Strong approximations in a charged-polymer model*

Yueyun Hu
Université Paris 13

Davar Khoshnevisan
University of Utah

November 19, 2009

Abstract

We study the large-time behavior of the charged-polymer Hamiltonian H_n of Kantor and Kardar [Bernoulli case] and Derrida, Griffiths, and Higgs [Gaussian case], using strong approximations to Brownian motion. Our results imply, among other things, that in one dimension the process $\{H_{[nt]}\}_{0\le t\le 1}$ behaves like a Brownian motion, time-changed by the intersection local-time process of an independent Brownian motion. Chung-type LILs are also discussed.

Keywords: Charged polymers, strong approximation, local time.
AMS 2000 subject classification: Primary: 60K35; Secondary: 60K37.
Running Title: Polymer measures, strong approximations.

1 Introduction

Consider a sequence $\{q_i\}_{i=1}^{\infty}$ of independent, identicallly-distributed mean-zero random variables, and let $S := \{S_i\}_{i=0}^{\infty}$ denote an independent simple random walk on $\mathbf{Z}^d$ starting from 0. For $n \ge 1$, define

$$H_n := \sum\sum_{1\le i<j\le n} q_i q_j \mathbf{1}_{\{S_i=S_j\}}; \tag{1.1}$$

*Research supported in part by the NSF grant DMS-0706728

this is the Hamiltonian of a socalled "charged polymer model." See Kantor and Kadar [12] in the case that the q_i's are Bernoulli, and Derrida, Griffiths, and Higgs [8] for the case of Gaussian random variables. Roughly speaking, $q_1, q_2, \ldots$ are random charges that are placed on a polymer path modeled by the trajectories of S; and one can construct a Gibbs-type polymer measure from the Hamiltonian H_n.

We follow Chen [4] (LIL and moderate deviations), Chen and Khoshnevisan [5] (comparison between H_n and the random walk in random scenery model), and Asselah [1] (large deviations in high dimensional case), and continue the analysis of the Hamiltonian H_n. We assume here and in the sequel that

$$\mathrm{E}(q_1^2) = 1 \text{ and } \mathrm{E}\left(|q_1|^{p(d)}\right) < \infty \quad \text{where} \quad p(d) := \begin{cases} 6 & \text{if } d = 1, \\ 4 & \text{if } d \geq 2. \end{cases} \tag{1.2}$$

Theorem 1.1. *On a possibly-enlarged probability space, we can define a version of* $\{H_n\}_{n=1}^\infty$ *and a one-dimensional Brownian motion* $\{\gamma(t)\}_{t\geq 0}$ *such that the following holds almost surely:*

$$H_n = \begin{cases} \dfrac{1}{\sqrt{2}}\,\gamma\left(\displaystyle\int_{-\infty}^{\infty} (\ell_n^x)^2 \, \mathrm{d}x\right) + o(n^{\frac{3}{4}-\epsilon}) & \text{if } d = 1 \text{ and } 0 < \epsilon < \frac{1}{24}, \\ \dfrac{1}{\sqrt{2\pi}}\,\gamma(n \log n) + O(n^{\frac{1}{2}} \log\log n) & \text{if } d = 2, \\ \sqrt{\kappa}\,\gamma(n) + o(n^{\frac{1}{2}-\epsilon}) & \text{if } d \geq 3 \text{ and } 0 < \epsilon < \frac{1}{8}, \end{cases} \tag{1.3}$$

where $\{\ell_t^x\}_{t\geq 0, x\in\mathbf{R}}$ *denotes the local times of a linear Brownian motion* B *independent of* γ, *and* $\kappa := \sum_{k=1}^\infty \mathrm{P}\{S_k = 0\}$.

It was shown in [5] that when $d = 1$ the distribution of H_n converges, after normalization, to the "random walk in random scenery." The preceding shows that the stochastic process $\{H_{[nt]}\}_{0\leq t\leq 1}$ does *not* converge weakly to the random walk in random scenery; rather, we have the following consequence of Brownian scaling for all $T > 0$: As $n \to \infty$,

$$\left\{ \frac{H_{[nt]}}{n^{3/4}} \right\}_{0 \le t \le T} \xrightarrow{D([0,T])} \left\{ \frac{1}{\sqrt{2}} \gamma \left(\int_{-\infty}^{\infty} (\ell_t^x)^2 \, \mathrm{d}x \right) \right\}_{0 \le t \le T}. \tag{1.4}$$

With a little bit more effort, we can also obtain strong limit theorems. Let us state the following counterpart to the LILs of Chen [4], as it appears to have novel content.

Theorem 1.2. *Almost surely: (i) If $d = 1$, then*

$$\liminf_{n \to \infty} \left(\frac{\log \log n}{n} \right)^{3/4} \max_{0 \le k \le n} |H_k| = (a^*)^{3/4} \frac{\pi}{4},$$

where $a^* = 2.189 \pm 0.0001$ is a numerical constant [11, (0.6)]*;*

(ii) If $d = 2$, then

$$\liminf_{n \to \infty} \sqrt{\frac{\log \log n}{n \log n}} \max_{0 \le k \le n} |H_k| = \frac{\sqrt{\pi}}{4};$$

(iii) If $d \ge 3$, then

$$\liminf_{n \to \infty} \sqrt{\frac{\log \log n}{n}} \max_{0 \le k \le n} |H_k| = \pi \sqrt{\frac{\kappa}{8}},$$

where κ was defined in Theorem 1.1.

Theorems 1.1 and 1.2 are proved respectively in Sections 2 and 3.

2 Proof of Theorem 1.1

Let W be a one-dimensional Brownian motion starting from 0. By the Skorohod embedding theorem, there exists a sequence of stopping times $\{T_n\}_{n=1}^{\infty}$ such that $\{T_n - T_{n-1}\}_{n=1}^{\infty}$ (with $T_0 = 0$) are i.i.d., and:

$$\begin{aligned} &\mathrm{E}(T_1) = \mathrm{E}(q_1^2) = 1, \quad \mathrm{Var}(T_1) \le \mathrm{const} \cdot \mathrm{E}(q_1^4) < \infty, \quad \text{and} \\ &\{W(T_n) - W(T_{n-1})\}_{n=1}^{\infty} \stackrel{\text{law}}{=} \{q_n\}_{n=1}^{\infty}. \end{aligned} \tag{2.1}$$

Throughout this paper, we take the following special construction of the charges $\{q_i\}_{i=1}^{\infty}$:

$$q_n := W(T_n) - W(T_{n-1}) \qquad \text{for } n \geq 1. \tag{2.2}$$

Next, we describe how we choose a special construction of the random walk S, depending on d.

If $d = 1$, then on a possibly-enlarged probability space let B be another one-dimensional Brownian motion, independent of W. By using a theorem of Révész [14], we may construct a one-dimensional simple symmetric random walk $\{S_i\}_{i=1}^{\infty}$ from B such that almost surely,

$$\sup_{x \in \mathbf{Z}} |L_n^x - \ell_n^x| = n^{\frac{1}{4}+o(1)} \quad \text{as } n \to \infty, \text{ where } L_n^x := \sum_{i=1}^{n} \mathbf{1}_{\{S_i = x\}}, \tag{2.3}$$

and ℓ_n^x denotes the local times of B at x up to time n.

If $d \geq 2$, then we just choose an independent simple symmetric random walk $\{S_n\}_{n=1}^{\infty}$, after enlarging the probability space, if we need to.

Now we define the Hamiltonians $\{H_n\}_{n=1}^{\infty}$ via the preceding constructions of $\{q_i\}_{i=1}^{\infty}$ and $\{S_n\}_{n=1}^{\infty}$. That is,

$$\begin{aligned} H_n &= \sum\sum_{1 \leq i < j \leq n} (W(T_i) - W(T_{i-1}))(W(T_j) - W(T_{j-1}))\mathbf{1}_{(S_i = S_j)} \\ &= \int_0^{T_n} G_n \, \mathrm{d}W, \end{aligned} \tag{2.4}$$

where, for all integers $n \geq 1$ and reals $s \geq 0$,

$$G_n(s) := \sum\sum_{1 \leq i < j \leq n} \mathbf{1}_{(S_i = S_j)} (W(T_i) - W(T_{i-1})) \mathbf{1}_{(T_{j-1} \leq s < T_j)}. \tag{2.5}$$

By the Dambis, Dubins–Schwarz representation theorem [15, Theorem 1.6, p. 170], after possibly enlarging the underlying probability space, we can find a one-dimensional Brownian motion γ such that $\int_0^t G_n \, \mathrm{d}W$ is equal to $\gamma(\int_0^t |G_n(s)|^2 \, \mathrm{d}s)$ for $t \geq 0$. We stress the fact that if $d = 1$, then γ *is independent* of B. This is so, because the bracket between the two continuous martingales vanishes: $\langle\int_0^\bullet G_n\,\mathrm{d}W\,,\,B\rangle_t=0$ for $t\geq 0$. Consequently, the following holds for all $n\geq 1$: Almost surely,

$$H_n=\gamma(\Xi_n),\quad \text{where}\quad \Xi_n:=\int_0^{T_n}|G_n(s)|^2\,\mathrm{d}s. \tag{2.6}$$

Proposition 2.1. *The following holds almost surely:*

$$\Xi_n=\begin{cases}\dfrac{1}{2}\displaystyle\int_{-\infty}^{\infty}(\ell_n^x)^2\,\mathrm{d}x+O(n^{\frac{3}{2}-\epsilon}) & \text{if } d=1 \text{ and } 0<\epsilon<\frac{1}{12},\\ \dfrac{1}{2\pi}\,n\log n+O(n\log\log n) & \text{if } d=2,\\ \kappa n+O(n^{1-\epsilon}) & \text{if } d\geq 3 \text{ and } 0<\epsilon<\frac{1}{4}.\end{cases} \tag{2.7}$$

We prove this proposition later. First, we show that in case $d=1$, the preceding proposition estimates Ξ_n correctly to leading term.

Lemma 2.2. *If $d=1$, then a.s., $\int_{-\infty}^{\infty}(\ell_n^x)^2\,\mathrm{d}x=n^{\frac{3}{2}+o(1)}$ as $n\to\infty$.*

Proof. This is well known; we include a proof for the sake of completeness.

Because $\int_{-\infty}^{\infty}\ell_n^x\,\mathrm{d}x=n$, we have $\int_{-\infty}^{\infty}(\ell_n^x)^2\,\mathrm{d}x\leq n\sup_{-\infty<x<\infty}\ell_n^x$, and this is $n^{\frac{3}{2}+o(1)}$ [13]. For the converse bound we apply the Cauchy–Schwarz inequality to find that $n^2=(\int_{-\infty}^{\infty}\ell_n^x\,\mathrm{d}x)^2\leq\int_{-\infty}^{\infty}(\ell_n^x)^2\,\mathrm{d}x\cdot\mathrm{Osc}_{[0,n]}B$, where $\mathrm{Osc}_{[0,n]}B:=\sup_{[0,n]}B-\inf_{[0,n]}B=n^{\frac{1}{2}+o(1)}$ by Khintchine's LIL. This completes the proof. $\square$

Let us complete the proof of Theorem 1.1, first assuming Proposition 2.1. That proposition will then be proved subsequently.

Proof of Theorem 1.1. We shall consider only the case $d=1$; the cases $d=2$ and $d\geq 3$ are proved similarly. We apply the Csörgő–Révész modulus of continuity of Brownian motion [7, Theorem 1.2.1] to $H_n=\gamma(\Xi_n)$—see (2.6)—with the changes of variables, $t=n^{\frac{3}{2}+o(1)}$ and $a(t)=n^{\frac{3}{2}-\epsilon}$; then apply Lemma 2.2 to see that $|\gamma(\Xi_n)-\gamma(\frac{1}{2}\int_{-\infty}^{\infty}(\ell_n^x)^2\,\mathrm{d}x)|=O(n^{\frac{3}{4}-\epsilon})$ as $n\to\infty$ a.s. $\square$

Lemma 2.3. *The following holds almost surely:*

$$\sum\sum_{1\le i<k\le n} \mathbf{1}_{\{S_i=S_k\}} = \begin{cases} \dfrac{1}{2}\displaystyle\int_{-\infty}^{\infty} (\ell_n^x)^2\,\mathrm{d}x + n^{\frac{5}{4}+o(1)} & \textit{if } d=1, \\ \dfrac{1}{2\pi}\, n\log n + O(n\log\log n) & \textit{if } d=2, \\ \kappa n + n^{\frac{1}{2}+o(1)} & \textit{if } d\ge 3. \end{cases} \tag{2.8}$$

Proof. In the case that $d=2$, this result follows from Bass, Chen and Rosen [2]; and in the case $d\ge 3$, from Chen [4, Theorem 5.2]. Therefore, we need to only check the case $d=1$.

We begin by writing $\sum\sum_{1\le i<k\le n}\mathbf{1}_{\{S_i=S_k\}} = \frac{1}{2}\sum_{x\in\mathbf{Z}}(L_n^x)^2 - \frac{n}{2}$. According to Bass and Griffin [3, Lemma 5.3], $\sup_{x\in\mathbf{Z}}\sup_{y\in[x,x+1]}|\ell_n^x - \ell_n^y| = n^{\frac{1}{4}+o(1)}$ as $n\to\infty$ a.s. This and (2.3) together imply that $|L_n^x - \ell_n^y| = n^{\frac{1}{4}+o(1)}$ uniformly over all $y\in[x\,,x+1]$ and $x\in\mathbf{Z}$ [a.s.], whence

$$\left|\sum_{x\in\mathbf{Z}}(L_n^x)^2 - \int_{-\infty}^{\infty}(\ell_n^y)^2\,\mathrm{d}y\right| \le n^{\frac{1}{4}+o(1)}\cdot\sum_{x\in\mathbf{Z}}\int_x^{x+1}(L_n^x+\ell_n^y)\,\mathrm{d}y. \tag{2.9}$$

Since the latter sum is equal to $2n$, the lemma follows. □

Lemma 2.4. *The following holds a.s.: As $n\to\infty$,*

$$\sum\sum_{1\le i<k\le n}\mathbf{1}_{\{S_i=S_k\}}\left(q_i^2-1\right) = \begin{cases} n^{\frac{7}{6}+o(1)} & \textit{if } d=1, \\ n^{\frac{3}{4}+o(1)} & \textit{if } d\ge 2. \end{cases} \tag{2.10}$$

Proof. We can let M_n denote the double sum in the lemma, and check directly that $M_n = \sum_{1\le i\le n-1}(L_n^{S_i} - L_i^{S_i})(q_i^2-1)$. Let $\mathcal{S}$ denote the σ-algebra generated by the entire process S. Then, conditionally on $\mathcal{S}$, each M_n is a sum of independent random variables. By Burkholder's inequality

[10, Theorem 2.10, p. 34], for all even integers $p \geq 2$,

$$\mathrm{E}\left(|M_n|^p\right) \leq \mathrm{const} \cdot \mathrm{E}\left(\left|\sum_{1\leq i\leq n-1}\left(L_n^{S_i} - L_i^{S_i}\right)^2 (q_i^2-1)^2\right|^{p/2}\right). \tag{2.11}$$

According to the generalized Hölder inequality,

$$\mathrm{E}\left(\prod_{k=1}^{p/2}(q_{i_k}^2-1)^2\right) \leq \prod_{k=1}^{p/2}\left\{\mathrm{E}\left(\left|q_{i_k}^2-1\right|^p\right)\right\}^{2/p} = \mathrm{E}\left(\left|q_1^2-1\right|^p\right). \tag{2.12}$$

Another application of the generalized Hölder inequality, together with an appeal to the Markov property, yields

$$\begin{aligned}\mathrm{E}\left(\prod_{k=1}^{p/2}\left(L_n^{S_{i_k}} - L_{i_k}^{S_{i_k}}\right)^2\right) &\leq \prod_{k=1}^{p/2}\left\{\mathrm{E}\left(\left|L_n^{S_{i_k}} - L_{i_k}^{S_{i_k}}\right|^p\right)\right\}^{2/p} \\ &= \prod_{k=1}^{p/2}\left\{\mathrm{E}\left(\left|L_{n-i_k}^0\right|^p\right)\right\}^{2/p}.\end{aligned} \tag{2.13}$$

Therefore, we can apply the local-limit theorem to find that

$$\mathrm{E}\left(|M_n|^p\right) \leq \mathrm{const} \cdot \begin{cases} n^p & \text{if } d=1, \\ n^{\frac{p}{2}+o(1)} & \text{if } d\geq 2. \end{cases} \tag{2.14}$$

The lemma follows from this and the Borel–Cantelli lemma. □

Lemma 2.5. *The following holds almost surely: As $n \to \infty$,*

$$\sum\sum_{1\leq i<k\leq n} \mathbf{1}_{\{S_i=S_k\}} q_i^2\left(T_k - T_{k-1} - 1\right) = \begin{cases} n^{1+o(1)} & \text{if } d=1, \\ n^{\frac{1}{2}+o(1)} & \text{if } d\geq 2. \end{cases} \tag{2.15}$$

Proof. Let N_n denote the double sum in the lemma, and note that

$$
\begin{aligned}
N_n &= \sum_{2\le k\le n} \beta_{k-1}\left(T_k - T_{k-1} - 1\right), \text{ where}\\
\beta_{k-1} &:= \sum_{1\le i\le k-1} q_i^2 \mathbf{1}_{\{S_i=S_k\}}.
\end{aligned}
\tag{2.16}
$$

Recall that $\mathcal{S}$ denotes the σ-algebra generated by the entire process S and observe that, conditionally on $\mathcal{S}$, $\{N_n\}_{n=1}^{\infty}$ is a mean-zero martingale with

$$
\begin{aligned}
\mathrm{E}(N_n^2 \,|\, \mathcal{S}) &= \mathrm{Var}(T_1)\cdot \sum_{2\le k\le n} \mathrm{E}\left(\beta_{k-1}^2 \,\middle|\, \mathcal{S}\right)\\
&= \mathrm{Var}(T_1)\cdot \sum_{2\le k\le n} \left(\mathrm{Var}(q_1^2) + \left|\mathrm{E}(q_1^2)\right|^2 \cdot L_{k-1}^{S_k}\right) L_{k-1}^{S_k}.
\end{aligned}
\tag{2.17}
$$

This and Doob's inequality together show that

$$
\mathrm{E}\left(\max_{1\le k\le n} N_k^2\right) \le \mathrm{const}\cdot n\left(\max_{a\in\mathbf{Z}} \mathrm{E}(L_n^a) + \max_{a\in\mathbf{Z}} \mathrm{E}(|L_n^a|^2)\right). \tag{2.18}
$$

By the local-limit theorem, the preceding is at most a constant multiple of $n(\sum_{1\le i\le n} i^{-d/2})^2$. The Borel–Cantelli lemma finishes the proof. □

Proof of Proposition 2.1. Recall the definition of each q_i. With that in mind, we can decompose Ξ_n as follows:

$$
\Xi_n = \sum_{1\le k\le n} \int_{T_{k-1}}^{T_k} \mathrm{d}s \left(\sum_{1\le i<k} \mathbf{1}_{\{S_i=S_k\}} q_i\right)^2 = \Xi_n^{(1)} + \Xi_n^{(2)}, \tag{2.19}
$$

where,

$$
\Xi_n^{(1)} := \sum\sum_{1\le i<k\le n} \mathbf{1}_{\{S_i=S_k\}} q_i^2 (T_k - T_{k-1}), \tag{2.20}
$$

and

$$
\Xi_n^{(2)} := 2\sum\sum\sum_{1\le i<j<k\le n} \mathbf{1}_{\{S_i=S_j=S_k\}} q_i q_j (T_k - T_{k-1}). \tag{2.21}
$$

Since

$$\Xi_n^{(1)} = \sum\sum_{1\le i<k\le n} \mathbf{1}_{\{S_i=S_k\}} + \sum\sum_{1\le i<k\le n} \mathbf{1}_{\{S_i=S_k\}} \left(q_i^2 - 1\right) \\ + \sum\sum_{1\le i<k\le n} \mathbf{1}_{\{S_i=S_k\}} q_i^2 \left(T_k - T_{k-1} - 1\right), \tag{2.22}$$

Lemmas 2.2, 2.3, 2.4, and 2.5 together imply that $\Xi_n^{(1)}$ has the large-n asymptotics that is claimed for Ξ_n. In light of Lemma 2.2, it suffices to show that almost surely the following holds as $n \to \infty$:

$$\Xi_n^{(2)} = \begin{cases} n^{\frac{17}{12}+o(1)} & \text{if } d = 1\ , \\ n^{\frac{3}{4}+o(1)} & \text{if } d \ge 2. \end{cases} \tag{2.23}$$

We can write $\Xi_n^{(2)} = 2\sum_{k=3}^n \tau_{k-1}(S_k)(T_k - T_{k-1})$, where

$$\tau_{k-1}(z) := \sum\sum_{1\le i<j\le k-1} \mathbf{1}_{\{S_i=S_j=z\}} q_i q_j \qquad \text{for } z \in \mathbf{Z} \text{ and } k > 1. \tag{2.24}$$

In particular, we can write

$$\Xi_n^{(2)} := 2\left(a_n + b_n\right), \qquad \text{where} \tag{2.25}$$

$$a_n := \sum_{k=3}^n \tau_{k-1}(S_k)\left(T_k - T_{k-1} - 1\right) \quad \text{and} \quad b_n := \sum_{k=3}^n \tau_{k-1}(S_k). \tag{2.26}$$

Recall that $\mathcal{S}$ denotes the σ-algebra generated by the process S. It follows that, conditional on $\mathcal{S}$, the process $\{a_n\}_{n=1}^\infty$ is a mean-zero martingale, and

$$\mathrm{E}\left(a_n^2 \,\middle|\, \mathcal{S}\right) = \mathrm{Var}(T_1) \cdot \sum_{k=3}^n \mathrm{E}\left(|\tau_{k-1}(S_k)|^2 \,\middle|\, \mathcal{S}\right). \tag{2.27}$$

The latter conditional expectation is also computed by a martingale computation. Namely, we write $\tau_{k-1}(z) = \sum_{j=2}^{k-1}(\sum_{i=1}^{j-1} \mathbf{1}_{\{S_i=S_j=z\}} q_i) q_j$ for all

$z \in \mathbf{Z}$ in order to deduce that

$$\mathrm{E}\left(|\tau_{k-1}(z)|^2 \,\middle|\, \mathcal{S}\right) = \sum_{j=2}^{k-1} \mathbf{1}_{\{S_j = z\}} L_{j-1}^z \leq \left(L_{k-1}^z\right)^2. \tag{2.28}$$

It follows from Doob's maximal inequality that

$$\mathrm{E}\left(\max_{1\leq k\leq n} a_k^2\right) \leq 4\mathrm{Var}(T_1)\cdot \mathrm{E}\left(\sum_{k=3}^{n} (L_{k-1}^{S_k})^2\right). \tag{2.29}$$

By time reversal, we can replace $L_{k-1}^{S_k}$ by L_{k-1}^0. Therefore, the local-limit theorem implies that $\mathrm{E}(\max_{1\leq k\leq n} a_k^2) \leq \mathrm{const}\cdot n(\sum_{i=1}^n i^{-d/2})^2$, and hence almost surely as $n\to\infty$, (2.23) is satisfied with $\Xi_n^{(2)}$ replaced by a_n [the Borel–Cantelli lemma]. It suffices to prove that (2.23) holds if Ξ_n is replaced by b_n.

We can write $b_n := b_{n,n}$, where

$$b_{n,k} = \sum_{j=2}^{n-1} \theta_{j-1,k}\, q_j \quad \text{for} \quad \theta_{j-1,k} := \sum_{i=1}^{j-1} \mathbf{1}_{\{S_i = S_j\}} q_i \left(L_k^{S_i} - L_j^{S_i}\right). \tag{2.30}$$

For each fixed integer $k \geq 1$, $\{b_{n,k}\}_{n\geq 3}$ is a mean-zero martingale, conditional on $\mathcal{S}$. Therefore, Burkholder's inequality yields

$$\mathrm{E}\left(|b_{n,k}|^p \,\middle|\, \mathcal{S}\right) \leq \mathrm{const}\cdot \mathrm{E}\left(\left[\sum_{j=2}^{n-1} \theta_{j-1,k}^2\, q_j^2\right]^{p/2} \,\middle|\, \mathcal{S}\right), \tag{2.31}$$

where the implied constant is nonrandom and depends only on p. Since $|\sum_{j=1}^{n-1} x_j|^{p/2} \leq n^{\frac{p}{2}-1}\sum_{j=1}^{n-1} |x_j|^{p/2}$ for all real $x_1,\ldots,x_{n-1}$, we can apply the preceding with $k := n$ to obtain

$$\mathrm{E}\left(|b_n|^p \,\middle|\, \mathcal{S}\right) \leq \mathrm{const}\cdot \mathrm{E}(|q_1|^p)\cdot n^{\frac{p}{2}-1} \sum_{j=2}^{n-1} \mathrm{E}\left(|\theta_{j-1,n}|^p \,\middle|\, \mathcal{S}\right). \tag{2.32}$$

Yet another application of Burkholder's inequality yields

$$\begin{aligned}\mathrm{E}\left(|\theta_{j-1,n}|^p \;\middle|\; \mathcal{S}\right) &\le \mathrm{const}\cdot\mathrm{E}\left(\left|\sum_{i=1}^{j-1}\mathbf{1}_{\{S_i=S_j\}}q_i^2\left(L_n^{S_i}-L_j^{S_i}\right)^2\right|^{p/2}\right)\\ &\le \mathrm{const}\cdot\mathrm{E}(|q_1|^p)\cdot\left(L_{j-1}^{S_j}\right)^{p/2}\left(L_n^{S_j}-L_j^{S_j}\right)^p,\end{aligned}\tag{2.33}$$

since $\mathrm{E}(q_{i_1}^2\cdots q_{i_{p/2}}^2)\le\mathrm{E}(|q_1|^p)$ for all $1\le i_1,\ldots,i_{p/2}<j$. We take expectations and apply the Markov property and time reversal to find that

$$\mathrm{E}\left(|\theta_{j-1,n}|^p\right)\le\mathrm{const}\cdot\mathrm{E}(|q_1|^p)\cdot\mathrm{E}\left[\left(L_{j-1}^0\right)^{p/2}\right]\mathrm{E}\left[\left(L_{n-j}^0\right)^p\right].\tag{2.34}$$

It follows readily that

$$\mathrm{E}\left(|b_n|^p\right)\le\mathrm{const}\cdot\mathrm{E}(|q_1|^p)\cdot n^{p/2}\mathrm{E}\left[\left(L_n^0\right)^{p/2}\right]\mathrm{E}\left[\left(L_n^0\right)^p\right]\tag{2.35}$$

This, the local-limit theorem, and the Borel–Cantelli lemma together imply that (2.23) holds with b_n in place of $\Xi_n^{(2)}$. The proposition follows. □

3 Proof of Theorem 1.2

In view of Theorem 1.1 and the LIL for the Brownian motion, it suffices to consider only the case $d=1$, and to establish the following:

$$\liminf_{n\to\infty}\left(\frac{\text{Ł}_2 n}{n}\right)^{3/4}\max_{0\le k\le n}|\gamma(\alpha(k))|=(a^*)^{3/4}\frac{\pi}{\sqrt{8}},\tag{3.1}$$

where

$$\text{Ł}_2 x:=\log\log(x\vee 1)\quad\text{and}\quad\alpha(t):=\int_{-\infty}^{\infty}(\ell_t^x)^2\,\mathrm{d}x.\tag{3.2}$$

It is known that [6, Theorem 3], $\sup_{x\in\mathbf{R}}\sup_{1\le k\le n}(\ell_k^x-\ell_{k-1}^x)=o((\log n)^{-1/2})$ almost surely [P]. This implies readily that $\max_{0\le k\le n}(\alpha(k+1)-\alpha(k-1))=O(n\sqrt{\log n})$ a.s. Therefore, it follows from [7, Theorem 1.2.1] that (3.1) is equivalent to the following:

$$\liminf_{t\to\infty}\left(\frac{Ł_2 t}{t}\right)^{3/4}\sup_{0\le s\le t}|\gamma(\alpha(s))| = (a^*)^{3/4}\frac{\pi}{\sqrt{8}}. \tag{3.3}$$

Brownian scaling implies that $\alpha(t)$ and $t^{3/2}\alpha(1)$ have the same distribution. On one hand, Proposition 1 of [11] tells us that the limit $C := \lim_{\lambda\to\infty}\exp\{a^*\,\lambda^{2/3}\}\,\mathrm{E}\exp(-\lambda\alpha(1))$ exists and is positive and finite. On the other hand, we can write $\gamma^*(t) := \sup_{0\le s\le t}|\gamma(s)|$ and appeal to Lemma 1.6.1 of [7] to find that for all $t, y > 0$,

$$\frac{2}{\pi}\exp\left(-\frac{\pi^2 t}{8y^2}\right) \le \mathrm{P}\left\{\gamma^*(t) < y\right\} \le \frac{4}{\pi}\exp\left(-\frac{\pi^2 t}{8y^2}\right). \tag{3.4}$$

Therefore uniformly for all $t > 0$ and $x \in (0\,,1]$,

$$\begin{aligned}\mathrm{P}\left\{\sup_{0\le s\le t}|\gamma(\alpha(s))| < xt^{3/4}\right\} &= \mathrm{P}\left\{\gamma^*(\alpha(1)) < x\right\} \le \frac{4}{\pi}\mathrm{E}\,\mathrm{e}^{-\pi^2\alpha(1)/(8x^2)}\\ &\le \mathrm{const}\cdot\exp\left(-\frac{a^*}{x^{4/3}}\left(\frac{\pi^2}{8}\right)^{2/3}\right).\end{aligned} \tag{3.5}$$

This and an application of the Borel–Cantelli lemma together yield one half of the (3.3); namely, (3.3) where "=" is replaced by "≥." In order to derive the other half we choose $t_n := n^n$ and $c > (a^*)^{3/4}\pi/\sqrt{8}$, and define

$$A_n := \left\{\omega : \sup_{0\le s\le t_n}|\gamma(\alpha(s))| < c\left(\frac{t_n}{Ł_2 t_n}\right)^{3/4}\right\}. \tag{3.6}$$

Every A_n is measurable with respect to $\mathcal{F}_{t_n} := \sigma\{\gamma(u) : u \le \alpha(t_n)\}\vee\sigma\{B_v : v \le t_n\}$. In light of the 0-1 law of Paul Lévy, and since $c > (a^*)^{3/4}\pi/\sqrt{8}$ is otherwise arbitrary, it suffices to prove that

$$\sum_{n=1}^{\infty}\mathrm{P}\left(A_n \mid \mathcal{F}_{t_{n-1}}\right) = \infty \qquad \text{a.s.} \tag{3.7}$$

The argument that led to (3.5) can be used to show that for all $v \geq 0$,

$$\mathrm{P}\left\{\gamma^*(v+\alpha(t)) < xt^{3/4}\right\} \geq \frac{2}{\pi}\exp\left(-\frac{\pi^2 v}{8x^2 t^{3/2}}\right)\mathrm{E}\,\mathrm{e}^{-\pi^2\alpha(1)/(8x^2)} \tag{3.8}$$
$$\geq \text{const}\cdot\exp\left(-\frac{\pi^2 v}{8x^2t^{3/2}} - \frac{a^*}{x^{4/3}}\left(\frac{\pi^2}{8}\right)^{2/3}\right).$$

In order to prove (3.7), let us choose and fix a large integer n temporarily. We might note that

$$\sup_{0\leq s\leq t_n}|\gamma(\alpha(s))| = \gamma^*(\alpha(t_n)) \leq \gamma^*(\alpha(t_{n-1})) + \widetilde{\gamma}^*(\alpha(t_n)-\alpha(t_{n-1})), \tag{3.9}$$

where $\widetilde{\gamma}(s) = \widetilde{\gamma}_n(s) := \gamma(s+\alpha(t_{n-1}))-\gamma(\alpha(t_{n-1}))$ and $\widetilde{\gamma}^*(s) := \sup_{0\leq v\leq s}|\widetilde{\gamma}(v)|$ for $s \geq 0$. Of course, $\widetilde{\gamma}$ is a Brownian motion independent of $\mathcal{F}_{t_{n-1}}$. Moreover, we can write $\ell_{t_n}^x = \ell_{t_{n-1}}^x + \widetilde{\ell}_{t_n-t_{n-1}}^{\,x-B_{t_{n-1}}}$, where $\widetilde{\ell}$ denotes the local time process of the Brownian motion $\widetilde{B}(s) := B(s+t_{n-1}) - B(t_{n-1}), s \geq 0$. Clearly, $(\widetilde{\gamma}\,,\widetilde{B})$ is a two-dimensional Brownian motion, independent of $\mathcal{F}_{t_{n-1}}$. Observe that

$$\alpha_{t_n} - \alpha_{t_{n-1}} = \int \mathrm{d}x\left[(\ell_{t_n}^x)^2 - (\ell_{t_{n-1}}^x)^2\right] = \int \mathrm{d}x\ \widetilde{\ell}_{t_n-t_{n-1}}^{\,x-B_{t_{n-1}}}\left(\ell_{t_{n-1}}^x + \widetilde{\ell}_{t_n-t_{n-1}}^{\,x-B_{t_{n-1}}}\right)$$
$$\leq (t_n - t_{n-1})\ell_{t_{n-1}}^* + \widetilde{\alpha}_{t_n-t_{n-1}}, \tag{3.10}$$

where $\ell_{t_{n-1}}^* := \sup_{x\in\mathbf{R}}\ell_{t_{n-1}}^x$. Therefore, we obtain

$$\gamma^*(\alpha_{t_n}) \leq \gamma^*(\alpha_{t_{n-1}}) + \widetilde{\gamma}^*\left((t_n-t_{n-1})\ell_{t_{n-1}}^* + \widetilde{\alpha}_{t_n-t_{n-1}}\right). \tag{3.11}$$

Let $\varepsilon > 0$ be such that $2\varepsilon < c - (a^*)^{3/4}\pi/\sqrt{8}$, and define

$$D_n := \left\{\gamma^*(\alpha_{t_{n-1}}) < \varepsilon\left(\frac{t_n}{\mathrm{L}_2 t_n}\right)^{3/4},\ \ell_{t_{n-1}}^* \leq \sqrt{3t_{n-1}\mathrm{L}_2 t_{n-1}}\right\}. \tag{3.12}$$

Clearly, D_n is $\mathcal{F}_{t_{n-1}}$-mesurable. Let $v_n := t_n\sqrt{3t_{n-1}\text{Ł}_2 t_{n-1}}$. Since

$$A_n \supset D_n \cap \left\{ \widetilde{\gamma}^*(v_n + \widetilde{\alpha}_{t_n}) \le (c-\varepsilon)\left(\frac{t_n}{\text{Ł}_2 t_n}\right)^{3/4} \right\}, \tag{3.13}$$

we can deduce from (3.8) that

$$\begin{aligned} &\mathrm{P}(A_n \,|\, \mathcal{F}_{t_{n-1}}) \\ &\ge \text{const} \cdot \mathbf{1}_{D_n} \exp\left(-\frac{\pi^2 v_n}{8}\left(\frac{\text{Ł}_2 t_n}{t_n}\right)^{3/2} - a^*\left(\frac{\pi^2}{8(c-\varepsilon)^2}\right)^{2/3} \text{Ł}_2 t_n \right) \\ &\ge \text{const} \cdot \mathbf{1}_{D_n} \, (n \ln n)^{-a^*(\pi^2/8)^{2/3}(c-\varepsilon)^{-4/3}}, \end{aligned} \tag{3.14}$$

where we have used the fact that $v_n/t_n^{3/2} \sim 1/n$. Because $a^*(\pi^2/8)^{2/3}(c-\varepsilon)^{-4/3} < 1$, (3.7) implies that almost surely, $\mathbf{1}_{D_n} = 1$ for all n large. Indeed, the LIL tells us that almost surely for all large n, $\ell^*_{t_{n-1}} \le \sqrt{3t_{n-1}\text{Ł}_2 t_{n-1}}$, and $\gamma^*(\alpha_{t_{n-1}}) \le \sqrt{3\alpha_{t_{n-1}}\text{Ł}_2\alpha_{t_{n-1}}}$. Since $\alpha(t) = \int(\ell_t^x)^2 dx \le t\,\ell_t^*$, we find that $\alpha_{t_{n-1}} \le t_{n-1}^{3/2}\sqrt{3\text{Ł}_2 t_{n-1}}$. Since $t_{n-1}/t_n \sim 1/n$, it follows that almost surely, $\ell^*_{t_{n-1}} \le \varepsilon(t_n/\text{Ł}_2 t_n)^{3/4}$ for all large n and prove that D_n realizes eventually for all large n. The proof of Theorem 1.2 is complete. □

Acknowledgements. We thank the University of Paris for generously hosting the second author (D.K.), during which time much of this research was carried out.

References

[1] Asselah, Amine, Annealed large deviation estimates for the energy of a polymer, *Preprint* (2009) *arXiv:0812.0443v2*.

[2] Bass, R.F., Chen, X. and Rosen, J., Moderate deviations and laws of the iterated logarithm for the renormalized self-intersection local times of planar random walks. *Electron. J. Probab.* **11** (2006), no. 37, 993–1030 (electronic).

[3] Bass, Richard F. and Griffin, Philip S., The most visited site of Brownian motion and simple random walk. *Z. Wahrsch. Verw. Gebiete* **70** (1985), no. 3, 417–436.

[4] Chen, X., Limit laws for the energy of a charged polymer. *Ann. Inst. Henri Poincaré: Probab. Stat.* **44** (2008), no. 4, 638–672.

[5] Chen, X. and Khoshnevisan, D., From charged polymers to random walk in random scenary. *IMS Lecture Notes. Optimality: The 3rd E.L. Lehmann Symposium* Vol. 57 (2009) 237–251.

[6] Csáki, E., Csörgő, M., Földes, A., and Révész, P., How big are the increments of the local time of a Wiener process? *Ann. Probab.* **11** (1983), no. 3, 593–608.

[7] Csörgő, M. and Révész, P., *Strong Approximations in Probability and Statistics,* Academic Press, Inc., New York-London, 1981.

[8] Derrida, B., Griffiths, B. and Higgs, P.G., A Model of Directed Walks with Random Self-Interactions. *Europhys. Lett.,* **18**, no. 4 (1992) pp. 361–366.

[9] Erdős, P.; and Taylor, S.J., Some problems concerning the structure of random walk paths. *Acta Math. Acad. Sci. Hungar.* **11** (1960) 137–162.

[10] Hall, P. and Heyde, C. C., *Martingale Limit Theory and Its Applications*, Academic Press, New York, 1980.

[11] van der Hofstad, R., den Hollander, F. and König, W., Central limit theorem for Edwards model. *Ann. Probab.* **25** (1997), no. 2 573–597.

[12] Kantor, Y. and Kardar, M., Polymers with Random Self-Interactions. *Europhys. Lett.* **14**, No. 5 (1991) 421–426.

[13] Kesten, Hary, An iterated logarithm law for local time. *Duke Math. J.* **32** (1965) 447–456.

[14] Révész, P., Local time and invariance. In: *Analytical Methods in Probability Theory* (Oberwolfach, 1980), pp. 128–145, Lecture Notes in Math., 861, Springer, Berlin-New York, 1981.

[15] Revuz, Daniel and Yor, Marc, *Continuous Martingales and Brownian Motion*, Springer-Verlag, Berlin, 1991.

Yueyun Hu. Département de Mathématiques, Université Paris XIII, 99 avenue J-B Clément, F-93430 Villetaneuse, France, *Email:* `yueyun@math.univ-paris13.fr`

Davar Khoshnevisan. Department of Mathematics, University of Utah, 155 South 1440 East, JWB 233, Salt Lake City, Utah 84112-0090, USA, *Email:* `davar@math.utah.edu`