From charged polymers
to random walk in random scenery

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Abstract

We prove that two seemingly-different models of random walk in random environment are generically quite close to one another. One model comes from statistical physics, and describes the behavior of a randomly-charged random polymer. The other model comes from probability theory, and was originally designed to describe a large family of asymptotically self-similar processes that have stationary increments.

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1 Introduction and the main results

The principal goal of this article is to show that two apparently-disparate models—one from statistical physics of disorder media [KK91, DGH92, DH94] and one from probability theory [KS79, Bol89]—are very close to one another.

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In order to describe the model from statistical physics, let us suppose that 
\( q := \{q_i\}_{i=1}^{\infty} \) is a collection of i.i.d. mean-zero random variables with finite 
variance \( \sigma^2 > 0 \). For technical reasons, we assume here and throughout that 
\[
\mu_6 := E(q_1^6) < \infty.
\] (1.1)

In addition, we let \( S := \{S_i\}_{i=0}^{\infty} \) denote a random walk on \( \mathbb{Z}^d \) with \( S_0 = 0 \) that 
is independent from the collection \( q \). We also rule out the trivial case that \( S_1 \) 
has only one possible value.

The object of interest to us is the random quantity
\[
H_n := \sum_{1 \leq i < j \leq n} q_i q_j \mathbf{1}_{\{S_i = S_j\}},
\] (1.2)

In statistical physics, \( H_n \) denotes a random Hamiltonian of spin-glass type that 
is used to build Gibbsian polymer measures. The \( q_i \)'s are random charges, and 
each realization of \( S \) corresponds to a possible polymer path; see the paper by 
Kantor and Kardar [KK91], its subsequent variations by Derrida et al [DGH92, 
DH94] and Wittmer et al [WJJ93], and its predecessors by Garel and Orland [GO88] 
and Obukhov [O86]. The resulting Gibbs measure then corresponds to a model for “random walk in random environment.” Although we do not 
consider continuous processes here, the continuum-limit analogue of \( H_n \) has 
also been studied in the literature [BP97, MP96].

Kesten and Spitzer [KS79] introduced a different model for “random walk in random environment,” which they call random walk in random scenery.\(^1\) We can describe that model as follows: Let \( Z := \{Z(x)\}_{x \in \mathbb{Z}^d} \) is a collection of i.i.d. 
random variables, with the same common distribution as \( q_1 \), and independent 
of \( S \). Define
\[
W_n := \sum_{i=1}^{n} Z(S_i).
\] (1.3)

The process \( W := \{W_n\}_{n=0}^{\infty} \) is called random walk in random scenery, and can 
be thought of as follows: We fix a realization of the \( d \)-dimensional random field 
\( Z \)—the “scenery”—and then run an independent walk \( S \) on \( \mathbb{Z}^d \). At time \( j \), 
the walk is at \( S_j \); we sample the scenery at that point. This yields \( Z(S_j) \), which is 
then used as the increment of the process \( W \) at time \( j \).

Our goal is to make precise the assertion that if \( n \) is large, then
\[
H_n \approx \gamma^{1/2} \cdot W_n \quad \text{in distribution},
\] (1.4)

where
\[
\gamma := \begin{cases} 
1 & \text{if } S \text{ is recurrent}, \\
\sum_{k=1}^{\infty} P\{S_k = 0\} & \text{if } S \text{ is transient}.
\end{cases}
\] (1.5)

Our derivation is based on a classification of recurrence vs. transience for random 
walks that appears to be new. This classification [Theorem 2.4] might be of 
independent interest.

\(^1\)Kesten and Spitzer ascribe the terminology to Paul Shields.
We can better understand (1.4) by considering separately the cases that $S$ is transient versus recurrent. The former case is simpler to describe, and appears next.

**Theorem 1.1.** If $S$ is transient, then

$$
\frac{W_n}{n^{1/2}} \xrightarrow{D} N(0, \sigma^2) \quad \text{and} \quad \frac{H_n}{n^{1/2}} \xrightarrow{D} N(0, \gamma \sigma^2).
$$

(Kesten and Spitzer [KS79] proved the assertion about $W_n$ under more restrictive conditions on $S$. Similarly, Chen [C07] proved the statement about $H_n$ under more hypotheses.

Before we can describe the remaining [and more interesting] recurrent case, we define

$$a_n := \left( n \sum_{k=0}^{n} P\{S_k = 0\} \right)^{1/2}.
$$

It is well known [P21, CF51] that $S$ is recurrent if and only if $a_n/n^{1/2} \to \infty$ as $n \to \infty$.

**Theorem 1.2.** If $S$ is recurrent, then for all bounded continuous functions $f : \mathbb{R}^d \to \mathbb{R}$,

$$
E \left[ f \left( \frac{W_n}{a_n} \right) \right] = E \left[ f \left( \frac{H_n}{a_n} \right) \right] + o(1),
$$

where $o(1)$ converges to zero as $n \to \infty$. Moreover, both $\{W_n/a_n\}_{n \geq 1}$ and $\{H_n/a_n\}_{n \geq 1}$ are tight.

We demonstrate Theorems 1.1 and 1.2 by using a variant of the replacement method of Liapounov [Lia00, pp. 362–364]; this method was rediscovered later by Lindeberg [Lin22], who used it to prove his famous central limit theorem for triangular arrays of random variables.

It can be proved that when $S$ is in the domain of attraction of a stable law, $W_n/a_n$ converges in distribution to an explicit law [KS79, Bol89]. Consequently, $H_n/a_n$ converges in distribution to the same law in that case. This fact was proved earlier by Chen [C07] under further [mild] conditions on $S$ and $q_1$.

We conclude the introduction by describing the growth of $a_n$ under natural conditions on $S$.

**Remark 1.3.** Suppose $S$ is strongly aperiodic, mean zero, and finite second moments, with a nonsingular covariance matrix. Then, $S$ is transient iff $d \geq 3$, and by the local central limit theorem, as $n \to \infty$,

$$
\sum_{k=1}^{n} P\{S_k = 0\} \sim \text{const} \times \begin{cases} n^{1/2} & \text{if } d = 1, \\ \log n & \text{if } d = 2. \end{cases}
$$

See, for example Spitzer [S76, P9 on p. 75]. Consequently,

$$a_n \sim \text{const} \times \begin{cases} n^{3/4} & \text{if } d = 1, \\ (n \log n)^{1/2} & \text{if } d = 2. \end{cases}
$$
This agrees with the normalization of Kesten and Spitzer [KS79] when \( d = 1 \), and Bolthausen [Bol89] when \( d = 2 \).

2 Preliminary estimates

Consider the local times of \( S \) defined by

\[
L_n^x := \sum_{i=1}^{n} 1_{\{S_i = x\}}.
\]  

(2.1)

A little thought shows that the random walk in random scenery can be represented compactly as

\[
W_n = \sum_{x \in \mathbb{Z}^d} Z(x) L_n^x.
\]  

(2.2)

There is also a nice way to write the random Hamiltonian \( H_n \) in local-time terms. Consider the “level sets,”

\[
\mathcal{L}_n^x := \{ i \in \{1, \ldots, n\} : S_i = x \}.
\]  

(2.3)

It is manifest that if \( j \in \{2, \ldots, n\} \), then \( L_j^x > L_{j-1}^x \) if and only if \( j \in \mathcal{L}_n^x \). Thus, we can write

\[
H_n = \frac{1}{2} \left( \sum_{x \in \mathbb{Z}^d} \left| \sum_{i=1}^{n} q_i 1_{\{S_i = x\}} \right|^2 - \sum_{i=1}^{n} q_i^2 \right)
\]  

\[
= \sum_{x \in \mathbb{Z}^d} h_n^x,
\]  

(2.4)

where

\[
h_n^x := \frac{1}{2} \left( \sum_{i \in \mathcal{L}_n^x} q_i^2 - \sum_{i \in \mathcal{L}_n^x} q_i^2 \right).
\]  

(2.5)

We denote by \( \hat{P} \) the conditional measure, given the entire process \( S \); \( \hat{E} \) denotes the corresponding expectation operator. The following is borrowed from Chen [C07, Lemma 2.1].

**Lemma 2.1.** Choose and fix some integer \( n \geq 1 \). Then, \( \{h_n^x\}_{x \in \mathbb{Z}^d} \) is a collection of i.i.d. random variables under \( \hat{P} \), and

\[
\hat{E} h_n^x = 0 \quad \text{and} \quad \hat{E} \left( |h_n^x|^2 \right) = \frac{\sigma^2}{2} L_n^x (L_n^x - 1) \quad \text{P.-a.s.}
\]  

(2.6)

Moreover, there exists a nonrandom positive and finite constant \( C = C(\sigma) \) such that for all \( n \geq 1 \) and \( x \in \mathbb{Z}^d \),

\[
\hat{E} \left( |h_n^x|^3 \right) \leq C \mu_6 |L_n^x (L_n^x - 1)|^{3/2} \quad \text{P.-a.s.}
\]  

(2.7)
Next we develop some local-time computations.

**Lemma 2.2.** For all \( n \geq 1 \),

\[
\sum_{x \in \mathbb{Z}^d} EL^x = n \quad \text{and} \quad \sum_{x \in \mathbb{Z}^d} E \left( \left| L^x \right|^2 \right) = n + 2 \sum_{k=1}^{n-1} \langle n - k \rangle P\{S_k = 0\}. \tag{2.8}
\]

Moreover, for all integers \( k \geq 1 \),

\[
\sum_{x \in \mathbb{Z}^d} E \left( \left| L^x \right|^k \right) \leq k! \sum_{j=0}^{n} P\{S_j = 0\} \right|_{k-1}. \tag{2.9}
\]

**Proof.** Since \( EL^x = \sum_{i=1}^{n} P\{S_i = x\} \) and \( \sum_{x \in \mathbb{Z}^d} P\{S_j = x\} = 1 \), we have \( \sum_{x} EL^x = n \). For the second-moment formula we write

\[
E \left( \left| L^x \right|^2 \right) = \sum_{1 \leq i \leq n} P\{S_i = x\} + 2 \sum_{1 \leq i < j \leq n} P\{S_i = x\} P\{S_j - S_i = 0\}. \tag{2.10}
\]

We can sum this expression over all \( x \in \mathbb{Z}^d \) to find that

\[
\sum_{x \in \mathbb{Z}^d} E \left( \left| L^x \right|^2 \right) = n + 2 \sum_{1 \leq i < j \leq n} P\{S_{j-i} = x\}. \tag{2.11}
\]

This readily implies the second-moment formula. Similarly, we write

\[
E \left( \left| L^x \right|^k \right) \leq k! \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} P\{S_{i_1} = \cdots = S_{i_k} = x\}
\]

\[
= k! \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} P\{S_{i_1} = x\} P\{S_{i_2} = 0\} \cdots P\{S_{i_k} = 0\} \tag{2.12}
\]

\[
\leq k! \sum_{i=1}^{n} P\{S_i = x\} \left| \sum_{j=1}^{n} P\{S_j = 0\} \right|^{k-1}
\]

Add over all \( x \in \mathbb{Z}^d \) to finish. \( \square \)

Our next lemma provides the first step in a classification of recurrence [versus transience] for random walks.

**Lemma 2.3.** It is always the case that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{x \in \mathbb{Z}^d} E \left( \left| L^x \right|^2 \right) = 1 + 2 \sum_{k=1}^{\infty} P\{S_k = 0\}. \tag{2.13}
\]

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Proof. Thanks to Lemma 2.2, for all \( n \geq 1 \),

\[
\frac{1}{n} \sum_{x \in \mathbb{Z}^d} \mathbb{E} \left( |L_n^x|^2 \right) = 1 + 2 \sum_{k=1}^{n-1} \left( 1 - \frac{k}{n} \right) \mathbb{P}\{S_k = 0\}. \quad (2.14)
\]

If \( S \) is transient, then the monotone convergence theorem ensures that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{x \in \mathbb{Z}^d} \mathbb{E} \left( |L_n^x|^2 \right) = 1 + 2 \sum_{k=1}^{\infty} \mathbb{P}\{S_k = 0\}. \quad (2.15)
\]

This proves the lemma in the transient case.

When \( S \) is recurrent, we note that (2.14) readily implies that for all integers \( m \geq 2 \),

\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{x \in \mathbb{Z}^d} \mathbb{E} \left( |L_n^x|^2 \right) \geq 1 + \sum_{1 \leq k \leq m/2} \mathbb{P}\{S_k = 0\}. \quad (2.16)
\]

Let \( m \uparrow \infty \) to deduce the lemma. \( \square \)

Next we “remove the expectation” from the statement of Lemma 2.3.

**Theorem 2.4.** As \( n \to \infty \),

\[
\frac{1}{n} \sum_{x \in \mathbb{Z}^d} (L_n^x)^2 \to 1 + 2 \sum_{k=1}^{\infty} \mathbb{P}\{S_k = 0\} \quad \text{in probability.} \quad (2.17)
\]

**Remark 2.5.** The quantity \( I_n := \sum_{x \in \mathbb{Z}^d} (L_n^x)^2 \) is the so-called self-intersection local time of the walk \( S \). This terminology stems from the following elementary calculation: For all integers \( n \geq 1 \),

\[
I_n = \sum_{1 \leq i < j \leq n} \mathbb{1}_{\{S_i = S_j\}}. \quad (2.18)
\]

Consequently, Theorem 2.4 implies that a random walk \( S \) on \( \mathbb{Z}^d \) is recurrent if and only if its self-intersection local time satisfies \( I_n/n \to \infty \) in probability. \( \square \)

**Proof.** First we study the case that \( \{S_i\}_{i=0}^{\infty} \) is transient.

Define

\[
Q_n := \sum_{1 \leq i < j \leq n} \mathbb{1}_{\{S_i = S_j\}}. \quad (2.19)
\]

Then it is not too difficult to see that

\[
\sum_{x \in \mathbb{Z}^d} (L_n^x)^2 = 2Q_n + n \quad \text{for all } n \geq 1. \quad (2.20)
\]
This follows immediately from (2.18), for example. Therefore, it suffices to prove that, under the assumption of transience,

\[ \frac{Q_k}{k} \to \sum_{j=1}^{\infty} P\{S_j = 0\} \quad \text{in probability as } k \to \infty. \]  

(2.21)

Lemma 2.3 and (2.20) together imply that

\[ \lim_{k \to \infty} \frac{E Q_k}{k} = \sum_{j=1}^{\infty} P\{S_j = 0\}. \]  

(2.22)

Hence, it suffices to prove that \( \text{Var} Q_n = o(n^2) \) as \( n \to \infty \). In some cases, this can be done by making an explicit [though hard] estimate for \( \text{Var} Q_n \); see, for instance, Chen [C07, Lemma 5.1], and also the technique employed in the proof of Lemma 2.4 of Bolthausen [Bol89]. Here, we opt for a more general approach that is simpler, though it is a little more circuitous. Namely, in rough terms, we write \( Q_n \) as \( Q_n^{(1)} + Q_n^{(2)} \), where \( E Q_n^{(1)} = o(n) \), and \( \text{Var} Q_n^{(2)} = o(n^2) \). Moreover, we will soon see that \( Q_n^{(1)}, Q_n^{(2)} \geq 0 \), and this suffices to complete the proof.

For all \( m := m_n \in \{1, \ldots, n - 1\} \) we write

\[ Q_n = Q_n^{1,m} + Q_n^{2,m}, \]  

(2.23)

where

\[ Q_n^{1,m} := \sum_{\substack{1 \leq i < j \leq n: \ \ j \geq i + m}} 1_{\{S_i = S_j\}} \quad \text{and} \quad Q_n^{2,m} := \sum_{\substack{1 \leq i < j \leq n: \ \ j < i + m}} 1_{\{S_i = S_j\}}. \]  

(2.24)

Because \( n > m \), we have

\[ \text{E} Q_n^{1,m} \leq n \sum_{k=m}^{\infty} P\{S_k = 0\}. \]  

(2.25)

We estimate the variance of \( Q_n^{2,m} \) next. We do this by first making an observation.

Throughout the remainder of this proof, define for all subsets \( \Gamma \) of \( \mathbb{N}^2 \),

\[ \Upsilon(\Gamma) := \sum_{(i,j) \in \Gamma} 1_{\{S_i = S_j\}}. \]  

(2.26)

Suppose \( \Gamma_1, \Gamma_2, \ldots, \Gamma_\nu \) are finite disjoint sets in \( \mathbb{N}^2 \), with common cardinality, and the added property that whenever \( 1 \leq a < b \leq \nu \), we have \( \Gamma_a < \Gamma_b \) in the sense that \( i < k \) and \( j < l \) for all \( (i,j) \in \Gamma_a \) and \( (k,l) \in \Gamma_b \). Then, it follows that

\( \{\Upsilon(\Gamma_\nu)\}_{\mu=1}^\nu \) is an i.i.d. sequence.  

(2.27)

For all integers \( p \geq 0 \) define

\[ B_p^m := \{(i,j) \in \mathbb{N}^2 : (p-1)m < i < j \leq pm\}, \]

\[ W_p^m := \{(i,j) \in \mathbb{N}^2 : (p-1)m < i \leq pm < j \leq (p+1)m\}. \]  

(2.28)
In Figure 1, \( \{B^m_p\}_{p=1}^\infty \) denotes the collection black and \( \{W^m_p\}_{p=1}^\infty \) the white triangles that are inside the slanted strip.

We may write

\[
Q_{2m}^{(n-1)m} = \sum_{p=1}^{n-1} \Upsilon(B^m_p) + \sum_{p=1}^{n-1} \Upsilon(W^m_p). \tag{2.29}
\]

Consequently,

\[
\text{Var} Q_{(n-1)m}^{2m} \leq 2 \text{Var} \sum_{p=1}^{n-1} \Upsilon(B^m_p) + 2 \text{Var} \sum_{p=1}^{n-1} \Upsilon(W^m_p). \tag{2.30}
\]

If \( 1 \leq a < b \leq m - 1 \), then \( B^m_a < B^m_b \) and \( W^m_a < W^m_b \). Consequently, (2.27) implies that

\[
\text{Var} Q_{(n-1)m}^{2m} \leq 2(n - 1) [\text{Var} \Upsilon(B^m_1) + \text{Var} \Upsilon(W^m_1)]. \tag{2.31}
\]

Because \( \Upsilon(B^m_1) \) and \( \Upsilon(W^m_1) \) are individually sums of not more than \( \binom{m}{2} \)-many ones,

\[
\text{Var} Q_{(n-1)m}^{2m} \leq 2(n - 1)m^2. \tag{2.32}
\]

Let \( Q_n^{(1)} := Q_n^{1m} \) and \( Q_n^{(2)} := Q_n^{2m} \), where \( m = m_n := n^{1/4} \) [say]. Then, \( Q_n = Q_n^{(1)} + Q_n^{(2)} \), and (2.25) and (2.32) together imply that \( \text{E} Q_{(n-1)m}^{(1)} = o((n - 1)m) \). Moreover, \( \text{Var} Q_{(n-1)m}^{(2)} = o((nm)^2) \). This gives us the desired
decomposition of $Q_{(n-1)m}$. Now we complete the proof: Thanks to (2.22),

$$EQ^{(2)}_{(n-1)m} \sim nm \cdot \sum_{j=1}^{\infty} P\{S_j = 0\} \quad \text{as } n \to \infty. \quad (2.33)$$

Therefore, the variance of $Q^{(2)}_{(n-1)m}$ is little-o of the square of its mean. This and the Chebyshev inequality together imply that $Q^{(2)}_{(n-1)m}/(nm)$ converges in probability to $\sum_{j=1}^{\infty} P\{S_j = 0\}$. On the other hand, we know also that $Q^{(1)}_{(n-1)m}/(nm)$ converges to zero in $L^1(P)$ and hence in probability. Consequently, we can change variables and note that as $n \to \infty$,

$$Q_{nm}/nm \to \sum_{j=1}^{\infty} P\{S_j = 0\} \quad \text{in probability.} \quad (2.34)$$

If $k$ is between $(n-1)m$ and $nm$, then

$$\frac{Q_{(n-1)m}}{nm} \leq \frac{Q_k}{k} \leq \frac{Q_{nm}}{(n-1)m}. \quad (2.35)$$

This proves (2.21), and hence the theorem, in the transient case.

In order to derive the recurrent case, it suffices to prove that $Q_n/n \to \infty$ in probability as $n \to \infty$.

Let us choose and hold an integer $m \geq 1$—so that it does not grow with $n$—and observe that $Q_n \geq Q^{2,m}_n$ as long as $n$ is sufficiently large. Evidently,

$$EQ^{2,m}_n = \sum_{1 \leq i < j \leq n} \sum_{j \leq i+m} P\{S_j = S_i\}$$

$$= (n-1) \sum_{k=1}^{m-1} P\{S_k = 0\}. \quad (2.36)$$

We may also observe that (2.32) continues to hold in the present recurrent setting. Together with the Chebyshev inequality, these computations imply that as $n \to \infty$,

$$\frac{Q^{2,m}_{n(m-1)}}{n} \to \sum_{k=1}^{m-1} P\{S_k = 0\} \quad \text{in probability.} \quad (2.37)$$

Because $Q_{n(m-1)} \geq Q^{2,m}_{n(m-1)}$, the preceding implies that

$$\lim_{n \to \infty} P\left\{\frac{Q_{n(m-1)}}{n} \geq \frac{1}{2} \sum_{k=1}^{m} P\{S_k = 0\}\right\} = 1. \quad (2.38)$$

A monotonicity argument shows that $Q_{n(m-1)}$ can be replaced by $Q_n$ without altering the end-result; see (2.35). By recurrence, if $\lambda > 0$ is any pre-described positive number, then we can choose [and fix] our integer $m$ such that $\sum_{k=1}^{m} P\{S_k = 0\} \geq 2\lambda$. This proves that $\lim_{n \to \infty} P\{Q_n/n \geq \lambda\} = 1$ for all $\lambda > 0$, and hence follows the theorem in the recurrent case.
3 Proofs of the main results

Now we introduce a sequence \( \{\xi_x\} \) of random variables, independent \([\text{under } P]\) of \( \{q_i\}_{i=1}^\infty \) and the random walk \( \{S_i\}_{i=0}^\infty \), such that

\[
E\xi_0 = 0, \quad E(\xi_0^2) = \sigma^2, \quad \text{and} \quad \hat{\mu}_3 := E(|\xi_0|^3) < \infty. \tag{3.1}
\]

Define

\[
\hat{h}_n^x := \left| \frac{L_n^x(L_n^x - 1)}{2} \right|^{1/2} \xi_x \quad \text{for all } n \geq 1 \text{ and } x \in \mathbb{Z}^d. \tag{3.2}
\]

Evidently, \( \{\hat{h}_n^x\} \) is a sequence of \([\text{conditionally } \text{i.i.d.}]\) random variables, under \( \hat{P} \), and has the same \([\text{conditional}]\) mean and variance as \( \{h_n^x\} \).

**Lemma 3.1.** There exists a positive and finite constant \( C_* = C_*(\sigma) \) such that if \( f : \mathbb{R}^d \to \mathbb{R} \) is three time continuously differentiable, then for all \( n \geq 1 \),

\[
\left| \mathbb{E}f \left( \sum_{x \in \mathbb{Z}^d} \hat{h}_n^x \right) - \mathbb{E}f(H_n) \right| \leq C_* M_f(\hat{\mu}_3 + \mu_6) n \sum_{j=0}^n P\{S_j = 0\} \left| h_n^y \right|^2 \tag{3.3}
\]

with \( M_f := \sup_{x \in \mathbb{R}^d} |f'''(x)| \).

**Proof.** Temporarily choose and fix some \( y \in \mathbb{Z}^d \), and notice that

\[
f(H_n) = f \left( \sum_{x \in \mathbb{Z}^d \setminus \{y\}} h_n^x \right) - f' \left( \sum_{x \in \mathbb{Z}^d \setminus \{y\}} h_n^x \right) h_n^y - \frac{1}{2} f'' \left( \sum_{x \in \mathbb{Z}^d \setminus \{y\}} h_n^x \right) |h_n^y|^2 + R_n, \tag{3.4}
\]

where \( |R_n| \leq \frac{1}{8} \|f'''\|_\infty |h_n^y|^3 \). It follows from this and Lemma 2.1 that

\[
\hat{E}f(H_n) = \hat{E}f \left( \sum_{x \in \mathbb{Z}^d \setminus \{y\}} h_n^x \right) - \frac{\sigma^2}{2} L_n^y(L_n^y - 1) \hat{E}f'' \left( \sum_{x \in \mathbb{Z}^d \setminus \{y\}} h_n^x \right) + R_n^{(1)}, \tag{3.5}
\]

where

\[
\left| R_n^{(1)} \right| \leq \frac{CM_f \mu_6}{12} |L_n^y(L_n^y - 1)|^{3/2} \quad \text{P-a.s.} \leq \frac{CM_f \mu_6}{12} |L_n^y|^3. \tag{3.6}
\]
We proceed as in (3.4) and write
\[
\begin{align*}
& f \left( \hat{h}_n^y + \sum_{x \in \mathbb{Z}^d \setminus \{y\}} h_n^x \right) \\
= & f \left( \sum_{x \in \mathbb{Z}^d \setminus \{y\}} h_n^x \right) - f' \left( \sum_{x \in \mathbb{Z}^d \setminus \{y\}} h_n^x \right) \hat{y}_n^y - \frac{1}{2} f'' \left( \sum_{x \in \mathbb{Z}^d \setminus \{y\}} h_n^x \right) \left| \hat{y}_n^y \right|^2 \tag{3.7}
\end{align*}
\]
\[+ \hat{R}_n,\]

where \(|\hat{R}_n| \leq \frac{1}{6} M_f |\hat{y}_n^y|^3 \leq \frac{1}{12 \sqrt{2}} M_f |L_n^y|^3 |\xi|^3\). It follows from this and Lemma 2.1 that
\[
\hat{E} f \left( \hat{h}_n^y + \sum_{x \in \mathbb{Z}^d \setminus \{y\}} h_n^x \right) - \hat{E} f \left( \sum_{x \in \mathbb{Z}^d \setminus \{y\}} h_n^x \right) - \frac{\sigma^2}{2} L_n^y (L_n^y - 1) \hat{E} f'' \left( \sum_{x \in \mathbb{Z}^d \setminus \{y\}} h_n^x \right) + R_n^{(2)}, \tag{3.8}
\]

where \(|R_n^{(2)}| \leq \frac{1}{12 \sqrt{2}} \hat{\mu}_3 M_f |L_n^y|^3\). Define \(C_* := (C + 1)/2\) to deduce from the preceding and (3.5) that P-a.s.,
\[
\left| \hat{E} f \left( \hat{h}_n^y + \sum_{x \in \mathbb{Z}^d \setminus \{y\}} h_n^x \right) - \hat{E} f \left( \sum_{x \in \mathbb{Z}^d} h_n^x \right) \right| \leq \frac{A}{6} |L_n^y|^3, \tag{3.9}
\]

where \(A := C_* M_f (\hat{\mu}_3 + \mu_0)\). Now we can readily iterate this inequality to find that P-a.s.,
\[
\left| \hat{E} f \left( \sum_{x \in \mathbb{Z}^d} \hat{h}_n^x \right) - \hat{E} f (H_n) \right| \leq \frac{A}{6} \sum_{y \in \mathbb{Z}^d} |L_n^y|^3. \tag{3.10}
\]

We take expectations and appeal to Lemma 2.2 to finish. \(\square\)

Next, we prove Theorem 1.1.

**Proof of Theorem 1.1.** We choose, in Lemma 3.1, the collection \(\{\xi_x\}_{x \in \mathbb{Z}^d}\) to be i.i.d. mean-zero normals with variance \(\sigma^2\). Then, we apply Lemma 3.1 with \(f(x) := g(x/n^{1/2})\) for a smooth bounded function \(g\) with bounded derivatives. This yields,
\[
\left| \hat{E} g(H_n/n^{1/2}) - \hat{E} g \left( \frac{1}{n^{1/2}} \sum_{x \in \mathbb{Z}^d} \hat{h}_n^x \right) \right| \leq \frac{\text{const.}}{n^{1/2}}. \tag{3.11}
\]
In this way,
\[
\sum_{x \in \mathbb{Z}^d} \hat{h}_n^x \overset{D}{=} \frac{\sigma}{\sqrt{2}} \left| \sum_{x \in \mathbb{Z}^d} L_n^x (L_n^x - 1) \right|^{1/2} N(0, 1) \quad \text{under } \hat{P}
\]
\[
= \frac{\sigma}{\sqrt{2}} \left| -n + \sum_{x \in \mathbb{Z}^d} (L_n^x)^2 \right|^{1/2} N(0, 1),
\]
where \(D\) denotes equality in distribution, and \(N(0, 1)\) is a standard normal random variable under \(\hat{P}\) as well as \(P\). Therefore, in accord with Theorem 2.4,
\[
\frac{1}{n^{1/2}} \sum_{x \in \mathbb{Z}^d} \hat{h}_n^x \overset{D}{=} \frac{\sigma}{\sqrt{2}} \left| -1 + \frac{1}{n} \sum_{x \in \mathbb{Z}^d} (L_n^x)^2 \right|^{1/2} N(0, 1)
\]
\[
= o_{\hat{P}}(1) + \gamma^{1/2} \cdot N(0, \sigma^2),
\]
where \(o_{\hat{P}}(1)\) is a term that converges to zero as \(n \to \infty\) in \(\hat{P}\)-probability a.s. \([P]\). Equation (3.11) then completes the proof in the transient case.

Theorem 1.2 relies on the following “coupled moderate deviation” result.

**Proposition 3.2.** Suppose that \(S\) is recurrent. Consider a sequence \(\{\epsilon_j\}_{j=1}^\infty\) of nonnegative numbers that satisfy the following:
\[
\lim_{n \to \infty} \epsilon_n^3 \left| \sum_{k=1}^n P\{S_k = 0\} \right|^2 = 0.
\]
Then for all compactly supported functions \(f : \mathbb{R}^d \to \mathbb{R}\) that are infinitely differentiable,
\[
\lim_{n \to \infty} \left| E[f(\epsilon_n W_n)] - E[f(\epsilon_n H_n)] \right| = 0.
\]

**Proof.** We apply Lemma 3.1 with the \(\xi_x\)'s having the same common distribution as \(q_1\), and with \(f(x) := g(\epsilon_n x)\) for a smooth and bounded function \(g\) with bounded derivatives. This yields,
\[
\left| E \left[ g \left( \epsilon_n \sum_{x \in \mathbb{Z}^d} \left| L_n^x (L_n^x - 1) \right|^{1/2} Z(x) \right) \right] - E [g(\epsilon_n H_n)] \right| \leq 2C_s M g \mu_6 \mu_n \delta \epsilon_n \left| \sum_{k=0}^n P\{S_k = 0\} \right|^2 (3.16)
\]
\[
= o(1),
\]
owing to Lemma (3.4).
According to Taylor's formula,

\[
g \left( \epsilon_n \sum_{x \in \mathbb{Z}^d} |L_n^x (L_n^x - 1)|^{1/2} Z(x) \right) = g \left( \epsilon_n \sum_{x \in \mathbb{Z}^d} Z(x)L_n^x \right) + \epsilon_n \sum_{x \in \mathbb{Z}^d} \left( |L_n^x (L_n^x - 1)|^{1/2} - L_n^x \right) Z(x) \cdot R,
\]

(3.17)

where \(|R| \leq \sup_{x \in \mathbb{R}^d} |g'(x)|\). Thanks to (2.2), we can write the preceding as follows:

\[
g \left( \epsilon_n \sum_{x \in \mathbb{Z}^d} |L_n^x (L_n^x - 1)|^{1/2} Z(x) \right) - g(\epsilon_n W_n)
= \epsilon_n \sum_{x \in \mathbb{Z}^d} \left( |L_n^x (L_n^x - 1)|^{1/2} - L_n^x \right) Z(x) \cdot R.
\]

(3.18)

Consequently, P-almost surely,

\[
\left| \widehat{E} \left[ g \left( \epsilon_n \sum_{x \in \mathbb{Z}^d} |L_n^x (L_n^x - 1)|^{1/2} Z(x) \right) \right] - \widehat{E} [g(\epsilon_n W_n)] \right| \\
\leq \sup_{x \in \mathbb{R}^d} |g'(x)| \sigma \epsilon_n \left\{ \widehat{E} \left( \sum_{x \in \mathbb{Z}^d} \left( |L_n^x (L_n^x - 1)|^{1/2} - L_n^x \right)^2 \right) \right\}^{1/2}.
\]

(3.19)

We apply the elementary inequality \((a^{1/2} - b^{1/2})^2 \leq |a - b|\)—valid for all \(a, b \geq 0\)—to deduce that P-almost surely,

\[
\left| \widehat{E} \left[ g \left( \epsilon_n \sum_{x \in \mathbb{Z}^d} |L_n^x (L_n^x - 1)|^{1/2} Z(x) \right) \right] - \widehat{E} [g(\epsilon_n W_n)] \right| \leq \sup_{x \in \mathbb{R}^d} |g'(x)| \sigma \epsilon_n \left\{ \widehat{E} \left( \sum_{x \in \mathbb{Z}^d} L_n^x \right) \right\}^{1/2}
\]

(3.20)

We take E-expectations and apply Lemma (3.4) to deduce from this and (3.16) that

\[
|E [g(\epsilon_n W_n)] - E [g(\epsilon_n H_n)]| = o(1).
\]

(3.21)

This completes the proof. \(\square\)

Our proof of Theorem 1.2 hinges on two more basic lemmas. The first is an elementary lemma from integration theory.
Lemma 3.3. Suppose \( X := \{X_n\}_{n=1}^{\infty} \) and \( Y := \{Y_n\}_{n=1}^{\infty} \) are \( \mathbb{R}^d \)-valued random variables such that: (i) \( X \) and \( Y \) each form a tight sequence; and (ii) for all bounded infinitely-differentiable functions \( g : \mathbb{R}^d \to \mathbb{R} \),

\[
\lim_{n \to \infty} \left| E_g(X_n) - E_g(Y_n) \right| = 0. \tag{3.22}
\]

Then, the preceding holds for all bounded continuous functions \( g : \mathbb{R}^d \to \mathbb{R} \).

Proof. The proof uses standard arguments, but we repeat it for the sake of completeness.

Let \( K_m := [-m, m]^d \), where \( m \) takes values in \( \mathbb{N} \). Given a bounded continuous function \( g : \mathbb{R}^d \to \mathbb{R} \), we can find a bounded infinitely-differentiable function \( h_m : \mathbb{R}^d \to \mathbb{R} \) such that \( h_m = g \) on \( K_m \). It follows that

\[
\left| E_g(X_n) - E_g(Y_n) \right| \leq \left| E h_m(X_n) - E h_m(Y_n) \right| + 2 \sup_{x \in \mathbb{R}^d} |g(x)| \left( P\{X_n \notin K_m\} + P\{Y_n \notin K_m\} \right). \tag{3.23}
\]

Consequently,

\[
\limsup_{n \to \infty} \left| E_g(X_n) - E_g(Y_n) \right| \leq 2 \sup_{x \in \mathbb{R}^d} |g(x)| \sup_{j \geq 1} \left( P\{X_j \notin K_m\} + P\{Y_j \notin K_m\} \right). \tag{3.24}
\]

Let \( m \) diverge and appeal to tightness to conclude that the left-hand side vanishes.

The final ingredient in the proof of Theorem 1.1 is the following harmonic-analytic result.

Lemma 3.4. If \( \epsilon_n := 1/a_n \), then (3.14) holds.

Proof. Let \( \phi \) denote the characteristic function of \( S_1 \). Our immediate goal is to prove that \( |\phi(t)| < 1 \) for all but a countable number of \( t \in \mathbb{R}^d \). We present an argument, due to Firas Rassoul-Agha, that is simpler and more elegant than our original proof.

Suppose \( S'_1 \) is an independent copy of \( S_1 \), and note that whenever \( t \in \mathbb{R}^d \) is such that \( |\phi(t)| = 1 \), \( D := \exp\{it \cdot (S_1 - S'_1)\} \) has expectation one. Consequently, \( E(|D - 1|^2) = E(D^2) - 1 = 0 \), whence \( D = 1 \) a.s. Because \( S_1 \) is assumed to have at least two possible values, \( S_1 \neq S'_1 \) with positive probability, and this proves that \( t \in 2\pi \mathbb{Z}^d \). It follows readily from this that

\[
\{ t \in \mathbb{R}^d : |\phi(t)| = 1 \} = 2\pi \mathbb{Z}^d, \tag{3.25}
\]

and in particular, \( |\phi(t)| < 1 \) for almost all \( t \in \mathbb{R}^d \).

By the inversion theorem \([S76, P3(b), p. 57]\), for all \( n \geq 0 \),

\[
P\{S_n = 0\} = \frac{1}{(2\pi)^d} \int_{(-\pi,\pi)^d} \{\phi(t)\}^n \, dt. \tag{3.26}
\]
This and the dominated convergence theorem together tell us that \( P\{S_n = 0\} = o(1) \) as \( n \to \infty \), whence it follows that

\[
\sum_{k=1}^{n} P\{S_k = 0\} = o(n) \quad \text{as} \quad n \to \infty.
\] (3.27)

For our particular choice of \( \epsilon_n \) we find that

\[
\epsilon_n^3 \left| \sum_{k=1}^{n} P\{S_k = 0\} \right|^2 = \left( \frac{1}{n} \sum_{k=1}^{n} P\{S_k = 0\} \right)^{1/2},
\] (3.28)

and this quantity vanishes as \( n \to \infty \) by (3.27). This proves the lemma. \( \square \)

**Proof of Theorem 1.2.** Let \( \epsilon_n := 1/a_n \). In light of Proposition 3.2, and Lemmas 3.3 and 3.4, it suffices to prove that the sequences \( n \mapsto \epsilon_n W_n \) and \( n \mapsto \epsilon_n H_n \) are tight. Lemma 2.2, (2.2), and recurrence together imply that for all \( n \) large,

\[
E\left( |\epsilon_n W_n|^2 \right) = \sigma^2 \epsilon_n^2 \sum_{x \in \mathbb{Z}^d} E\left( |L^x_n|^2 \right)
\]

\[
\leq \text{const} \cdot \epsilon_n^2 n \sum_{k=1}^{n} P\{S_k = 0\}
\]

\[
= \text{const}.
\] (3.29)

Thus, \( n \mapsto \epsilon_n W_n \) is bounded in \( L^2(P) \), and hence is tight.

We conclude the proof by verifying that \( n \mapsto \epsilon_n H_n \) is tight. Thanks to (2.4) and recurrence, for all \( n \) large,

\[
E\left( |\epsilon_n H_n|^2 \right) \leq \text{const} \cdot \epsilon_n^2 E \sum_{x \in \mathbb{Z}^d} (L^x_n)^2
\]

\[
\leq \text{const} \cdot \epsilon_n^2 n \sum_{k=1}^{n} P\{S_k = 0\}
\]

\[
= \text{const}.
\] (3.30)

Confer with Lemma 2.2 for the penultimate line. Thus, \( n \mapsto \epsilon_n H_n \) is bounded in \( L^2(P) \) and hence is tight, as was announced. \( \square \)

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References


