SÉMINAIRE DE PROBABILITÉS XXXIV, Lec. Notes in Math., 151-156 (2000)

On Sums of iid Random Variables Indexed by N Parameters*

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Summary. Motivated by the works of J.L. Doob and R. Cairoli, we discuss reverse N–parameter inequalities for sums of i.i.d. random variables indexed by N parameters. As a corollary, we derive Smythe's law of large numbers.

1. Introduction

For any integer $N \geqslant 1$, let us consider $\mathbb{Z}_+^N \triangleq \{1, 2, \cdots\}^N$ and endow it with the following partial order: for all $\mathbf{n}, \mathbf{m} \in \mathbb{Z}_+^N$,

$$\mathbf{n} \preccurlyeq \mathbf{m} \quad \Longleftrightarrow \quad n_i \leqslant m_i, \qquad \text{for all } 1 \leqslant i \leqslant N.$$

Suppose $\{X, X(\mathbf{k});\ \mathbf{k} \in \mathbb{Z}_+^N\}$ is a sequence of independent, identically distributed random variables, indexed by $\mathbb{Z}_+^N$. The corresponding random walk S is given by:

$$S(\mathbf{n}) \triangleq \sum_{\mathbf{k} \preccurlyeq \mathbf{n}} X(\mathbf{k}), \qquad \mathbf{n} \in \mathbb{Z}_+^N.$$

According to Cairoli and Dalang [CD], for all $p > 1$,

$$\begin{aligned} \mathbb{E}\sup_{\mathbf{n}} \Big| \frac{S(\mathbf{n})}{\langle \mathbf{n} \rangle} \Big| < \infty \quad &\Longleftrightarrow \quad \mathbb{E}\big[|X| \big(\log_+ |X| \big)^N \big] < \infty, \\ \mathbb{E}\sup_{\mathbf{n}} \Big| \frac{S(\mathbf{n})}{\langle \mathbf{n} \rangle} \Big|^p < \infty \quad &\Longleftrightarrow \quad \mathbb{E}|X|^p < \infty. \end{aligned} \tag{1.1}$$

Here and throughout, for all $x > 0$,

$$\log_+ x \triangleq \begin{cases} \ln(x), & \text{if } x > e \\ 1, & \text{if } 0 < x \leqslant e \end{cases},$$

* Research partially supported by NSA and NSF

and for all $\mathbf{n} \in \mathbb{Z}_+^N$, $\langle \mathbf{n} \rangle \triangleq \prod_{j=1}^N n_j$. When $N = 1$, this is classical. In this case, J.L. Doob has given a more probabilistic interpretation of this fact by observing that $S(n)/n$ is a reverse martingale; cf. Chung [Ch] for this and more. The goal of this note is to show how a quantitative version of the method of Doob can be carried out, even when $N > 1$. Our approach involves projection arguments which are reminiscent of some old ideas of R. Cairoli; see Cairoli [Ca], Cairoli and Dalang [CD] and Walsh [W].

Perhaps the best way to explain the proposed approach is by demonstrating the following result which may be of independent interest. For related results and a wealth of further references, see [CD], Shorack and Smythe [S1] and Smythe [S2].

Theorem 1. *For all $p > 1$,*

$$\mathbb{E} \sup_{\mathbf{n} \in \mathbb{Z}_+^N} \left| \frac{S(\mathbf{n})}{\langle \mathbf{n} \rangle} \right|^p \leqslant \left(\frac{p}{p-1} \right)^{Np} \mathbb{E}|X|^p. \tag{1.2}$$

Moreover, the corresponding L^1 norm has the following bound:

$$\mathbb{E} \sup_{\mathbf{n} \in \mathbb{Z}_+^N} \left| \frac{S(\mathbf{n})}{\langle \mathbf{n} \rangle} \right| \leqslant \left(\frac{e}{e-1} \right)^N \Big\{ N + \mathbb{E}\big[|X| \big(\log_+ |X| \big)^N \big] \Big\}. \tag{1.3}$$

Theorem 1 implies the "hard" half of both displays in eq. (1.1). The easy half is obtained upon observing that for all $p \geqslant 1$,

$$\mathbb{E} \sup_{\mathbf{n}} \left| \frac{S(\mathbf{n})}{\langle \mathbf{n} \rangle} \right|^p \geqslant 2^{-p} \mathbb{E} \sup_{\mathbf{n}} \left| \frac{X(\mathbf{n})}{\langle \mathbf{n} \rangle} \right|^p,$$

and directly calculating the above.

An enhanced version of Theorem 1 is stated and proved in Section 2. There, we also demonstrate how to use Theorem 1 together with Banach space arguments to obtain the law of large numbers for $S(\mathbf{n})$ due to Smythe [S2].

2. Proof of Theorem 1

I will prove (1.3) of Theorem 1. Eq. (1.2) follows along similar lines. In fact, it turns out to be alot simpler to prove more. Define for all $p \geqslant 0$,

$$\Psi_p(x) \triangleq x \big(\log_+ x \big)^p, \qquad x > 0.$$

I propose to prove the following extension of Theorem 1:

Theorem 1-bis. *For all $p \geqslant 0$,*

$$\mathbb{E} \sup_{\mathbf{n} \in \mathbb{Z}_+^N} \Psi_p \left(\frac{S(\mathbf{n})}{\langle \mathbf{n} \rangle} \right) \leqslant (p+1)^N \left(\frac{e}{e-1} \right)^N \Big\{ N + \mathbb{E} \Psi_{p+N} \big(|X| \big) \Big\}.$$

Setting $p \equiv 0$ in Theorem 1-bis, we arrive at Theorem 1.

Let us recall the following elementary fact:

Lemma 2.1. *Suppose $\{M_n; n \geqslant 1\}$ is a reverse martingale. Then for all $p > 1$,*

$$\mathbb{E} \sup_{n \geqslant 1} |M_n|^p \leqslant \Big(\frac{p}{p-1}\Big)^p \mathbb{E}|M_1|^p. \tag{2.1}$$

For any $p \geqslant 0$,

$$\mathbb{E} \sup_{n \geqslant 1} \Psi_p\big(|M_n|\big) \leqslant (p+1)\Big(\frac{e}{e-1}\Big)\Big\{1 + \mathbb{E}\Psi_{p+1}(|X|)\Big\}. \tag{2.2}$$

Proof. Eq. (2.1) follows from integration by parts and the maximal inequality of Doob. Likewise, one shows that

$$\mathbb{E} \sup_{n \geqslant 1} \Psi_p\big(|M_n|\big) \leqslant \Big(\frac{e}{e-1}\Big)\Big\{1 + \mathbb{E}\Big[\Psi_p(|M_1|) \ln_+ \Psi_p(|M_1|)\Big]\Big\}.$$

For all $x > 0$, $\ln_+ \Psi_p(x) \leqslant \ln_+ x + p \ln_+ \ln_+ x$. Eq. (2.2) follows easily. ◇

Now, each $\mathbf{n} \in \mathbb{Z}_+^N$ can be thought of as $\mathbf{n} = \big(\widehat{\mathbf{n}}, n_N\big)$, where $\widehat{\mathbf{n}}$ is defined by $\widehat{\mathbf{n}} \triangleq \big(n_1, \cdots, n_{N-1}\big) \in \mathbb{Z}_+^{N-1}$. For all $\mathbf{n} \in \mathbb{Z}_+^N$ and all $1 \leqslant j \leqslant n_N$, define

$$Y(\widehat{\mathbf{n}}, j) \triangleq \frac{1}{\prod_{j=1}^{N-1} n_j} \sum_{i_1=1}^{n_1} \cdots \sum_{i_{N-1}=1}^{n_{N-1}} X(\widehat{\mathbf{i}}, j).$$

Clearly,

$$\frac{S(\mathbf{n})}{\langle \mathbf{n} \rangle} = \frac{1}{n_N} \sum_{j=1}^{n_N} Y(\widehat{\mathbf{n}}, j), \qquad \mathbf{n} \in \mathbb{Z}_+^N. \tag{2.3}$$

Let

$$\mathcal{R}(k) \triangleq \sigma\big\{X(\mathbf{m}); m_N > k\big\} \vee \sigma\big\{S(\mathbf{m}); m_N = k\big\}, \qquad k \geqslant 1,$$

where $\sigma\{\cdots\}$ represents the ($\mathbb{P}$–completed) σ–field generated by $\{\cdots\}$.

Lemma 2.2. *$\{\mathcal{R}(k); r \geqslant 1\}$ is a reverse filtration indexed by $\mathbb{Z}_+^1$.*

Proof. This means that $\mathcal{R}(k) \supset \mathcal{R}(k+1)$ — a simple fact. ◇

Lemma 2.3. *For all $\mathbf{n} \in \mathbb{Z}_+^N$,*

$$\frac{S(\mathbf{n})}{\langle \mathbf{n} \rangle} = \mathbb{E}\big[Y(\widehat{\mathbf{n}}, 1) \bigm| \mathcal{R}(n_N)\big].$$

Assuming Lemma 2.3 for the moment, let us prove Theorem 1.

Proof of Theorem 1-bis. Without loss of generality, we can and will assume that

$$\mathbb{E}\Psi_{p+N}(|X|) < \infty. \tag{2.4}$$

Otherwise, there is nothing to prove. When $N = 1$, the result follows immediately from Lemma 2.1. Our proof proceeds by induction over N. Suppose Theorem 1-bis holds for all sums of iid random variables indexed by $\mathbb{Z}_+^{N-1}$ whose incremental distribution is the same as that of X. We will prove it holds for N. By Lemma 2.3,

$$\mathbb{E} \sup_{\mathbf{n}\in\mathbb{Z}_+^N} \Psi_p\Big(\frac{S(\mathbf{n})}{\langle\mathbf{n}\rangle}\Big) \leqslant \mathbb{E} \sup_{k\geqslant 1} \Psi_p\Big(\mathbb{E}\big[W \bigm| \mathcal{R}(k)\big]\Big),$$

where

$$W \triangleq \sup_{n_1,\cdots,n_{N-1}\geqslant 1} \big|Y(\widehat{\mathbf{n}},1)\big|.$$

However, $\big\{Y(\widehat{\mathbf{n}},1); \widehat{\mathbf{n}}\in\mathbb{Z}_+^{N-1}\big\}$ is the average of a random walk indexed by $\mathbb{Z}_+^{N-1}$ with the same increments as S. Therefore, by the induction assumption,

$$\mathbb{E}\Psi_p(W) \leqslant (p+1)^{N-1}\Big(\frac{e}{e-1}\Big)^{N-1}\Big\{N-1+\mathbb{E}\Psi_{p+N}(|X|)\Big\}. \tag{2.5}$$

In particular, $\mathbb{E}W < \infty$. Together with with Lemma 2.1's eq. (2.2), this implies that $M_k \triangleq \mathbb{E}\big[W \bigm| \mathcal{R}(k)\big]$ is a reverse martingale, By eq. (2.2) of Lemma 2.1,

$$\mathbb{E}\Big[\sup_{\mathbf{n}\in\mathbb{Z}_+^N} \Psi_p\Big(\frac{S(\mathbf{n})}{\langle\mathbf{n}\rangle}\Big)\Big] \leqslant (p+1)\Big(\frac{e}{e-1}\Big)\Big\{1+\mathbb{E}\big[\Psi_p(W)\big]\Big\}.$$

Note that $(p+1)e(e-1)^{-1} \geqslant 1$. Therefore, applying (2.5) to this inequality, we obtain Theorem 1-bis. $\diamondsuit$

Proof of Lemma 2.3. Recall (2.3). It remains to show that for $1 \leqslant j \leqslant n_N$,

$$\mathbb{E}\big[Y(\widehat{\mathbf{n}},j) \bigm| \mathcal{R}(n_N)\big] = \mathbb{E}\big[Y(\widehat{\mathbf{n}},1) \bigm| \mathcal{R}(n_N)\big]. \tag{2.6}$$

To this end, we observe that $\big\{Y(\widehat{\mathbf{n}},j); 1\leqslant j\leqslant n_N\big\}$ is a sequence of iid random variables. By exchangeability,

$$\mathbb{E}\big[Y(\widehat{\mathbf{n}},j) \bigm| \mathcal{B}(\mathbf{n})\big] = \mathbb{E}\big[Y(\widehat{\mathbf{n}},1) \bigm| \mathcal{B}(\mathbf{n})\big], \tag{2.7}$$

where for all $\mathbf{n}\in\mathbb{Z}^N$,,

$$\mathcal{B}(\mathbf{n}) \triangleq \sigma\big\{S(\mathbf{k}); \mathbf{k}\in\mathbb{Z}_+^N \text{ with } k_N = n_N \text{ and } k_j \leqslant n_j, \text{ for all } 1\leqslant j\leqslant N-1\big\}.$$

Let $\mathcal{C}_0(n_N)$ denote the sigma-field generated by $\big\{X(\mathbf{k});\ k_N > n_N\big\}$ and define

$$\mathcal{C}(n_N) \triangleq \mathcal{C}_0(n_N) \vee \sigma\big\{X(\mathbf{k});\ k_N = n_N \text{ and for some } 1\leqslant j\leqslant N-1,\ k_j > n_j\big\}.$$

It is easy to see that $\mathcal{B}(\mathbf{n})$ is independent of $\mathcal{C}(n_N)$ and

$$\mathcal{R}(n_N) = \mathcal{C}(n_N) \vee \mathcal{B}(\mathbf{n}). \tag{2.8}$$

Eq. (2.6) follows from (2.7), (2.8) and the elementary fact that the collection $\{Y(\widehat{\mathbf{n}}, j); 1 \leqslant j \leqslant n_N\}$ is independent of $\mathcal{C}(n_N)$. ♢

Open Problem.⋆ Motivated by the proof of Theorem 1-bis — and in the notation of that proof — consider:

$$T(n_N)(\widehat{\mathbf{n}}) \triangleq \frac{1}{n_N} \sum_{j=1}^{n_N} Y(\widehat{\mathbf{n}}, j).$$

It is easy to see that $T(n_N)$ is a reverse martingale which takes its values in the space of all sequences indexed by $\mathbb{Z}_+^{N-1}$. For all $\mathbf{n} \in \mathbb{Z}_+^N$ and any two reals $a < b$, define $U_{a,b}(n_N)(\widehat{\mathbf{n}})$ to be the total number of upcrossings of the interval $[a, b]$ before time n_N of the (real valued) reverse martingale $k \mapsto T(k)(\widehat{\mathbf{n}})$. Is it true that there exist constants C_1 and C_2 (which depend **only** on N) such that

$$\mathbb{E}\Big[\sup_{\widehat{\mathbf{n}} \in \mathbb{Z}_+^{N-1}} U_{a,b}(n_N)(\widehat{\mathbf{n}})\Big] \leqslant C_1 \frac{\mathbb{E}\big[\sup_{\widehat{\mathbf{n}} \in \mathbb{Z}_+^{N-1}} |T(1)(\widehat{\mathbf{n}}) - a|\big]}{(b-a)^{C_2}}? \tag{2.9}$$

Note that when $N = 1$, the supremum is vacuous. In this case, the above holds with $C_1 = C_2 = 1$ and is DOOB's upcrossing inequality for the reversed martingale T. If it holds, (2.9) and Theorem 1 together imply SMYTHE's strong law of large numbers; cf. [S2]. The main part of the aforementioned result is the following:

Theorem 2. ([S2]) *Suppose*

$$\mathbb{E}\Big[|X|\big(\log_+ |X|\big)^{N-1}\Big] < \infty \quad \textit{and} \quad \mathbb{E}X = 0. \tag{2.10}$$

Then almost surely,

$$\lim_{\langle \mathbf{n} \rangle \to \infty} \frac{S(\mathbf{n})}{\langle \mathbf{n} \rangle} = 0.$$

Remark. Classical arguments show that condition (2.10) is necessary as well.

Proof. I will first prove Theorem 2 for $N = 2$. Let c_0 denote the collection of all bounded functions $a : \mathbb{Z}_+^1 \mapsto \mathbb{R}$ such that $\lim_{k \to \infty} |a(k)| = 0$. Topologize c_0 with the supremum norm: $\|a\| \triangleq \sup_k |a(k)|$. Then, c_0 is a separable Banach space. Let

$$\xi_j(k) \triangleq \frac{1}{k} \sum_{i=1}^{k} X(i, j).$$

⋆ **Added Note.** Since this article was accepted for publication, we have found the answer to the open problem above to be affirmative.

Note that ξ_j are i.i.d. random functions from $\mathbb{Z}_+^1$ to $\mathbb{R}$. By Theorem 1, for all $j \geqslant 1$, $\mathbb{E}\|\xi_j\| \leqslant e^2(e-1)^{-2}\big\{2 + \mathbb{E}\big[|X|\log_+|X|\big]\big\} < \infty$. By the classical strong law of large numbers, $\xi_1, \xi_2, \cdots$ are i.i.d. elements of c_0. The most elementary law of large numbers on Banach spaces will show that as elements of c_0, almost surely,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \xi_j = 0.$$

See Ledoux and Talagrand [LT; Corollary 7.10] for this and much more. In other words, almost surely

$$\lim_{n_1\to\infty} \frac{1}{n_1} \sum_{i_1=1}^{n_1} X(i_1, i_2) = 0,$$

uniformly over all $i_2 \geqslant 1$. Plainly, this implies the desired result and much more when $N = 2$. The general case follows by inductive reasoning; the details are omitted. ♢

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