$.

Step 2. Truncation. Now we prove Theorem 1.4 under the conditions give there; that is, $\int_{-\infty}^\infty x\,\nu(dx)=0$ and $\int_{-\infty}^\infty x^2\,\nu(dx)=1$.

For any $c>0$ define $X_k^c(u)=X_k(u)\mathbf{1}_{\{|X_k(u)|\le c\}}-\int_{-c}^c x\,\nu(dx)$. Also define $S_n^c(u)=\sum_{k=1}^n X_k^c(u)$. It is easy to see that $\{S_n^c\}_{n=1}^\infty$ and $\{S_n-S_n^c\}_{n=1}^\infty$ define two independent, centered, dynamical random walks. According to Step 1, $\sigma(c)\mathscr{S}_n^c\Rightarrow\mathscr{U}$ as $n\to\infty$, where: (a) $\mathscr{S}^c$ is defined as $\mathscr{S}$, but in terms of the X^c's instead of the X's; and (b) $\sigma^{-2}(c)=\mathrm{Var}(X_1(0);|X_1(0)|\le c)$. Because $\lim_{c\to\infty}\sigma(c)=1$ and $\mathscr{U}$ is continuous it suffices to prove that for all $\varepsilon>0$,

$$\lim_{c\to\infty}\sup_{n\ge1}\mathrm{P}\left\{\sup_{s,t\in[0,1]}|\mathscr{S}_n(s,t)-\mathscr{S}_n^c(s,t)|\ge\varepsilon\right\}=0. \tag{5.26}$$

But we can change scale and apply Lemma 5.5 to deduce that

$$\mathrm{E}\left[\max_{k\in\{1,\dots,n\}}\sup_{u\in[0,1]}|S_k(u)-S_k^c(u)|^2\right]\le 64\mathrm{Var}\left(X_1(0);|X_1(0)|\ge c\right)n, \tag{5.27}$$

for all integers $n\ge1$. Equation (5.26) follows from the preceding and the Chebyshev inequality; Theorem 1.4 follows. □

6. PROOF OF THEOREM 1.6

First, we develop some estimates for general random walks. Thus, for the time being, let $\{s_n\}_{n=1}^\infty$ denote a random walk on $\mathbf{Z}$ with increments $\{\xi_n\}_{n=1}^\infty$. As is customary, let P_x denote the law of $\{x+s_n\}_{n=1}^\infty$ for any $x\in\mathbf{R}$, and introduce s_0 so that $P_z\{s_0=z\}=1$ for all $z\in\mathbf{Z}$; note that $\mathrm{P}=P_0$. We assume, for the time being, that the set of possible points of $\{s_n\}_{n=1}^\infty$ generates the entire additive group $\mathbf{Z}$. Thanks to the free abelian-group theorem this is a harmless assumption. See Khoshnevisan (2002, p. 78) for details. Define

$$G(n)=\sum_{i=1}^n P_0\{s_i=0\}\qquad\forall n\ge1. \tag{6.1}$$

Lemma 6.1. *For all $n \geq 1$ and $z \in \mathbf{Z}$,*

$$P_z\{T(0) > n\} \leq \frac{1}{G(n)P_0\{T(z) \leq T(0)\}}. \tag{6.2}$$

Proof. We start with a last-exit decomposition. Because the following are disjoint events,

$$\begin{aligned} 1 &\geq \sum_{j=1}^{n} P_0\left\{s_j = 0, s_{j+1} \neq 0, \ldots, s_{j+n} \neq 0\right\} \\ &= \sum_{j=1}^{n} P_0\left\{s_j = 0, s_{j+1} - s_j \neq 0, \ldots, s_{j+n} - s_j \neq 0\right\} \\ &= \sum_{j=1}^{n} P_0\{s_j = 0\}P_0\{T(0) > n\} \\ &= G(n)P_0\{T(0) > n\}. \end{aligned} \tag{6.3}$$

By the strong Markov property,

$$P_0\{T(0) > n\} \geq P_0\{T(z) \leq T(0)\}P_z\{T(0) > n\}. \tag{6.4}$$

The result follows from this and the preceding display. □

Consider the local times,

$$L_n^x = \sum_{j=0}^{n} \mathbf{1}_{\{s_j = x\}} \qquad \forall x \in \mathbf{Z},\ n \geq 0. \tag{6.5}$$

Evidently, $G(n) = E_0[L_n^0] - 1$, where E_z denotes the expectation operator under P_z.

Lemma 6.2. *For all $z \in \mathbf{Z}$ and $n \geq 1$, $P_z\{T(0) > n\} \leq E_0[L^0_{T(z)}]/G(n)$.*

Proof. If $z = 0$, then $L^0_{T(z)} = 2$, and the lemma follows from Lemma 6.1. From now on, we assume that $z \neq 0$. We can apply the strong Markov property to the return times to z, and deduce that for all non-negative integers k,

$$P_0\left\{L^0_{T(z)} = k+1\right\} = \left[P_0\left\{T(0) < T(z)\right\}\right]^k P_0\{T(z) < T(0)\}. \tag{6.6}$$

[The $k+1$ is accounted for by the fact that $L_0^0 = 1$.] Therefore, the P_0-law of $L^0_{T(z)}$ is geometric with mean

$$E_0\left[L^0_{T(z)}\right] = \frac{1}{P_0\left\{T(z) < T(0)\right\}}. \tag{6.7}$$

This and Lemma 6.1 together prove the lemma. □

Lemma 6.3. *If $\{s_n\}_{n=1}^{\infty}$ is recurrent, then for all non-zero integers z and all $n \geq 1$,*

$$\begin{aligned} &P_z\left\{T(0) > n\right\} \leq \frac{2\{1 + G(\theta(z))\}}{G(n)}, \ \textit{where} \\ &\theta(z) = \inf\left\{n \geq 1 :\ P_0\left\{T(z) > n\right\} \leq \frac{1}{8}\right\}. \end{aligned} \tag{6.8}$$

Proof. Recurrence insures that $\theta(z)$ is finite for all $z \in \mathbf{Z}$. Now for any positive integer m,

$$E_0\left[L^0_{T(z)}\right] \le E_0\left[L^0_m\right] + E_0\left[L^0_{T(z)}; T(z) > m\right]$$
$$\le 1 + G(m) + \sqrt{E_0\left[\left(L^0_{T(z)}\right)^2\right] P_0\left[T(z) > m\right]}. \tag{6.9}$$

Since $L^0_{T(z)}$ has a geometric distribution [see (6.6)], $E_0[(L^0_{T(z)})^2] \le 2\{E_0[L^0_{T(z)}]\}^2$. Thus,

$$E_0\left[L^0_{T(z)}\right] \le 1 + G(m) + E_0\left[L^0_{T(z)}\right]\sqrt{2P_0\left[T(z) > m\right]}. \tag{6.10}$$

Choose $m = \theta(z)$ to find that the square root is at most $\frac{1}{2}$. Solve for $E_0[L^0_{T(z)}]$ to finish. □

Lemma 6.4. *Suppose* $\mathrm{E}[\xi_1] = 0$ *and* $\sigma^2 = \mathrm{E}[\xi_1^2] < \infty$. *Then we can find a finite constant* $A_{6.11} > 1$ *such that*

$$P_z\{T(0) > n\} \le A_{6.11}\frac{1+|z|}{\sqrt{n}} \qquad \forall z \in \mathbf{Z},\ n \ge 1. \tag{6.11}$$

Proof. First of all, we claim that there exists $A_{6.12} > 1$ such that for all $n \ge 1$,

$$A_{6.12}^{-1}\sqrt{n} \le G(n) \le A_{6.12}\sqrt{n}. \tag{6.12}$$

When $\{s_n\}_{n=1}^\infty$ is strongly aperiodic this follows from the local central limit theorem (Spitzer, 1976, II.7.P9). In the general case, consider the random walk $\{s'_n\}_{n=1}^\infty$ whose increment-distribution is $\frac{1}{2}(\nu + \delta_0)$. The walk $\{s'_n\}_{n=1}^\infty$ has the same law as $\{s_{c(n)}\}_{n=1}^\infty$ where $c(n) = \min\{m : \lambda_0 + \cdots + \lambda_m \ge n\}$ for an i.i.d. sequence $\{\lambda_n\}_{n=1}^\infty$ of mean-($\frac{1}{2}$) geometric random variables that are totally independent of $\{s_n\}_{n=1}^\infty$. Because $\sum_{i=0}^n \mathbf{1}_{\{s'_i=0\}} = \sum_{i=0}^n \lambda_i \mathbf{1}_{\{s_i=0\}}$, it follows that $G'(n) = 2G(n)$ where $G'(n) = \sum_{i=1}^n P\{s'_i = 0\}$. Because $\{s'_n\}_{n=1}^\infty$ is strongly aperiodic, (6.12) follows. In light of this and Lemmas 6.1 and 6.3, it suffices to prove that

$$\theta(z) = O(z^2) \quad \text{as } |z| \to \infty \text{ in } \mathbf{Z}. \tag{6.13}$$

If $\beta > 0$ is fixed, then

$$P_z\left\{T(0) > \lfloor \beta z^2 \rfloor\right\} = P_0\left\{L^{-z}_{\lfloor \beta z^2 \rfloor} = 0\right\} \le P_0\left\{L^{-z}_{\lfloor \beta z^2 \rfloor} \le \sqrt{|z|}\right\}$$
$$= P_0\left\{\ell^{-1}_{\sigma\beta} \le \sigma\right\} + o(1) \quad \text{as } |z| \to \infty. \tag{6.14}$$

Here ℓ^{-1}_t denotes the local time of Brownian motion at -1 by time t. [The preceding display follows from the local-time invariance principle of Borodin (1981).] Recurrence of Brownian motion implies that there exist $\beta, z_0 > 0$ such that whenever $|z| \ge z_0$, $P_z\{T(0) > \beta z^2\} \le \frac{1}{8}$; i.e., $\theta(z) \le \beta z^2$ as long as $|z| \ge z_0$. This verifies (6.13) and completes our proof. □

Proof of Theorem 1.6. We can appeal to the free abelian-group theorem again to assume without loss of generality that the possible values of $\{s_n\}_{n=1}^\infty$ generate the entire additive group $\mathbf{Z}$.

Apply (6.10) with $m = \theta(z)$ to find that $E_0[L^0_{T(z)}] \le 2\{1 + G(\theta(z))\}$. Combine this with (6.12) and (6.13) to find that $E_0[L^0_{T(z)}] \le A\sqrt{1+z^2}$ for some constant A

that does not depend on $z \in \mathbf{Z}$. This and (6.7) together imply the lower bound of Theorem 1.6.

To obtain the other bound let $\tau = \inf\{n : s_n \leq 0\}$. Because $T(0) \geq \tau$, Lemma 6.4 and (6.4) together prove that

$$\frac{A_{6.11}}{\sqrt{n}} \geq P_0\{T(z) \leq T(0)\} P_z\{\tau > n\}. \tag{6.15}$$

Thanks to Pemantle and Peres (1995, Lemma 3.3), as long as $|z| \leq A'\sqrt{n}$ for a fixed A', $P_z\{\tau > n\} \geq A''|z|/\sqrt{n}$ for a fixed A''. The result follows. □

Remark 6.5. The last portion of the preceding proof shows also that $P_z\{T(0) > n\} \geq A''|z|/\sqrt{n}$. This proves that the bound in Lemma 6.4 is sharp up to a multiplicative constant.

7. PROOF OF THEOREM 1.5

The basic outline of our proof follows the same general line of thought as the derivation of (3.1) of Penrose (1990). However, as was noted by Benjamini et al. (2003), the present discrete set-up contains inherent technical difficulties that do not arise in the continuous setting of Penrose (1990).

Choose and fix a large positive integer M, and define

$$\gamma = \frac{3}{6+2\varepsilon}, \qquad q_n = \left\lfloor \frac{n}{M} \right\rfloor, \qquad {}^\forall n = 1, 2, \ldots. \tag{7.1}$$

Within $[n/2, n]$ we can find $\lfloor n/(4q_n) \rfloor$-many closed intervals $\{I_k^n\}_{k=1}^{\lfloor n/(4q_n)\rfloor}$, of length q_n each, such that the distance between I_i^n and I_j^n is at least q_n if $i \neq j$. Motivated by §5 of Benjamini et al. (2003), let $E_n(t)$ denote the event that

(7.2) $\{S_k(t)\}_{k=0}^\infty$ takes both (strictly) positive and (strictly) negative values in every one of $I_1^n, \ldots, I_{\lfloor n/(4q_n)\rfloor}^n$.

Also let $F_n(t)$ denote the event that

(7.3) $\{S_k(t)\}_{k=0}^\infty$ does not return to zero in $[n/2, n]$.

Lemma 7.1. *Uniformly for all* $t \geq 0$,

$$\limsup_{n\to\infty} \frac{\ln \mathrm{P}(E_n(t) \cap F_n(t))}{\ln n} \leq -\frac{M\gamma}{12}. \tag{7.4}$$

Proof. The uniformity assertion holds tautologically since $\mathrm{P}(E_n(t) \cap F_n(t))$ does not depend on $t \geq 0$. Without loss of generality, we may and will work with $t = 0$.

Let f_i^n denote the smallest value in I_i^n. Also define

$$c_i^n = \inf\left\{\ell \in I_i^n \setminus \{f_i^n\} : \ S_{\ell-1}(0) S_\ell(0) < 0\right\}, \tag{7.5}$$

where $\inf \varnothing = \infty$. Finally, define A_i^n to be the event that c_i^n is finite, but $S_k(0) \neq 0$ for all $k = c_i^n + 1, \ldots, c_i^n + q_n$. A little thought shows that for any integer $j \geq 1$,

$$\begin{aligned} &\mathrm{P}\left(A_{j+1}^n \,\middle|\, A_1^n, \ldots, A_j^n\right) \\ &\leq \mathrm{P}\left\{\max_{1\leq i\leq n} |X_i(0)| \geq n^\gamma\right\} + \sup_{|x|\leq n^\gamma} P_x\left\{S_k(0) \neq 0, \ {}^\forall k = 1, \ldots, q_n\right\}. \end{aligned} \tag{7.6}$$

To estimate the first term we note that $(2+\varepsilon)\gamma - 1 = \gamma/3$. Therefore,

$$\begin{aligned}\mathrm{P}\left\{\max_{1\le i\le n}|X_i(0)|\ge n^\gamma\right\} &\le n\mathrm{P}\left\{|X_1(0)|\ge n^\gamma\right\} \le \frac{\mathrm{E}\left\{|X_1(0)|^{2+\varepsilon}\right\}}{n^{-(2+\varepsilon)\gamma-1}}\\ &= O\left(n^{-\gamma/3}\right) \qquad \text{as } n\to\infty.\end{aligned} \tag{7.7}$$

See (7.1). On the other hand, by Lemma 6.4 and (7.1),

$$\sup_{|x|\le n^\gamma} P_x\left\{S_k(0)\neq 0,\ ^\forall k=1,\ldots,q_n\right\} \le A_{6.11}\frac{n^\gamma}{\sqrt{q_n}} = O\left(n^{-\gamma/3}\right), \tag{7.8}$$

as $n\to\infty$. These remarks, together with (7.6) imply that

$$\sup_{j\ge 1}\mathrm{P}\left(A_{j+1}^n \,\middle|\, A_1^n,\ldots,A_j^n\right) = O\left(n^{-\gamma/3}\right). \tag{7.9}$$

Thus, as $n\to\infty$,

$$\begin{aligned}\mathrm{P}(E_n(t)\cap F_n(t)) &\le \mathrm{P}\left(\bigcap_{i=1}^{\lfloor n/(4q_n)\rfloor} A_i^n\right)\\ &= O\left(n^{-\gamma\lfloor n/(4q_n)\rfloor/3}\right) \le n^{o(1)-M\gamma/12}.\end{aligned} \tag{7.10}$$

This proves the lemma. □

Lemma 7.2. *There exists* $M_0 = M_0(\varepsilon)$ *such that whenever* $M > M_0$,

$$\sum_{n=1}^\infty \mathrm{P}\left(\bigcap_{s\in[0,1]}[E_n(s)\cap F_n(s)]\right) < \infty. \tag{7.11}$$

Proof. By Lemma 7.1 and the strong Markov property,

$$\int_0^\infty \mathrm{P}\left(\bigcap_{s\in[0,t]}[E_n(s)\cap F_n(s)]\right)e^{-t}\,dt \le n^{o(1)+1-M\gamma/12} \qquad (n\to\infty). \tag{7.12}$$

See the proof of Lemma 5.3 of Benjamini et al. (2003). On the other hand,

$$\begin{aligned}&\int_0^\infty \mathrm{P}\left(\bigcap_{s\in[0,t]}[E_n(s)\cap F_n(s)]\right)e^{-t}\,dt\\ &\ge \frac{1}{e}\int_0^1 \mathrm{P}\left(\bigcap_{s\in[0,t]}[E_n(s)\cap F_n(s)]\right)dt\\ &\ge \frac{1}{e}\mathrm{P}\left(\bigcap_{s\in[0,1]}[E_n(s)\cap F_n(s)]\right).\end{aligned} \tag{7.13}$$

Therefore, $\mathrm{P}(\cap_{s\in[0,1]}[E_n(s)\cap F_n(s)]) \le n^{o(1)+1-M\gamma/12}$. The lemma follows with $M_0 = 24/\gamma$. □

The following is essentially Lemma 5.4 of Benjamini et al. (2003). To prove it, go through their derivation, and replace their I_i^n's by ours.

Lemma 7.3. *Suppose $M > M_0$. Then,*

$$\mathrm{P}\left(\bigcap_{t\geq 0} \limsup_n E_n(t)\right) = 1. \tag{7.14}$$

Proof of Theorem 1.5. Choose and fix some $M > M_0$. Then follow along the proof of Benjamini et al. (2003, Theorem 1.11), but replace their τ by one, and the respective applications of their Lemmas 5.3 and 5.4 by our Lemmas 7.2 and 7.3. □

8. Applications to the OU Process on Classical Wiener Space

Let β denote a two-parameter Brownian sheet and consider once more the construction (5.2). In addition, recall from §5.2 the process $\{O_t^u; u \in [0,1]\}_{t\geq 0}$, which can be written in terms of the Brownian sheet β as follows:

$$O_t^u = \frac{\beta(u, e^{2t})}{e^t} \qquad {}^\forall t \geq 0, u \in [0,1]. \tag{8.1}$$

This proves readily that the process $\{O_t^\bullet\}_{t\geq 0}$ is an infinite-dimensional stationary diffusion on $C([0,1])$ whose invariant measure is the Wiener measure on $C([0,1])$. The process $O = \{O_t^\bullet\}_{t\geq 0}$ is a fundamental object in infinite-dimensional analysis. See, for example, Kuelbs (1973), Malliavin (1979), and Walsh (1986). These furnish three diverse theories in each of which O plays a central role.

An interesting artifact of our Theorem 1.4 is that it gives the coin-tosser a chance to understand some of this infinite-dimensional theory. For example, note that for any fixed $u \geq 0$, the process $\{O_t^u\}_{t\geq 0}$ is an ordinary one-dimensional Ornstein–Uhlenbeck process. Therefore, Corollary 1.3 can be stated, equivalently, as follows:

Corollary 8.1. *Let E and H be as in Theorem 1.2. Then there exists a finite constant $A_{8.2} > 1$ such that for all $z \geq 1$ and $u \geq 0$,*

$$A_{8.2}^{-1} \mathrm{K}_E\left(\frac{1}{z^2}\right)\bar{\Phi}(z) \leq \mathrm{P}\left\{\sup_{t\in E} O_t^u \geq z\right\} \leq A_{8.2}\mathrm{K}_E\left(\frac{1}{z^2}\right)\bar{\Phi}(z). \tag{8.2}$$

Similarly, the methods of this paper yield the following. We omit the details.

Corollary 8.2. *If E and H are as in Theorem 2.1,*

$$\begin{aligned} &\sup_{t\in E}\limsup_{u\to\infty}\left[O_t^u - H(u)\sqrt{u}\right] > 0 \iff \Psi_H(E) = +\infty \\ &\dim_{\mathscr{H}}\left\{t \in [0,1] :\ \limsup_{u\to\infty}\left[O_t^u - H(u)\sqrt{u}\right] \geq 0\right\} = \min\left(\frac{4-\delta(H)}{2}\ ,\ 1\right). \end{aligned} \tag{8.3}$$

This is a multi-fractal extension of the main result of Mountford (1992) and extends some of the latter's infinite-dimensional potential theory. The results of this section seem to be new.

9. Concluding Remarks and Open Problems

The single-most important problem left open here is to remove the normality assumption in Theorems 1.1 and 1.2. For instance, these theorems are not known to hold in the most important case where the increments are Rademacher variables.

Problem 9.1. *Do Theorems 1.1 and 1.2 hold for all incremental distributions ν that have mean zero, variance one, and $2+\varepsilon$ moments for some $\varepsilon > 0$?*

We suspect the answer is yes, but have no proof in any but the Gaussian case. As regards our invariance principles, we cannot resolve the following:

Problem 9.2. *Does Theorem 1.5 hold for $\varepsilon = 0$?*

We do not have a plausible conjecture in either direction.

There is a large literature on tails of highly-oscillatory Gaussian random fields. See, for instance, Pickands (1967) and Qualls and Watanabe (1971); see Berman (1992) for a pedagogic account as well as further references. In their simplest non-trivial setting, these works seek to find good asymptotic estimates for the tails of the distribution of $\sup_{t\in E} g(t)$ where g is a stationary centered Gaussian random field that satisfies $\mathrm{E}\{|g(0)-g(t)|^2\} = 1 + c(1+o(1))|t|^\alpha$ as $|t| \to 0$. The "time-set" E is often an interval or, more generally, a hyper-cube. What if E is a fractal set? More generally, one can ask:

Problem 9.3. *Do the results of §8 have analogues for more general Gaussian random fields?*

There are a number of other interesting a.s. properties of random walks one of which is the following due to Chung (1948): Suppose $\{\xi_i\}_{i=1}^\infty$ are i.i.d., mean-zero variance-one, and $\xi_1 \in L^3(\mathrm{P})$. Then $s_n = \xi_1 + \cdots + \xi_n$ satisfies

$$\liminf_{n\to\infty} \max_{1\le j\le n} \frac{|s_j|}{\sqrt{n/\ln\ln n}} = \frac{\pi}{\sqrt{8}} \quad \text{a.s.} \tag{9.1}$$

Chung (1948) contains the corresponding integral test. In the context of dynamical walks let us state, without proof, the following: If, in addition, $\xi_1 \in L^4(\mathrm{P})$, then

(9.2) Chung's LIL is dynamically stable.

That is, with probability one,

$$\liminf_{n\to\infty} \max_{1\le j\le n} \frac{|S_j(t)|}{\sqrt{n/\ln\ln n}} = \frac{\pi}{\sqrt{8}} \qquad \forall t \ge 0. \tag{9.3}$$

Problem 9.4. *What can one say about the set of times $t \in [0,1]$ at which $\{S_n(t)\}_{n=1}^\infty$ is below $\sqrt{n}/H(n)$ infinitely often?*

This is related to finding sharp estimates for the "lower tail" of $\max_{1\le j\le n}|S_j(t)|$. At this point, we have only partial results along these directions. For instance, when ν is standard normal, we can prove the existence of a constant A such that for all compact $E \subseteq [0,1]$,

$$\frac{e^{\pi^2/(8\varepsilon_n^2)}}{A\varepsilon_n^2} \le \mathrm{P}\left\{ \inf_{t\in[0,1]} \max_{1\le j\le n} |S_j(t)| \le \varepsilon_n\sqrt{n} \right\} \le \frac{Ae^{\pi^2/(8\varepsilon_n^2)}}{\varepsilon_n^6}, \tag{9.4}$$

for any $(0,1)$-valued $\{\varepsilon_n\}_{n=1}^\infty$ that tends to zero and $\liminf_n n\varepsilon_n^8 > \pi/\sqrt{2}$. The solution to the preceding problem would require, invariably, a tightening of this bound. In a companion article (Khoshnevisan et al., 2004a) we prove that the right-hand side of (9.4) is tight for the continuum-limit of dynamical walks. The said theorem uses a second-order eigenvalue estimate of Lifshits and Shi (2003) which is not yet available in the context of dynamical random walks. Thus it is natural to end the paper with the following open problem.

Problem 9.5. *Is the right-hand side of* (9.4) *is sharp up to a multiplicative constant?*

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