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WEIGHT FUNCTIONS AND PATHWISE LOCAL CENTRAL LIMIT THEOREMS

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Summary. This paper is concerned with weak convergence together with convergence rates in weighted almost sure local central limit theorems for random walks. The main tools are stochastic calculus and strong approximations.

1. INTRODUCTION.

Let $X, X_1, X_2, \cdots$ be i.i.d. random variables with $\mathbb{E}X = 0$ and $\sigma^2 = \mathbb{E}X^2 \in (0, \infty)$. Let $S_n = \sum_{1 \le j \le n} X_j$ be the corresponding random walk. Assuming that $\mathbb{E}|X|^{2+\delta} < \infty$ for some $\delta > 0$, Brosamler (1988) and Schatte (1988, 1990, 1991) have shown that,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{1 \le k \le n} \frac{1}{k} G\left(\frac{S_k}{k^{1/2}\sigma}\right) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} G(t) d\Phi(t), \qquad \text{a.s. }, \tag{1.1}$$

if $|G(x)| \le \exp(cx^2)$, for some $c < 1/4$. Here and throughout, Φ is the usual standard normal distribution function and $\log t = \log_e(t \vee e)$ where $\log_e$ denotes the natural logarithm. In particular, if $G(x) = I\{x \le x_0\}$ for some fixed point, $x_0 \in \mathbb{R}$, then (1.1) is the so–called central limit theorem for logarithmic averages, because the limiting value in (1.1) is simply $\Phi(x_0)$. Lévy (1937) and Erdős and Hunt (1953) obtained the first results for logarithmic averages of signs of random walks. For further results in logarithmic averages, we refer to Lacey and Phillip (1990), Révész (1990), Lacey (1991), Berkes, Dehling and Móri (1991), Berkes and Dehling (1993), Csáki, Földes and Révész (1993) and Csáki and Földes (1994). Weigl (1986) (cf. also Révész (1990)), Csörgő and Horváth (1992) and Horváth and Khoshnevisan (1994) have investigated the rate of convergence in (1.1) via strong approximations. A consequence of this development is that for a large class of functions, G,

$$\frac{1}{\sigma_0(G)\big(\log n\big)^{1/2}} \left(\sum_{1 \le k \le n^t} \frac{1}{k} G\left(\frac{S_k}{k^{1/2}\sigma}\right) - t \log n \int_{-\infty}^{\infty} G(s) d\Phi(s) \right), \qquad 0 \le t \le 1$$

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converges weakly (i.e., in $\mathcal{D}\big([0,1]\big)$) to Brownian motion, for some determinable positive constant, $\sigma_0(G)$.

Viewing (1.1) as a strong law for the central limit theorem, it is natural to ask whether (1.1) has a local version. Such local versions of (1.1) play an integral role in the study of return times for random walks; see Erdős and Taylor (1960) and Section 2 below. To state such a result, let us assume that there is a lattice, $\mathbb{L} \subset \mathbb{R}$, such that $\mathbb{P}(X \in \mathbb{L}) = 1$. By Csáki, Földes and Révész (1993), (cf. also Chung and Erdős (1951) and Erdős and Taylor (1960)), if we further assume that $\mathbb{E}|X|^3 < \infty$, then for any fixed $x_0 \in \mathbb{L}$,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{1\le k\le n} \frac{1}{k^{1/2}} I\big\{S_k = x_0\big\} = \frac{1}{(2\pi)^{1/2}\sigma}, \qquad \text{a.s. .} \tag{1.2}$$

Calling (1.2) an "almost sure local central limit theorem", Csáki et al. (1993) have posed the problem of finding the rate of convergence in (1.2). On the other hand, Arcones and Klass (personal communications) have asked whether the $k^{-1/2}$ term in (1.2) can be replaced by other weight functions. It is the goal of this paper, to give a simultaneous answer to both questions mentioned above.

We begin with the following dichotomy which identifies the appropriate class of weight functions.

Proposition 1.1. *For $F : \mathbb{Z} \mapsto \mathbb{R}_+$, define $\mathcal{S}(F) = \sum_{k\ge1} k^{-1/2} F(k)$. Suppose $\mathbb{P}(X \in \mathbb{L}) = 1$ and $\mathbb{E}X = 0$ and $\sigma^2 = \mathbb{E}X^2 \in (0,\infty)$. Then for any $x_0 \in \mathbb{L}$,*

$$\mathbb{P}\big(\textstyle\sum_k F(k) I\big\{S_k = x_0\big\} = \infty\big) = \begin{cases} 1, & \text{if } \mathcal{S}(F) = \infty \\ 0, & \text{if } \mathcal{S}(F) < \infty \end{cases}.$$

We shall see later (see Theorem 1.6 below) that in effect, the only interesting weight functions are of the form, $F(k) = k^{-1/2}\mu(k)$, where $\mu(e^t)$ is regularly varying. Moreover, by Proposition 1.1, for there to be a result of type (1.2) with this more general weight function, we need, $\sum_{k\ge1} k^{-1}\mu(k) = \infty$. With this in mind, define for any $\mu : [1,\infty) \mapsto \mathbb{R}_+$, $t \ge 1$ and $x_0 \in \mathbb{L}$,

$$A_\mu(t) = \sum_{1\le k\le t} \frac{\mu(k)}{k^{1/2}} I\big\{S_k = x_0\big\} - \frac{1}{(2\pi)^{1/2}\sigma} \int_0^{\log t} \mu(e^s) ds. \tag{1.3}$$

(As an example, take $\mu(t) = 1$. If $\mathbb{E}|X|^3 < \infty$, then (1.2) is equivalent to the statement that as $n \to \infty$, $A_\mu(n) = o\big(\log n\big)$, almost surely.)

The main result of this paper is the following strong approximation theorem:

Theorem 1.2. *Suppose $\mathbb{P}(X \in \mathbb{L}) = 1$, $\mathbb{E}X = 0$, $\sigma^2 = \mathbb{E}X^2 \in (0,\infty)$ and $\mathbb{E}|X|^{2+\delta} < \infty$, for some $\delta > 0$. Suppose further that $\mu : [1,\infty) \mapsto \mathbb{R}$ is monotone, possesses a continuous derivative and*

$$\lim_{t\to\infty} \frac{\log \mu(t)}{\log t} = 0. \tag{1.4}$$

Then on a suitable probability space, there exists a reconstruction of A_μ, together with a Brownian motion, B^*, such that for all $\varepsilon > 0$,

$$\left| A_\mu(n) - \left(\frac{2\log 2}{\sigma^2\pi}\right)^{1/2} \int_0^{\log n} \mu(e^s) dB^*(s) \right| = O\big(1 + |\mu(n)| (\log n)^{(1/4)+\varepsilon}\big), \qquad \text{a.s. },$$

where A_μ is defined in (1.3).

Theorem 1.2 has a number of interesting consequences. First, let us see that one can indeed obtain (1.2) together with its rate of convergence under the nearly minimal condition that $\mathbb{E}|X|^{2+\delta} < \infty$. Let $\mu(t) = 1$ to see that by Theorem 1.2,

$$\left| A_\mu(n) - \left(\frac{2\log 2}{\sigma^2\pi}\right)^{1/2} B^*(\log n) \right| = O\big((\log n)^{(1/4)+\varepsilon}\big), \qquad \text{a.s. }.$$

In particular, standard facts about the Brownian motion, B^*, show the following:

Corollary 1.3. *Under the assumptions of Theorem 1.2, (1.2) holds and*

$$\left(\frac{\sigma^2\pi}{2\log n \cdot \log 2}\right)^{1/2} \left(\sum_{1 \le k \le n^t} \frac{1}{k^{1/2}} I\{S_k = x_0\} - \frac{t \log n}{(2\pi)^{1/2}\sigma} \right), \qquad 0 \le t \le 1,$$

converges in $\mathcal{D}[0,1]$ to Brownian motion. Furthermore,

$$\limsup_{n\to\infty} \frac{A_\mu(n)}{\big(\log n \cdot \log\log\log n\big)^{1/2}} = \left(\frac{4\log 2}{\sigma^2\pi}\right)^{1/2}, \qquad \text{a.s.} \tag{1.5}$$

and

$$\liminf_{n\to\infty} \left(\frac{\log\log\log n}{\log n}\right)^{1/2} \max_{1\le k\le n} |A_\mu(k)| = \frac{1}{2\sigma}\big(\pi \log 2\big)^{1/2}, \qquad \text{a.s. }. \tag{1.6}$$

One can also obtain the functional version of (1.5) (in the sense of Strassen (1964)) as well as Lévy classes corresponding to both (1.5) and (1.6). For the Brownian motion, B^*, the latter results can be found in Révész (1990), for example.

An interesting class of μ's is when $\mu(e^t)$ is regularly varying of index $\alpha > -1/2$. For such μ's, (1.4) holds (cf. Bingham et al. (1987, p.26)) Define for all $t \ge 1$,

$$\gamma(t) = \left(\int_0^{\log t} \mu^2(e^s) ds \right)^{1/2}. \tag{1.7}$$

Theorem 1.4. *Suppose X satisfies the conditions of Theorem 1.2. Suppose also, that $\mu(e^t)$ is regularly varying of index $\alpha > -1/2$ and possesses a continuous derivative. If γ^{-1} is the right continuous inverse function to γ (see (1.7)), then*

(i) $\{\sqrt{\sigma^2\pi/(2n\log 2)}A_\mu(\gamma^{-1}(\sqrt{nt}));\ 0 \le t \le 1\}$ *converges in* $\mathcal{D}[0,1]$ *to Brownian motion;*

(ii) $\limsup_{n\to\infty} |\mu(n)|^{-1}\big(\log n \cdot \log\log|\mu(n)|\big)^{-1/2} A_\mu(n) = \big(\log(2)/((2\alpha+1)\sigma^2\pi)\ \big)^{1/2}$, *a.s.*

(iii) $\liminf_{n\to\infty} |\mu(n)|^{-1}\big(\log\log|\mu(n)|\big)^{1/2}(\log n)^{-1/2}\max_{1\le k\le n}|A_\mu(k)| = c$, *a.s., where* $c = \big(\pi\log(2)/(4\sigma^2(2\alpha+1))\big)^{1/2}$.

Next, we investigate the class of functions of form, $\mu(t) = (\log t)^\alpha = \big(\log_e(t \vee e)\big)^\alpha$. By Proposition 1.1, it is enough to consider $\alpha \ge -1$ only. Then we have the following:

Theorem 1.5. *Suppose X satisfies the conditions of Theorem 1.2. Let $\mu(t) = (\log t)^\alpha$, $\alpha_0 = 2\alpha + 1$ and $\nu_n = \big(\sigma^2\pi/(2n\log 2)\big)^{1/2}$.*

(i) If $\alpha > -1/2$, then $\{\nu_n A_\mu\big(\exp((\alpha_0 nt)^{1/\alpha_0})\big);\ 0 \le t \le 1\}$ converges in $\mathcal{D}[0,1]$ to Brownian motion.

(ii) If $\alpha = -1/2$, then $\{\nu_n A_\mu\big(\exp(e^{nt})\big);\ 0 \le t \le 1\}$ converges in $\mathcal{D}[0,1]$ to Brownian motion.

(iii) If $-1 \le \alpha < -1/2$, then almost surely, $\sup_n |A_\mu(n)| < \infty$.

In particular, the above theorem says that when $\mu(t) = (\log t)^\alpha$, the rate of convergence in the weighted version of (1.2) goes through a phase transition at $\alpha = -1/2$. Consequently, $R(n)A_\mu(n) \xrightarrow{D}$ N(0,1), where,

$$R(n) = \begin{cases} \left(\dfrac{\sigma^2\pi(2\alpha+1)}{2(\log n)^{2\alpha+1}\log 2}\right)^{1/2}, & \text{if } \alpha > -1/2 \\[2ex] \left(\dfrac{\sigma^2\pi}{\log 2 \cdot \log\log n}\right)^{1/2}, & \text{if } \alpha = -1/2 \end{cases}.$$

Furthermore, when $\alpha \in [-1, -1/2)$, there cannot be such a central limit theorem.

Finally, we come back to the issue of what happens when $\mu(e^t)$ is not regularly varying. In this case, it is essentially impossible to obtain almost sure principles. We illustrate this by considering functions of type: $\mu(t) = t^\theta$ where $\theta \ge 0$. (The case $\theta < 0$ is trivial by Proposition 1.1.)

Theorem 1.6. *Under the conditions of Theorem 1.2 for X, for each $\theta \ge 0$,*

$$n^{-\theta}\sum_{1\le k\le nt} k^{\theta-(1/2)} I\{S_k = x_0\}, \qquad 0 \le t \le 1,$$

converges in $\mathcal{D}[0,1]$ *to*

$$\int_0^t s^{\theta-(1/2)} dL(s), \qquad 0 \le t \le 1,$$

where L *is the process of local times for standard Brownian motion at zero.*

Remark 1.6.1. That $\int_0^t s^{\theta-(1/2)} dL(s)$ is a finite process is part of the assertion of the theorem.

The key idea behind the proof of the above results is a detailed analysis (via stochastic calculus) of the process of local times, ℓ, of the standard Ornstein–Uhlenbeck process. Our analysis leads to a strong improvement of the Chacon–Ornstein ergodic theorem for ℓ, which is of independent interest. For the statement of the latter ergodic theorem, see Revuz and Yor (1991). Our improvement appears in Section 3, together with other related facts about the Ornstein–Uhlenbeck process. An application of Theorem 1.2 to the study of return times of random walks appears in Section 2 while the proof of the latter theorem appears in Section 4. Theorems 1.4 through 1.6 are proved in Section 5. Proposition 1.1 is a routine generalization of the well–known fact that a random walk is recurrent if and only if the expected number of returns to the origin is infinite. Viewed as such, the proof of Proposition 1.1 is well–known. For the sake of completeness, we have included it in an appendix.

2. AN APPLICATION TO RETURN TIMES.

Using the notation of the previous sections, define the return times of the random walk as: $\rho_0 = 0$ and for all $k \ge 1$, $\rho_k = \min\{j > \rho_{k-1} : \ S_k = x_0\}$. It is well–known (cf. Feller (1957) for simple walks, for example) that $\mathbb{E}\rho_1 = \infty$. Moreover, by the strong Markov property, $\rho_n = \sum_{1\le k\le n} \left(\rho_k - \rho_{k-1}\right)$ is an increasing random walk. Hence, there cannot possibly be a strong law of large numbers for ρ_k's. On the other hand, Erdős and Taylor (1960) have shown that a suitable re-interpretation of such a result does exist. Namely, if S_n is the simple symmetric random walk on $\mathbb{L}$, then

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{1\le k\le n} \frac{1}{\rho_k^{1/2}} = \left(\frac{2}{\pi\sigma^2}\right)^{1/2}, \qquad \text{a.s.,} \tag{2.1}$$

for $\sigma = 1$. In Csáki et al. (1993), the above has been shown to be true for any random walk on $\mathbb{L}$ satisfying: $\mathbb{E}X = 0$, $\sigma^2 = \mathbb{E}X^2 < \infty$ and $\mathbb{E}|X|^3 < \infty$. In this section, we discuss the existence and rate of convergence in (2.1) under the nearly minimal conditions of Theorem 1.2 on the distribution of X. To this end, let us define for all $t \ge 1$,

$$H(t) = \sum_{1\le k\le t} \frac{1}{\rho_k^{1/2}} - \left(\frac{2}{\pi\sigma^2}\right)^{1/2} \log t. \tag{2.2}$$

Then we have the following result:

Theorem 2.1. *Under the conditions of Theorem 1.2 for X, the reconstruction of Theorem 1.2 yields that for all $\varepsilon > 0$, almost surely,*

$$\left| H(n) - \sqrt{\frac{2\log 2}{\sigma^2 \pi}} B^*(2\log n) \right| = O\big((\log n)^{(1/4)+\varepsilon} \big).$$

Remark 2.1.1. One can use the techniques of Section 3, to extend the above to the case of more general weight functions.

Corollary 2.2. *Under the conditions of Theorem 1.2,*
(i) (2.1) holds, and
(ii) $\big\{ \big(\sigma^2\pi/(2n\log 2) \big)^{1/2} H\big(\exp(nt/2)\big);\ 0 \le t \le 1 \big\}$ converges in $\mathcal{D}\big([0,1]\big)$ to Brownian motion.

Proof of Theorem 2.1. Recalling (1.3), define $A(n) = A_\mu(n)$ for $\mu(t) = 1$. It is not hard to see that,

$$H(n) = A(\rho_n) - \frac{1}{(2\pi)^{1/2}\sigma} \cdot \log\big(n^2/\rho_n \big). \tag{2.3}$$

Recalling the integral tests of Khintchine (1938) and Breiman (1968) (cf. also Mijnheer (1975)), we can argue as in Horváth (1986, Example 6), to see that almost surely,

$$\big| \log\big(\rho_n/n^2 \big) \big| = O\big(\log\log n \big). \tag{2.4}$$

By (2.3) and Theorem 1.2, for all $\varepsilon > 0$,

$$\begin{aligned} \left| H(n) - \sqrt{\frac{2\log 2}{\sigma^2 \pi}} B^*(\log \rho_n) \right| &= O\big((\log \rho_n)^{(1/4)+\varepsilon} \big) \\ &= O\big((\log n)^{(1/4)+\varepsilon} \big). \end{aligned} \tag{2.5}$$

But,

$$\begin{aligned} \big| B^*(\log \rho_n) - B^*(\log n^2) \big| &\le \sup_{0 \le t \le \log n^2} \ \sup_{|s| \le |\log(\rho_n/n^2)|} \big| B^*(t+s) - B^*(t) \big| \\ &= O\big(\log\log n \big), \end{aligned}$$

by the modulus of continuity of Csörgő and Révész (1991, p.30). By (2.5), the result follows. ♢

3. THE LOCAL TIME OF THE ORNSTEIN–UHLENBECK PROCESS.

Let $\{W(t);\ t \ge 0\}$ be a Brownian motion. The local time of W at zero is denoted by $L(t)$. For the definition, existence and properties of $L(t)$, we refer to Revuz and Yor (1991, Chapter VI). By Tanaka's formula (cf. Revuz and Yor (1991, p. 207)), we have,

$$|W(t)| = \beta(t) + L(t), \tag{3.1}$$

where,

$$\beta(t) = \int_0^t \operatorname{sign}\big(W(s)\big) dW(s). \tag{3.2}$$

It is easy to see that the quadratic variation, $\langle \beta \rangle_t$, of $\beta(t)$ is t, a.s.. Moreover, β is a continuous local martingale with respect to the filtration of W. Hence, by Lévy's characterization theorem (cf. Revuz and Yor (1991, p. 141)), β is a Brownian motion.

Let

$$U(t) = e^{-t/2} W(e^t), \tag{3.3}$$

and

$$V(t) = e^{-t/2} \beta(e^t). \tag{3.4}$$

Evidently, U and V are Ornstein–Uhlenbeck processes, i.e., centered Gaussian processes with

$$\mathbb{E}U(s)U(t) = \mathbb{E}V(s)V(t) = \exp\big(-|s-t|/2\big).$$

It is well–known that U and V are linear diffusions (cf. Rogers and Williams (1987), for example). We begin with some elementary observations, many of which are well–known. Let $f(x,t) = xt^{-1/2}$. Applying Itô's formula (cf. Revuz and Yor (1991, p. 138)) to $f\big(\beta(t), t\big)$, we obtain the following for all $t \geq 1$:

$$\begin{aligned} \frac{\beta(t)}{t^{1/2}} &= \beta(1) + \int_1^t s^{-1/2} d\beta(s) - \frac{1}{2} \int_1^t s^{-3/2} \beta(s) ds \\ &= \beta(1) + \int_1^t s^{-1/2} d\beta(s) - \frac{1}{2} \int_0^{\log t} V(s) ds, \end{aligned}$$

by changing variables. A similar expression holds for $B(t)/t^{1/2}$. Using (3.3) and (3.4), the above development implies that

$$V(t) = V(0) + \Gamma_\beta(t) - \frac{1}{2} \int_0^t V(s) ds, \tag{3.5}$$

and

$$U(t) = U(0) + \Gamma_W(t) - \frac{1}{2} \int_0^t U(s) ds, \tag{3.6}$$

where,

$$\Gamma_\beta(t) = \int_1^{e^t} s^{-1/2} d\beta(s), \tag{3.7}$$

and

$$\Gamma_W(t) = \int_1^{e^t} s^{-1/2} dW(s). \tag{3.8}$$

It is easy to see that Γ_β and Γ_W are centered Gaussian processes for which,

$$\mathbb{E}\Gamma_\beta(s)\Gamma_\beta(t) = \mathbb{E}\Gamma_W(s)\Gamma_W(t) = s \wedge t.$$

Therefore, Γ_β and Γ_W are both Brownian motions.

Next, we compute the strong generator of U and V. First, we apply Itô's formula to $f\big(U(t)\big)$, where f is a twice continuously differentiable function. By (3.6), we obtain,

$$f\big(U(t)\big) - f\big(U(0)\big) - \int_0^t \mathcal{A}f\big(U(s)\big)ds = \int_0^t f'\big(U(s)\big)d\Gamma_W(s), \tag{3.9}$$

where,

$$\mathcal{A}f(x) = \frac{1}{2}f''(x) - \frac{1}{2}xf'(x). \tag{3.10}$$

The right hand side of (3.9) being a martingale, it follows from the martingale problem, that the generator of U is $\mathcal{A}$. Since U and V are equivalent in distribution, the same holds for V.

Let $\ell(t)$ denote the local time of U at zero. The existence of ℓ follows from Revuz and Yor (1991, p. 209). Moreover, ℓ can be written as,

$$\ell(t) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t I\big\{|U(s)| \le \varepsilon\big\}ds, \qquad \text{a.s.,} \tag{3.11}$$

and

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}\left|\ell(t) - \frac{1}{2\varepsilon}\int_0^t I\big\{|U(s)| \le \varepsilon\big\}ds\right|^p = 0, \tag{3.12}$$

for all $p > 0$.

Lemma 3.1. *If $D(t) = |U(t)| - V(t)$, then almost surely for all $t > 0$,*

$$\frac{1}{2}\int_0^t D(s)ds = \ell(t) + D(0) - D(t). \tag{3.13}$$

Proof. First, we use Tanaka's formula and (3.6) to see that

$$\begin{aligned} |U(t)| &= |U(0)| + \int_0^t \operatorname{sign}\big(U(s)\big)dU(s) + \ell(t) \\ &= |U(0)| + \int_0^t \operatorname{sign}\big(U(s)\big)d\Gamma_W(s) - \frac{1}{2}\int_0^t \operatorname{sign}\big(U(s)\big)U(s)ds + \ell(t) \\ &= |U(0)| + \int_0^t \operatorname{sign}\big(U(s)\big)d\Gamma_W(s) - \frac{1}{2}\int_0^t |U(s)|ds + \ell(t). \end{aligned} \tag{3.14}$$

Subtracting (3.5) from (3.14), we obtain

$$D(t) = D(0) + M(t) - \frac{1}{2}\int_0^t D(s)ds + \ell(t), \tag{3.15}$$

where,

$$M(t) = \int_0^t \operatorname{sign}\big(U(s)\big)d\Gamma_W(s) - \Gamma_\beta(t).$$

Since M is a continuous martingale, it follows that $M(t) = M(0) = 0$, if we can prove that the quadratic variation, $\langle M\rangle_t$, is zero for all t (cf. Revuz and Yor (1991, p. 119)). We note that

$$\begin{aligned}\langle M\rangle_t &= \langle \int_0^{\cdot} \operatorname{sign}(U(s))d\Gamma_W(s)\rangle_t + \langle \Gamma_\beta\rangle_t - 2\langle \int_0^{\cdot} \operatorname{sign}(U(s))d\Gamma_W(s), \Gamma_\beta\rangle_t \\ &= Z_1(t) + Z_2(t) - 2Z_3(t).\end{aligned}$$

Here $\langle N^1, N^2\rangle_t$ denotes the mutual variation of the semimartingales, N^1 and N^2.

Since Γ_W and Γ_β are Brownian motions, $Z_2(t) = t$, for all $t \geq 0$, almost surely. Furthermore,

$$Z_1(t) = \int_0^t \big(\operatorname{sign}U(s)\big)^2 ds = t, \qquad \text{a.s. .}$$

To finish the lemma, it remains to prove that $Z_3(t) = t$.

Writing $\Gamma_\beta(t) = \int_0^t d\Gamma_\beta(s)$, it follows that,

$$Z_3(t) = \int_0^t \operatorname{sign}\big(U(s)\big)d\langle \Gamma_W, \Gamma_\beta\rangle_s.$$

By (3.7) and (3.8),

$$\langle \Gamma_W, \Gamma_\beta\rangle_t = \int_1^{e^t} s^{-1} d\langle W, \beta\rangle_s,$$

almost surely. Therefore,

$$Z_3(t) = \int_0^t \operatorname{sign}\big(U(s)\big)d\langle W, \beta\rangle_{e^s}, \qquad \text{a.s. .}$$

By (3.2) and (3.3),

$$\begin{aligned}\langle W, \beta\rangle_t &= \langle \int_0^{\cdot} dW(u), \beta\rangle_t \\ &= \int_0^t \operatorname{sign}\big(W(s)\big)ds \\ &= \int_0^t \operatorname{sign}\big(U(\log s)\big)ds, \qquad \text{a.s. .}\end{aligned}$$

Hence,

$$Z_3(t) = \int_0^t \operatorname{sign}^2\big(U(s)\big)ds = t. \qquad \text{a.s..}$$

This proves the lemma. ♢

Lemma 3.2. *For all* $t \geq 0$,

$$\mathbb{E}\ell(t) = \frac{t}{(2\pi)^{1/2}}. \tag{3.16}$$

Moreover, as $t \to \infty$,

$$\mathit{Var}\;\ell(t) = \frac{2\log 2}{\pi} \cdot t + o(t). \tag{3.17}$$

Proof. Since $\mathbb{E}D(t) = (2/\pi)^{1/2}$, (3.16) follows from Lemma 3.1. To prove (3.17), define,

$$p(x,y) = \frac{1}{2\pi}\big(2 - \exp(-|x-y|)\big)^{-1/2}.$$

By (3.12),

$$\mathbb{E}\ell^2(t) = 2\int_0^t \int_0^y p(x,y)dxdy.$$

Elementary calculations reveal

$$\mathbb{E}\ell^2(t) = \frac{t^2}{2\pi} + \frac{1}{\pi}\int_0^t \log\big(2 + 2(1-e^{-s})^{1/2} + e^{-s}\big)ds.$$

This proves (3.17) and hence the lemma. ◊

Following Motoo (1959), define the stopping time, $\{\tau_k;\ t \geq 0\}$ by, $\tau_0 = 0$, and for all $j \geq 0$,

$$\tau_{2j+1} = \inf\big\{s > \tau_{2j} :\ U(s) = 1\big\},$$
$$\tau_{2j+2} = \inf\big\{s > \tau_{2j+1} :\ U(s) = 0\big\}.$$

Lemma 3.3. *The following are true:*

- *(i) The random variables,* $\{\tau_{2j+1} - \tau_{2j};\ j \geq 1\}$, *are i.i.d. with a finite moment generating function in a neighbourhood of zero.*
- *(ii) The random variables,* $\{\tau_{2j+2} - \tau_{2j+1};\ j \geq 0\}$, *are i.i.d. with a finite moment generating function in a neighbourhood of zero.*
- *(iii) The sequence,* $\{\tau_{2j+1} - \tau_{2j};\ j \geq 1\}$ *and* $\{\tau_{2j+2} - \tau_{2j+1};\ j \geq 0\}$, *are independent.*
- *(iv) The random variables,* $\big\{\big(\ell(\tau_{2j+2}) - \ell(\tau_{2j})\ ,\ \tau_{2j+2} - \tau_{2j}\big);\ j \geq 1\big\}$ *are i.i.d. with each co-ordinate possessing a finite moment generating function in a neighbourhood of the origin.*

Proof. That the increments are i.i.d. is a consequence of the strong Markov property of U. The finiteness of the moment generating function of τ_j is due to the Gaussian decay rate of the tails of the density of the speed measure of U. For more details, see Motoo (1959). For (iv), we need only to prove that $\ell(\tau_4) - \ell(\tau_3)$ has a finite moment generating function in a neighbourhood of zero. By the proof of Khoshnevisan (1993, Lemma 4.3), $\ell(\tau_4) - \ell(\tau_3)$ has an exponential distribution. This proves the lemma. $\diamondsuit$

By Lemma 3.3, the following is finite:

$$\nu = \mathbb{E}\big(\tau_4 - \tau_2\big). \tag{3.18}$$

The parameter, ν, can be calculated by the optional sampling theorem, (3.10), and (3.14) and indeed, $\nu = 2\big(\log(2)/\pi\big)^{1/2}$. However, we will not show these calculations, as we will not have need for the exact value of ν.

Lemma 3.4. *Fix $\varepsilon > 0$. As $k \to \infty$,*

$$\left|\ell(k\nu) - \ell(\tau_{2k}) - \frac{k\nu - \tau_{2k}}{(2\pi)^{1/2}}\right| = O\big(k^{(1/4)+\varepsilon}\big), \qquad \text{a.s..} \tag{3.19}$$

Furthermore,

$$\mathbb{E}\left|\ell(k\nu) - \ell(\tau_{2k}) - \frac{k\nu - \tau_{2k}}{(2\pi)^{1/2}}\right|^2 = o(k). \tag{3.20}$$

Proof. First, we show that almost surely, as $T \to \infty$,

$$\sup_{0\le s\le u(T)}\left|\ell(T+s) - \ell(T) - \frac{s}{(2\pi)^{1/2}}\right| = O\big(T^{(1/4)+\varepsilon} + \sqrt{u(T)\log T}\big), \tag{3.21}$$

where $u(T) \le T$. By Lemma 3.1, it is enough to consider,

$$\sup_{0\le s\le u(T)}\Big|D(T+s) - D(T)\Big| + \frac{1}{2}\sup_{0\le s\le u(T)}\left|\int_T^{T+s}\big(D(u) - (2/\pi)^{1/2}\nu\big)du\right|.$$

It is easy to see that

$$\sup_{0\le s\le T}|D(s)| \le \sup_{0\le s\le T}\big(|U(s)| + |V(s)|\big) = O\big((\log T)^{1/2}\big),$$

almost surely. Following the proof of Csörgő and Horváth (1992, Theorem 1.4), we can find a constant, $\gamma > 0$ and a Brownian motion, $\{\widetilde{W}(t);\ t \ge 0\}$, such that as $t \to \infty$,

$$\big|\textstyle\int_0^t\big(D(s) - (2/\pi)^{1/2}\nu\big)ds - \gamma\widetilde{W}(t)\big| = O\big(t^{(1/4)+\varepsilon}\big), \qquad \text{a.s. .}$$

By the modulus of continuity of $\widetilde{W}$ (cf. Csörgő and Révész (1991, p. 90)), (3.21) follows. To prove (3.19), write τ_{2k} as

$$\tau_{2k} = \sum_{2\le j\le k}\big(\tau_{2j} - \tau_{2j-2}\big) + \tau_2. \tag{3.22}$$

By (3.18), Lemma 3.3 and the law of the iterated logarithm,

$$|\tau_{2k} - k\nu| = O\big(\sqrt{k \log\log k}\big), \qquad \text{a.s..} \tag{3.23}$$

By (3.21) and (3.23), (3.19) follows.

To prove (3.20), we point out that by (3.22) and Lemma 3.3, for all $p \geq 1$ there exists some $c_p \in (0,\infty)$, such that

$$\mathbb{E}\tau_{2k}^p \leq c_p k^p. \tag{3.24}$$

Furthermore, for all $\vartheta > 0$, there exists some $\alpha = \alpha(\vartheta) \in (0,1)$, such that for all $k \geq 1$,

$$\mathbb{P}\big(E_k^c\big) \leq \alpha^{-1} \exp\big(-\alpha k^{\vartheta}\big), \tag{3.25}$$

where

$$E_k = \big\{|\tau_{2k} - k\nu| \leq k^{(1/2)+\vartheta}\big\}. \tag{3.26}$$

Write,

$$\begin{aligned}
\mathbb{E}\bigg|\ell(\tau_{2k}) - \ell(k\nu) - \frac{\tau_{2k} - k\nu}{(2\pi)^{1/2}}\bigg|^2 &\leq 2\mathbb{E}\big(\tau_{2k} + k\nu\big)^2 I\big\{E_k^c\big\} \\
&\quad + \mathbb{E}\bigg|\ell(\tau_{2k}) - \ell(k\nu) - \frac{\tau_{2k} - k\nu}{(2\pi)^{1/2}}\bigg|^2 I\big\{E_k\big\} \\
&\quad + 2\mathbb{E}\big(\ell(\tau_{2k}) - \ell(k\nu)\big)^2 I\big\{E_k^c\big\} \\
&= Q_{1,k} + Q_{2,k} + Q_{3,k}.
\end{aligned}$$

We shall show that $Q_{j,k} = o(k)$ for $j = 1,2,3$.

By (3.24)–(3.26), $Q_{1,k} = o(k)$. It is easy to see that

$$\mathbb{E}\bigg(\sup_{s:|s-k\nu|\leq k^{(1/2)+\vartheta}} \big|D(k\nu) - D(s)\big|\bigg)^2 = o(k).$$

Moreover, since $\big\{\int_0^t \big(D(s) - (2/\pi)^{1/2}\big)ds;\ t \geq 0\big\}$ is a mean zero martingale, by Doob's maximal inequality,

$$\begin{aligned}
&\mathbb{E}\bigg(\sup_{s:|s-k\nu|\leq k^{(1/2)+\vartheta}} \int_{k\nu}^s \big(D(t) - (2/\pi)^{1/2}\big)dt\bigg)^2 \\
&\qquad\leq 4 \sup_{s:|s-k\nu|\leq k^{(1/2)+\vartheta}} \mathbb{E}\bigg(\int_{k\nu}^s \big(D(t) - (2/\pi)^{1/2}\big)dt\bigg)^2 \\
&\qquad= o(k).
\end{aligned}$$

By Lemma 3.1 and the above two estimates, $Q_{2,k} = o(k)$. Finally, by Lemmas 3.1 and 3.3(iv), it follows that for every $p \geq 1$, there exists a $c_p^* \in (0,\infty)$, such that for all $k \geq 1$,

$$\mathbb{E}\ell^p(k\nu) + \mathbb{E}\ell^p(\tau_{2k}) \leq c_p^* k^p.$$

By (3.25) and lemma 3.1,

$$\begin{aligned}
Q_{3,k} &\leq 4\big(\mathbb{E}\ell^4(\tau_{2k})\mathbb{P}(E_k^c)\big)^{1/2} + 4\big(\mathbb{E}\ell^4(k\nu)\mathbb{P}(E_k^c)\big)^{1/2} \\
&= o(k).
\end{aligned}$$

This proves the lemma. $\diamondsuit$

Lemma 3.5. *There exists a sequence of i.i.d. random variables, $\{Y_j;\ j \geq 1\}$, such that for all $\varepsilon > 0$,*

$$\Big|\ell(k\nu) - \sum_{1\leq j\leq k} Y_j\Big| = O\big(k^{(1/4)+\varepsilon}\big), \qquad \text{a.s.}\ . \tag{3.27}$$

Moreover,

$$\mathbb{E}\bigg(\ell(k\nu) - \sum_{1\leq j\leq k} Y_j\bigg)^2 = o(k). \tag{3.28}$$

Finally, Y_1 has a finite moment generating function in a neighbourhood of zero and

$$\mathbb{E}Y_1 = \frac{\nu}{(2\pi)^{1/2}}, \tag{3.29}$$

$$Var\ Y_1 = \frac{\nu \log 4}{\pi}, \tag{3.30}$$

where ν is defined by (3.18).

Proof. Since $\tau_0 = 0$, we can write,

$$\begin{aligned}\ell(k\nu) &= \ell(\tau_{2k}) + \ell(k\nu) - \ell(\tau_{2k}) \\ &= \sum_{1\leq j\leq k}\bigg(\ell(\tau_{2j}) - \ell(\tau_{2j-2}) + \frac{\nu - (\tau_{2j} - \tau_{2j-2})}{(2\pi)^{1/2}}\bigg) \\ &\qquad + \ell(k\nu) - \ell(\tau_{2k}) - \frac{k\nu - \tau_{2k}}{(2\pi)^{1/2}}.\end{aligned}$$

By (3.18) and Lemma 3.2, we immediately get (3.29) and (3.30). ♢

We are now ready to state and prove the main result of this section.

Proposition 3.6. *On an appropriate probability space, there exists a reconstruction of ℓ together with a Brownian motion, $\{B^*(t);\ t \geq 0\}$, such that for all $\varepsilon > 0$,*

$$\Big|\ell(t) - \frac{t}{(2\pi)^{1/2}} - \sqrt{\frac{2\log 2}{\pi}}B^*(t)\Big| = o\big(t^{(1/4)+\varepsilon}\big), \qquad \text{a.s.}\ .$$

Proof. Fix an arbitrary $\varepsilon > 0$. By Lemma 3.5 and Komlós, Major and Tusnády (1975, 1976), on some probability space, we can reconstruct ℓ together with a Brownian motion, $\{B_1^*(t);\ t \geq 0\}$, such that

$$\Big|\ell(k\nu) - \frac{k\nu}{(2\pi)^{1/2}} - \sqrt{\frac{\nu\log 4}{\pi}}B_1^*(k)\Big| = o\big(k^{(1/4)+\varepsilon}\big), \qquad \text{a.s.}\ . \tag{3.31}$$

By the modulus of continuity of B_1^* (cf. Csörgő and Révész (1991, p.30)),

$$\sup_{k\le t\le k+1} \big|B_1^*(k) - B_1^*(t)\big| = O\big(\sqrt{\log k}\big), \qquad \text{a.s .} \tag{3.32}$$

Hence, by (3.21), (3.31) and (3.32) we have

$$\sup_{k\le t\le k+1} \big|\ell(k\nu) - \ell(t\nu)\big| = o\big(k^{(1/4)+\varepsilon}\big), \qquad \text{a.s. .} \tag{3.33}$$

By (3.31)–(3.33),

$$\left|\ell(t\nu) - \frac{t\nu}{(2\pi)^{1/2}} - \sqrt{\frac{\nu\log 4}{\pi}}B_1^*(t)\right| = o\big(t^{(1/4)+\varepsilon}\big), \qquad \text{a.s. .}$$

Since $B^*(t) = \nu^{1/2}B_1^*(t/\nu)$ is another Brownian motion, Proposition 3.6 follows. ◇

4. THE PROOF OF THEOREM 1.2.

Throughout this section, let $\mu : [1,\infty) \mapsto \mathbb{R}$ satisfy the conditions of Theorem 1.2. Without loss of generality, we may take $\mathbb{L} = \mathbb{Z}$ and $x_0 = 0$, for otherwise, one can rescale and relabel $\mathbb{L}$. Finally, we shall make all of the calculations for the case $\sigma = 1$. The modifications for general $\sigma > 0$ are routine.

Let η denote the local time of the random walk, i.e., for all $t \ge 1$,

$$\eta(t) = \sum_{1\le k\le t} I\big\{S_k = 0\big\}. \tag{4.1}$$

By Bass and Khoshnevisan (1992), on a suitable probability space one can reconstruct the random walk together with a Brownian motion, W, with local time, L, such that for some $\varepsilon = \varepsilon(\delta) > 0$,

$$\big|\eta(t) - L(t)\big| = o\big(t^{(1/2)-\varepsilon}\big), \qquad \text{a.s. .} \tag{4.2}$$

Lemma 4.1. *For the construction of (4.2),*

$$\sup_n \left| \sum_{1\le k\le n} \frac{\mu(k)}{k^{1/2}} I\big\{S_k = 0\big\} - \int_1^n \frac{\mu(t)}{t^{1/2}}dL(t)\right| < \infty, \qquad \text{a.s. .}$$

Proof. Since both η and L are increasing, integration by parts reveals that,

$$\begin{aligned}\sum_{1\le k\le n} &\frac{\mu(k)}{k^{1/2}} I\big\{S_k = 0\big\} - \int_1^n \frac{\mu(t)}{t^{1/2}}dL(t)\\
&= \int_1^n \frac{\mu(t)}{t^{1/2}} d\big(\eta(t) - L(t)\big)\\
&= \frac{\mu(n)}{n^{1/2}}\big(\eta(n) - L(n)\big) - \mu(1)\big(\eta(1) - L(1)\big) - \int_1^n \big(\eta(t) - L(t)\big)\frac{\mu'(t)}{t^{1/2}}dt\\
&\quad + \frac{1}{2}\int_1^n \big(\eta(t) - L(t)\big)\frac{\mu(t)}{t^{3/2}}dt.\end{aligned} \tag{4.3}$$

By (4.2), there exists some $\varepsilon = \varepsilon(\delta)$ such that as $n \to \infty$,

$$\frac{\mu(n)}{n^{1/2}}\big(\eta(n) - L(n)\big) = O\big(n^{-\varepsilon}|\mu(n)|\big) \tag{4.4}$$
$$= O(1),$$

by (1.4). Furthermore, since μ is monotone and μ' continuous, by (4.2) and integration by parts,

$$\int_1^n \big(\eta(t) - L(t)\big)\frac{\mu'(t)}{t^{1/2}}dt = O(1) \cdot \int_1^n t^{-\varepsilon}|\mu'(t)|dt$$
$$= O(1) \cdot \int_1^n \frac{|\mu(t)|}{t^{1+\varepsilon}}dt.$$

By (1.4), $\mu(t) = o(t^{\varepsilon/2})$. Therefore,

$$\int_1^n \big(\eta(t) - L(t)\big)\frac{\mu'(t)}{t^{1/2}}dt = O(1), \qquad \text{a.s. .} \tag{4.5}$$

Finally by another application of (4.2),

$$\int_1^n \big(\eta(t) - L(t)\big)\frac{\mu(t)}{t^{3/2}}dt = O(1) \cdot \int_1^n \frac{|\mu(t)|}{t^{1+\varepsilon}}dt \tag{4.6}$$
$$= O(1),$$

by (1.4). The lemma is a consequence of (4.3)–(4.6). ◇

Recall the process, D, from Lemma 3.1.

Lemma 4.2. *We have*

$$\left| \int_0^{\log n} \mu(e^s)dD(s)\right| = O\big(|\mu(n)|\sqrt{\log\log n} + 1\big), \qquad \text{a.s. .}$$

Proof. Integrating by parts,

$$\int_0^t \mu(e^s)dD(s) = \mu(e^t)D(t) - \mu(1)D(0) - \int_0^t e^s\mu'(e^s)D(s)ds.$$

As we have mentioned earlier, $|D(s)| = O\big(\sqrt{\log s}\big)$. The result follows from the above together with the following:

$$\int_0^t e^s|\mu'(e^s)|ds = \big|\mu(e^t) - \mu(1)\big|.$$

The above follows from the assumed monotonicity of μ together with the assumption that μ' is continuous. This proves the lemma. ◇

Lemma 4.3. *For the construction of (4.2),*

$$\left| \int_1^n \frac{\mu(t)}{t^{1/2}} dL(t) - \frac{1}{2} \int_0^{\log n} D(s)\mu(e^s) ds \right| = O\big(|\mu(n)| \sqrt{\log \log n}\big), \qquad \text{a.s. .}$$

Proof. By the definition of D (see Lemma 3.1) and by (3.1), (3.3) and (3.4),

$$D(t) = \frac{L(e^t)}{e^{t/2}}.$$

Integrating by parts twice,

$$\begin{aligned} \int_1^n \frac{\mu(t)}{t^{1/2}} dL(t) &= \mu(e^s)D(s)\Big|_0^{\log n} - \int_0^{\log n} e^s \mu'(e^s) D(s) ds + \frac{1}{2} \int_0^{\log n} D(s)\mu(e^s) ds \\ &= \int_0^{\log n} \mu(e^s) dD(s) + \frac{1}{2} \int_0^{\log n} D(s)\mu(e^s) ds. \end{aligned}$$

The result follows from Lemma 4.2. ♢

Recall (3.1), (3.3) and (3.11).

Lemma 4.4. *For the same construction as (4.2), almost surely,*

$$\left| \int_1^n \frac{\mu(t)}{t^{1/2}} dL(t) - \int_0^{\log n} \mu(e^s) d\ell(s) \right| = O\big(1 + |\mu(n)| \sqrt{\log \log n}\big).$$

Proof. Writing (3.13) in its differential form,

$$\frac{1}{2} D(s) ds = d\ell(s) - dD(s).$$

Hence,

$$\begin{aligned} \frac{1}{2} \int_0^{\log n} D(s)\mu(e^s) ds &= \int_0^{\log n} \mu(e^s) d\ell(s) - \int_0^{\log n} \mu(e^s) dD(s) \\ &= \int_0^{\log n} \mu(e^s) d\ell(s) + O\big(1 + |\mu(n)| \sqrt{\log \log n}\big), \end{aligned}$$

by Lemmas 4.1 and 4.2. Lemmas 4.1 and 4.3 together imply the result. ♢

Lemma 4.5. *Fix an arbitrary $\varepsilon > 0$. For the same construction as (4.2), almost surely,*

$$\left| \int_0^{\log n} \mu(e^s) d\ell(s) - \frac{1}{(2\pi)^{1/2}} \int_0^{\log n} \mu(e^s) ds - \sqrt{\frac{2\log 2}{\pi}} \int_0^{\log n} \mu(e^s) dB^*(s) \right| = O\big(1 + |\mu(n)| (\log n)^{(1/4)+\varepsilon}\big),$$

where B^* is the Brownian motion of Proposition 3.6.

Proof. By integration by parts,

$$\int_0^{\log n} \mu(e^s) d\ell(s) = \mu(e^s)\ell(s)\Big|_0^{\log n} - \int_0^{\log n} e^s \mu'(e^s)\ell(s) ds.$$

By Proposition 3.6, with probability one,

$$\ell(s) = \frac{s}{(2\pi)^{1/2}} + \sqrt{\frac{2\log 2}{\pi}} B^*(s) + o\big(s^{(1/4)+\varepsilon}\big).$$

By the assumptions on μ, one easily sees that for all $t > 0$,

$$\int_0^t s^{(1/4)+\varepsilon} e^s \mu'(e^s) ds \le t^{(1/4)+\varepsilon} \big|\mu(e^t) - \mu(1)\big|.$$

This proves the lemma. ♢

Proof of Theorem 1.2. Theorem 1.2 is a trivial consequence of Lemmas 4.1, 4.4 and 4.5, together with the triangle inequality. ♢

5. MORE PROOFS.

The Proof of Theorem 1.4. Theorem 1.4 is an immediate consequence of the following strong approximation result:

Theorem 5.1. *Under the conditions of Theorem 1.4, on a suitable probability space one can reconstruct A_μ, together with a Brownian motion, $\widetilde{B}$, such that for each $\varepsilon > 0$, almost surely,*

$$\Big| A_\mu\big(\gamma^{-1}(\sqrt{n})\big) - \widetilde{B}(n) \Big| = O\big(n^{1/2} (\log n)^{(-1/4)+\varepsilon}\big).$$

Proof. Since $\mu(e^t)$ is regularly varying of index $\alpha > -1/2$, by Karamata's Tauberian theorem (cf. Bingham et al. (1987, p.26)), as $t \to \infty$, we have that, $\gamma(t) \sim (2\alpha + 1)^{-1/2} |\mu(t)| \sqrt{\log t}$. Hence, (cf. Bingham et al. (1987, p.28)),

$$|\mu\big(\gamma^{-1}(t)\big)| \big(\log \gamma^{-1}(t)\big)^{(1/4)+\varepsilon} = O\big(t (\log t)^{-(1/4)+\varepsilon}\big), \tag{5.3}$$

for any $\varepsilon > 0$. Define $\widetilde{B}$ by,

$$\widetilde{B}\big(\gamma^2(t)\big) = \int_0^{\log t} \mu(e^s) dB^*(s),$$

where B^* is the Brownian motion of Theorem 1.2. Evidently, $\widetilde{B}$ is a Brownian motion. By Theorem 1.2, the modulus of continuity of $\widetilde{B}$ (cf. Csörgő and Révész (1991, p.90)) and the assumed properties of μ, for all $\varepsilon > 0$,

$$\left| A_\mu(t) - \sqrt{\frac{2\log 2}{\sigma^2 \pi}} \widetilde{B}(\gamma^2(t)) \right| = O\big(|\mu(t)|(\log t)^{(1/4)+\varepsilon}\big), \qquad \text{a.s .}$$

The result follows from changing variables, together with (5.3). ♢

The Proof of Theorem 1.5. When $\alpha > -1/2$, Theorem 1.5 is trivially a consequence of Theorem 1.4. The other cases are treated similar to the proof of Theorem 1.4. ♢

Proof of Theorem 1.6. The proof of Lemma 4.1 goes through with no changes to show that,

$$\text{(5.4)} \qquad \left| \sum_{1 \le k \le n} k^{\theta-(1/2)} I\big\{S_k = x_0\big\} - \int_1^n t^{\theta-(1/2)} dL(t) \right| = o(n^\theta), \qquad \text{a.s .}$$

Let $L_n(t) = n^{-1/2} L(nt)$. By Brownian scaling, L_n has the same distribution as L. In particular,

$$\left\{ \int_{1/n}^t s^{\theta-(1/2)} dL_n(s);\ t \ge 0 \right\} \stackrel{D}{=} \left\{ \int_{1/n}^t s^{\theta-(1/2)} dL(s);\ t \ge 0 \right\}.$$

Changing variables, we see that for all $n \ge 1$ and $t \ge (1/n)$,

$$n^{-\theta} \int_1^{nt} s^{\theta-(1/2)} dL(s) = \int_{1/n}^t s^{\theta-(1/2)} dL_n(s).$$

Since,

$$\lim_{n \to \infty} \int_{1/n}^t s^{\theta-(1/2)} dL(s) = \int_0^t s^{\theta-(1/2)} dL(s), \qquad \text{a.s. },$$

it remains to show that the above integral is a.s. finite and continuous. By (3.1), $\mathbb{E}L(t) = (2t/\pi)^{1/2}$. Integrating by parts, we see that

$$\mathbb{E} \int_0^t s^{\theta-(1/2)} dL(s) = \frac{t^\theta}{(2\pi)^{1/2}\theta} < \infty,$$

which gives the desired finiteness and continuity of the above integral. This and (5.4) together prove the theorem. ♢

STOCHASTIC PROCESSES AND THEIR APPLICATIONS **59**, 105–123.

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Appendix. The Proof of Proposition 1.1

By rescaling and relabelling $\mathbb{L}$, we might as well assume that $\mathbb{L} = \mathbb{Z}$ and that $x_0 = 0$. Moreover, it is easy to see that there is no loss in generality in assuming that F is positive and non-increasing. Define for all $n \geq 1$

$$\begin{aligned} \mathfrak{s}_n &= \sum_{1 \leq k \leq n} F(k) I\big\{S_k = 0\big\}, \\ p_n &= \mathbb{P}(S_n = 0). \end{aligned}$$

Evidently, $\mathbb{E}\mathfrak{s}_n = \sum_{1 \leq k \leq n} F(k) p_k$. By the local central limit theorem (cf. Révész (1990)), as $k \to \infty$, $p_k \sim (2\pi k)^{-1/2}$. Hence there exists some $c > 1$, such that for all $n \geq 1$,

$$c^{-1} \mathcal{S}_n(F) \leq \mathbb{E}\mathfrak{s}_n \leq c\, \mathcal{S}_n(F), \tag{A.1}$$

where $\mathcal{S}_n(F) = \sum_{1 \leq k \leq n} k^{-1/2} F(k)$. By the monotone convergence theorem, $\mathcal{S}(F) < \infty$ implies that $\mathbb{E} \sup_n \mathfrak{s}_n < \infty$ which, in turn, proves the sufficiency. To prove the necessity, assume that $\mathcal{S}(F) = \infty$, i.e., that $\mathcal{S}_n(F) \to \infty$. Since by (A.1), this implies that $\mathbb{E}\mathfrak{s}_n \to \infty$, Kolmogorov's 0–1 law together with a standard second moment argument show that it is sufficient to show the following:

$$\limsup_{n \to \infty} \frac{\mathbb{E}\mathfrak{s}_n^2}{\big(\mathbb{E}\mathfrak{s}_n\big)^2} \leq 2. \tag{A.2}$$

Indeed,

$$\mathbb{E}\mathfrak{s}_n^2 = \sum_{1 \leq k \leq n} F^2(k) p_k + 2 \sum_{k=2}^{n} \sum_{j=1}^{k-1} F(k) F(j) p_{k-j} p_j = T_{1,n} + T_{2,n}.$$

Recall that F is non-increasing, and that as $n \to \infty$, $\mathbb{E}\mathfrak{s}_n \to \infty$. Therefore, $T_{1,n} \leq F(1)\mathbb{E}\mathfrak{s}_n = o\big(\mathbb{E}^2 \mathfrak{s}_n\big)$. Moreover, since F is non-increasing and $\mathbb{E}\mathfrak{s}_n$ is non-decreasing,

$$T_{2,n} = 2 \sum_{j=1}^{n-1} \sum_{k=1}^{n-j} F(k+j) F(j) p_k p_j \leq 2 \sum_{j=1}^{n} \mathbb{E}\mathfrak{s}_{n-j} F(j) p_j \leq 2 \big(\mathbb{E}\mathfrak{s}_n\big)^2.$$

This proves (A.2) and hence Proposition 1.1. ◊

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STOCHASTIC PROCESSES AND THEIR APPLICATIONS **59**, 105–123.

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