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Bounds on gambler's ruin probabilities in terms of moments

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Abstract. Consider a wager that is more complicated than simply winning or losing the amount of the bet. For example, a pass line bet with double odds is such a wager, as is a bet on video poker using a specified drawing strategy. We are concerned with the probability that, in an independent sequence of identical wagers of this type, the gambler loses L or more betting units (i.e., the gambler is "ruined") before he wins W or more betting units. Using an idea of Markov, Feller established upper and lower bounds on the probability of ruin, bounds that are often very close to each other. However, his formulation depends on finding a positive nontrivial root of the equation $\phi(\rho) = 1$, where ϕ is the probability generating function for the wager in question. Here we give simpler bounds, which rely on the first few moments of the specified wager, thereby making such gambler's ruin probabilities more easily computable.

1. Introduction. Let X be an integer-valued random variable representing the result of a gambling opportunity, in betting units (positive, negative, and zero values correspond respectively to a win, loss, and tie for the gambler). We assume that

$$P\{-\nu \le X \le \mu\} = 1, \qquad P\{X = -\nu\} > 0, \qquad P\{X = \mu\} > 0, \tag{1.1}$$

where μ and ν are positive integers, and that

$$E[X] \neq 0. \tag{1.2}$$

Letting

$$\phi(\rho) := E[\rho^X] \tag{1.3}$$

denote the probability generating function, we note that $\phi(1) = 1$, $\phi'(1) = E[X]$, and, by (1.1), $\phi(\rho) > 1$ for sufficiently small $\rho \in (0, 1)$ and for sufficiently large $\rho \in (1, \infty)$. Since $X(X - 1) \ge 0$, it follows that ϕ is convex on $(0, \infty)$, and so there exists a unique $\rho_0 \in (0, 1) \cup (1, \infty)$ such that

$$\phi(\rho_0) = 1. \tag{1.4}$$

If $E[X] < 0$, then $\rho_0 > 1$. If $E[X] > 0$, then $\rho_0 < 1$.

Now let $X_1, X_2, \ldots$ be independent and identically distributed (i.i.d.) with common distribution that of X, representing the results of repeated independent trials of the given gambling opportunity. Then

$$S_n := X_1 + \cdots + X_n \tag{1.5}$$

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represents the gambler's profit after n such trials. We suppose that the gambler's final bet is on trial

$$N(-L,W) := \min\{n \geq 1 : S_n \leq -L \text{ or } S_n \geq W\}, \tag{1.6}$$

where W and L are positive integers, that is, he stops betting as soon as he wins at least W betting units or loses at least L betting units.

Following Markov (1912) and Uspensky (1937), Feller (1950) obtained bounds on the probability of ruin or, equivalently, the probability of success:

$$\frac{\rho_0^L - 1}{\rho_0^{L+W+\mu-1} - 1} \leq P\{S_{N(-L,W)} \geq W\} \leq \frac{\rho_0^{L+\nu-1} - 1}{\rho_0^{L+W+\nu-1} - 1}. \tag{1.7}$$

(See Feller (1968), Eq. (8.12) of Chapter XIV.) In particular, if $P\{X = 1\} = p > 0$, $P\{X = -1\} = q > 0$, and $P\{X = 0\} = r \geq 0$, where $p + q + r = 1$ and $p \neq q$, then $\rho_0 = q/p$ and $\mu = \nu = 1$, so (1.7) becomes

$$P\{S_{N(-L,W)} = W\} = \frac{(q/p)^L - 1}{(q/p)^{L+W} - 1}, \tag{1.8}$$

which is of course well known.

Uspensky (1937) treated the special case in which $P\{X = -\nu\} + P\{X = \mu\} = 1$. However, his formulation required that the gambler avoid overshooting the boundaries $-L$ and W. This is natural if one assumes, as did Uspensky, that initially the gambler has L units and his opponent has W units. On the other hand, Feller's model can be regarded as that of a gambler (perhaps in a casino) whose goal is to win W or more units before losing L or more units. We prefer the latter approach.

Notice that ρ_0 is a root of a polynomial of degree $\mu + \nu$, so, except in a few special cases, some numerical scheme is usually needed to evaluate ρ_0. Furthermore, when ρ_0 is close to 1, as it frequently is, one may need to use high-precision arithmetic to prevent serious roundoff error.

Our aim in this paper is to find bounds on the success probability $P\{S_{N(-L,W)} \geq W\}$, expressible solely in terms of the first four moments of X (and of course W, L, μ, and ν), that are often nearly as accurate as those in (1.7) and much easier to compute. We also bound $E[N(-L,W)]$, the expected duration of the session, in a similar way.

In addition, we relax Feller's assumptions a bit, still requiring that X be bounded, but no longer requiring that X be integer-valued or even discrete. The point is that, if a game has fractional payoffs (see Section 4 for examples of this), one should not be forced to rescale the basic monetary unit in order to apply these results.

Our original goal was to estimate the error in an approximation to the gambler's ruin formula due to Griffin (1981). That approximation can be described as follows. Given X as above, consider a random variable Y with

$$P\{Y = \alpha\} = p \quad \text{and} \quad P\{Y = -\alpha\} = 1 - p, \tag{1.9}$$

where $\alpha > 0$ and $p \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ are chosen so that

$$E[Y] = E[X] \quad \text{and} \quad E[Y^2] = E[X^2]. \tag{1.10}$$

Specifically, $\alpha(2p-1) = E[X]$ and $\alpha^2 = E[X^2]$, so

$$\alpha = \sqrt{E[X^2]} \quad \text{and} \quad p = \frac{1}{2} + \frac{E[X]}{2\sqrt{E[X^2]}}. \tag{1.11}$$

Let $Y_1, Y_2, \ldots$ be i.i.d. with common distribution that of Y, and put

$$S_n^* = Y_1 + \cdots + Y_n. \tag{1.12}$$

Because of (1.10), S_n and S_n^* should have similar distributions for large n. Therefore, if

$$N^*(-L, W) := \min\{n \geq 1 : S_n^* \leq -L \text{ or } S_n^* \geq W\}, \tag{1.13}$$

we would expect that

$$P\{S_{N(-L,W)} \geq W\} \approx P\{S^*_{N^*(-L,W)} \geq W\} \approx \frac{((1-p)/p)^{L/\alpha} - 1}{((1-p)/p)^{(L+W)/\alpha} - 1}, \tag{1.14}$$

where α and p are as in (1.11) and the second approximation is based on (1.8). (The second approximation could be made an equality by replacing L/α [resp., W/α] by the smallest integer greater than or equal to L/α [resp., W/α], but the given expression seems to be a more accurate approximation to the probability of interest.)

However, the first approximation in (1.14) is difficult to justify, so we take a different approach. Although our formulation lacks the simplicity of (1.14), it makes up for it in accuracy.

2. Bounds on ρ_0.

We replace (1.1) by

$$P\{-\nu \leq X \leq \mu\} = 1, \qquad P\{X < 0\} > 0, \qquad P\{X > 0\} > 0, \tag{2.1}$$

where μ and ν are positive numbers (not necessarily integers). It is no longer assumed that X is integer-valued or even discrete.

Lemma 1. Suppose that X satisfies (2.1) and (1.2), where μ and ν are positive numbers. Let ϕ be the probability generating function of X, i.e.,

$$\phi(\rho) := E[\rho^X], \qquad 0 < \rho < \infty. \tag{2.2}$$

Then there exists a unique $\rho_0 \in (0,1) \cup (1,\infty)$ such that (1.4) holds. If $E[X] < 0$, then $\rho_0 > 1$. If $E[X] > 0$, then $\rho_0 < 1$.

Proof. If X assumes no values in the interval $(0,1)$, then ϕ is convex and the argument of Section 1 works. If general, ϕ need not be convex, but the moment generating function M, defined by $M(t) = \phi(e^t)$, is always convex. Noting that $M(0) = 1$, $M'(0) = E[X]$, and, by (2.1), $M(t) > 1$ for $|t|$ sufficiently large, we see that there exists a unique $t_0 \neq 0$ such that $M(t_0) = 1$. Moreover, $t_0 > 0$ if $E[X] < 0$ and $t_0 < 0$ if $E[X] > 0$. We obtain the stated conclusions with $\rho_0 = e^{t_0}$. $\diamondsuit$

In this section we find upper and lower bounds on ρ_0 in terms of the first four moments of X, which we denote by

$$m_k := E[X^k], \qquad k = 1, 2, 3, 4. \tag{2.3}$$

By Taylor's theorem with integral remainder term, we have for all real x

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4\zeta(x), \tag{2.4}$$

where

$$\zeta(x) := 4\int_0^1 (1-t)^3 e^{tx}\,dt \in [e^x \wedge 1, e^x \vee 1]. \tag{2.5}$$

It follows that, for all $\rho > 0$,

$$\begin{aligned}\phi(\rho) = E[e^{(\log\rho)X}] = 1 + (\log\rho)m_1 + \frac{1}{2}(\log\rho)^2 m_2 + \frac{1}{6}(\log\rho)^3 m_3 \\ + \frac{1}{24}(\log\rho)^4 E[X^4\zeta((\log\rho)X)].\end{aligned} \tag{2.6}$$

Now suppose that $\rho_1 > 0$ is such that $\log\rho_1$ satisfies the quadratic equation

$$m_1 + \frac{1}{2}(\log\rho_1)m_2 + \frac{1}{6}(\log\rho_1)^2 m_3 = 0, \tag{2.7}$$

which is the case if

$$\rho_1 := \exp\Big\{-3\Big(m_2 - \sqrt{(m_2)^2 - (8/3)m_1m_3}\Big)\Big/(2m_3)\Big\} \tag{2.8}$$

and if $m_1m_3 \le 3m_2^2/8$. Then $\log\rho_1 \ne 0$ since $m_1 \ne 0$, so

$$\phi(\rho_1) = 1 + \frac{1}{24}(\log\rho_1)^4 E[X^4\zeta((\log\rho_1)X)] > 1. \tag{2.9}$$

Also, by (2.8) and regardless of the sign of m_3, $\rho_1 > 1$ if $m_1 < 0$ and $\rho_1 < 1$ if $m_1 > 0$. It follows from (2.9) that $1 < \rho_0 < \rho_1$ if $m_1 < 0$ and $\rho_1 < \rho_0 < 1$ if $m_1 > 0$.

Of course, in the rare situation in which $m_3 = 0$, we can replace (2.8) by $\rho_1 := e^{-2m_1/m_2}$, and the inequality (2.9) as well as the conclusions following it will remain true. In fact, even when $m_3 \ne 0$,

$$\rho_0 \approx e^{-2m_1/m_2} \tag{2.10}$$

is an elegantly simple, albeit somewhat crude, approximation. An alternative approximation based on two moments is

$$\rho_0 \approx \left(\frac{1 - m_1/\sqrt{m_2}}{1 + m_1/\sqrt{m_2}}\right)^{1/\sqrt{m_2}}, \tag{2.11}$$

which can be inferred from (1.14) and (1.11). (This has the advantage over (2.10) of being exact in the case of an even-money proposition.) Sileo (1992) observed that (2.10) follows from (2.11) and the fact that $\log(1+x) \approx x$ for small x. See Kozek (1995) and Canjar (2000) for further discussion of (2.10) and its relative in which m_2 is replaced by the variance.

Next, let

$$\gamma := m_4(\log\rho_1)^2/6 \tag{2.12}$$

and suppose that $\rho_2 > 0$ is such that $\log\rho_2$ satisfies the quadratic equation

$$m_1 + \frac{1}{2}(\log\rho_2)m_2 + \frac{1}{6}(\log\rho_2)^2 m_3 = -\frac{1}{2}(\log\rho_2)\gamma, \tag{2.13}$$

which is the case if

$$\rho_2 := \exp\Big\{-3\Big(m_2+\gamma-\sqrt{(m_2+\gamma)^2-(8/3)m_1m_3}\Big)\Big/(2m_3)\Big\}. \tag{2.14}$$

(Of course, $\rho_2 := e^{-2m_1/(m_2+\gamma)}$ if $m_3 = 0$.) Then, by (2.6),

$$\phi(\rho_2) = 1 - \frac{1}{12}m_4(\log\rho_1)^2(\log\rho_2)^2 + \frac{1}{24}(\log\rho_2)^4 E[X^4\zeta((\log\rho_2)X)] < 1, \tag{2.15}$$

provided $\zeta((\log\rho_2)X) < 2$ with probability 1, which by (2.1) is the case if $\rho_2^{-\nu}\vee\rho_2^{\mu} < 2$. Here we are using the facts that, by (2.8) and (2.14) and regardless of the sign of m_3, $1 < \rho_2 < \rho_1$ if $m_1 < 0$ and $\rho_1 < \rho_2 < 1$ if $m_1 > 0$, with the result that, in either case, $0 < (\log\rho_2)^2 < (\log\rho_1)^2$. It follows from (2.15) that $1 < \rho_2 < \rho_0$ if $m_1 < 0$ and $\rho_0 < \rho_2 < 1$ if $m_1 > 0$.

We summarize the main conclusions of this section in the form of a lemma.

Lemma 2. With the assumptions and notation of Lemma 1, suppose also that $m_1m_3 \le 3m_2^2/8$, where m_k is as in (2.3). Define ρ_1 by (2.8) and ρ_2 by (2.14) with γ as in (2.12). (If $m_3 = 0$, then $\rho_1 := e^{-2m_1/m_2}$ and $\rho_2 := e^{-2m_1/(m_2+\gamma)}$.) Assume that $\rho_2^{-\nu}\vee\rho_2^{\mu} < 2$.

If $m_1 < 0$, then $1 < \rho_2 < \rho_0 < \rho_1$. If $m_1 > 0$, then $\rho_1 < \rho_0 < \rho_2 < 1$.

Remark. Part of this lemma was independently discovered by Canjar (2000). In our notation, he showed that $\rho_1 < \rho_0 < 1$ if $m_1 > 0$ and $m_1m_3 \le 3m_2^2/8$.

3. Bounds on success probability and expected duration. Given positive numbers W and L (not necessarily integers), the function

$$f_{L,W}(\rho) := \frac{\rho^L - 1}{\rho^{L+W} - 1} \tag{3.1}$$

is strictly decreasing on $(0,1)$ as well as on $(1,\infty)$, because the inequality $f'_{L,W}(\rho) < 0$ is equivalent to

$$1 < \frac{L}{L+W}\rho^{-W} + \frac{W}{L+W}\rho^L, \tag{3.2}$$

which holds for all $\rho \in (0,1) \cup (1,\infty)$ by the strict convexity of the function $h(x) := \rho^x$. (See Kozek (1995) for an alternative argument.)

Before stating the main result, we need to introduce some new parameters. Let X satisfy (2.1) and (1.2). Let $X_1, X_2, \ldots$ be i.i.d. with common distribution that of X, define S_n for $n = 1, 2, \ldots$ by (1.5), and define $N(-L, W)$ by (1.6). Let $W^* \geq W$ and $L^* \geq L$ be positive numbers satisfying

$$P\{S_{N(-L,W)} \in [-L^*, -L] \cup [W, W^*]\} = 1. \tag{3.3}$$

W^* and L^* are the maximum win and loss amounts, taking overshoot into account. They should be chosen as close to W and L, respectively, as possible. To determine their values, let $\varepsilon \in [0, \mu \wedge W]$ and $\delta \in [0, \nu \wedge L]$ be chosen as large as possible so that

$$P\{S_n \in [-(L-\delta), W-\varepsilon] \text{ whenever } 1 \leq n < N(-L, W)\} = 1. \tag{3.4}$$

In words, ε and δ measure how close the gambler's accumulated profit can get to W and $-L$ without actually achieving these values. We can then take

$$L^* = L + \nu - \delta, \qquad W^* = W + \mu - \varepsilon. \tag{3.5}$$

If W and L are integers and X is integer-valued, then (3.4) holds with $\varepsilon = \delta = 1$, though this may not be the optimal choice for ε and δ (see Example 1 in Section 4).

Theorem. Under the assumptions and notation of Lemmas 1 and 2 and the preceding two paragraphs, we have the following conclusions. If $m_1 < 0$, then

$$f_{L,W^*}(\rho_1) < f_{L,W^*}(\rho_0) \leq P\{S_{N(-L,W)} \geq W\} \leq f_{L^*,W}(\rho_0) < f_{L^*,W}(\rho_2). \tag{3.6}$$

If $m_1 > 0$, then

$$f_{L,W^*}(\rho_2) < f_{L,W^*}(\rho_0) \leq P\{S_{N(-L,W)} \geq W\} \leq f_{L^*,W}(\rho_0) < f_{L^*,W}(\rho_1). \tag{3.7}$$

Remarks. 1. The inner (non-strict) inequalities reduce to Feller's bounds (1.7) if X is integer-valued, W and L are positive integers, and $\varepsilon = \delta = 1$ in (3.5). However, the inner bounds above involving ρ_0 have two advantages over Feller's bounds. First, they do not require that X be integer-valued or that W and L be integers, and second, they are *scale invariant*. This means that, if X, W, L, W^*, and L^* are all multiplied by the same positive constant, which merely amounts to a change in the monetary unit, the bounds remain unchanged. Of course, the same is true of the outer bounds involving ρ_1 and ρ_2.

2. We allow the possibility that $P\{X = 0\} > 0$. Some authors, however, prefer to regard a tie as a momentary delay in the resolution of the bet, in effect replacing the distribution of X by the conditional distribution of X given $X \neq 0$. Whether or not one adopts this practice, the constants ρ_0, ρ_1, and ρ_2, as well as the bounds in (3.6) and (3.7), are unaffected.

3. Suppose $m_1 < 0$ and the gambler has no loss limit and unlimited wealth or credit. Then $1 < \rho_2 < \rho_0 < \rho_1$, so we can let $L \to \infty$ in (3.6) to obtain

$$\rho_1^{-W^*} < \rho_0^{-W^*} \le P\{S_{N(-\infty,W)} \ge W\} \le \rho_0^{-W} < \rho_2^{-W}, \tag{3.8}$$

where $N(-\infty, W)$ is as in (1.6) with $\inf \emptyset = \infty$, and $S_\infty = -\infty$.

4. Suppose $m_1 > 0$ and the gambler has no win limit and an adversary with unlimited wealth or credit. Then $\rho_1 < \rho_0 < \rho_2 < 1$, so we can let $W \to \infty$ in (3.7) to obtain

$$\rho_1^{L^*} < \rho_0^{L^*} \le P\{S_{N(-L,\infty)} \le -L\} \le \rho_0^L < \rho_2^L, \tag{3.9}$$

where $N(-L, \infty)$ is as in (1.6) with $\inf \emptyset = \infty$, and $S_\infty = \infty$. It is often the case that $L^* = L$, and in such situations the bounds in (3.9) reduce to

$$\rho_1^L < P\{S_{N(-L,\infty)} = -L\} = \rho_0^L < \rho_2^L. \tag{3.10}$$

The equality in (3.10) is well known and is related to the formula for the extinction probability of a supercritical Galton–Watson branching process. It was recently rediscovered by Evgeny Sorokin and posted on the Internet (see Dunbar and B. (1999)).

5. The results of the preceding paragraph can be used by casino gamblers playing favorable games. If a gambler wants to ensure that the probability that he goes broke while betting at a constant rate is at most α, then it is sufficient that his bank-to-bet ratio L satisfy $\rho_0^L \le \alpha$, or

$$L \ge \frac{\log \alpha}{\log \rho_0}. \tag{3.11}$$

The slightly stronger condition $L \ge \log \alpha / \log \rho_2$ is of course also sufficient.

Proof of Theorem. The outer inequalities in (3.6) and (3.7) are immediate from Lemma 2 and the monotonicity in (3.1).

As for the inner inequalities, notice that $\{\rho_0^{S_n},\ n \ge 0\}$ is a martingale, hence by optional stopping

$$E[\rho_0^{S_{N(-L,W)}}] = 1. \tag{3.12}$$

Because of (3.3), (3.12) implies that, if $m_1 < 0$ (and hence $\rho_0 > 1$), then

$$\begin{aligned}\rho_0^{-L^*}(1 - P\{S_{N(-L,W)} \ge W\}) + \rho_0^W P\{S_{N(-L,W)} \ge W\}\\ \le 1 \le \rho_0^{-L}(1 - P\{S_{N(-L,W)} \ge W\}) + \rho_0^{W^*} P\{S_{N(-L,W)} \ge W\},\end{aligned} \tag{3.13}$$

while, if $m_1 > 0$ (and hence $\rho_0 < 1$), then the opposite inequalities hold in (3.13). In any case, we obtain (3.6) and (3.7). ◇

Next, using the notation of (3.1), we define

$$g_1(\rho) = m_1^{-1}\big((L^* + W) f_{L,W^*}(\rho) - L^*\big) \tag{3.14}$$

and

$$g_2(\rho) = m_1^{-1}\big((L + W^*) f_{L^*,W}(\rho) - L\big). \tag{3.15}$$

Corollary. Under the notation and assumptions of the theorem and the preceding paragraph, we have the following conclusions. If $m_1 < 0$, then

$$g_2(\rho_2) < g_2(\rho_0) \le E[N(-L,W)] \le g_1(\rho_0) < g_1(\rho_1). \tag{3.16}$$

If $m_1 > 0$, then

$$g_1(\rho_2) < g_1(\rho_0) \le E[N(-L,W)] \le g_2(\rho_0) < g_2(\rho_1). \tag{3.17}$$

Remarks. 1. These bounds, just like those in (3.6) and (3.7), are scale invariant. However, conditioning to eliminate ties effectively multiplies the bounds in (3.16) and (3.17) by $P\{X \ne 0\}$.

2. Limiting cases include the following. If $m_1 < 0$, then

$$|m_1|^{-1}L \le E[N(-L,\infty)] \le |m_1|^{-1}L^*. \tag{3.18}$$

If $m_1 > 0$, then

$$m_1^{-1}W \le E[N(-\infty,W)] \le m_1^{-1}W^*. \tag{3.19}$$

3. In the special case in which $P\{X = 1\} = p > 0$, $P\{X = -1\} = q > 0$, and $P\{X = 0\} = r \ge 0$, where $p + q + r = 1$ and $p \ne q$, and W and L are positive integers, we have $\rho_0 = q/p$, $W^* = W$, and $L^* = L$, so the inner inequalities in (3.16) and (3.17) become

$$E[N(-L,W)] = (p-q)^{-1}\left((L+W)\frac{(q/p)^L - 1}{(q/p)^{L+W} - 1} - L\right), \tag{3.20}$$

which is of course well known.

Proof of Corollary. Again, the outer inequalities in (3.16) and (3.17) are immediate from Lemma 2 and the monotonicity in (3.1).

As for the inner ones, $\{S_n - nm_1,\ n \ge 0\}$ is a martingale, hence by optional stopping

$$E[S_{N(-L,W)}] = m_1 E[N(-L,W)]. \tag{3.21}$$

(Actually, apply the optional stopping theorem with $N(-L,W)\wedge n$ in place of $N(-L,W)$, and then let $n \to \infty$.) Because of (3.3), (3.21) implies that

$$\begin{aligned}-L^*(1 - P\{S_{N(-L,W)} \ge W\}) &+ WP\{S_{N(-L,W)} \ge W\}\\ &\le m_1 E[N(-L,W)]\\ &\qquad \le -L(1 - P\{S_{N(-L,W)} \ge W\}) + W^*P\{S_{N(-L,W)} \ge W\},\end{aligned} \tag{3.22}$$

and the desired conclusions follows from these inequalities and the theorem. $\diamondsuit$

4. Examples. We consider two examples, one with negative expectation and one with positive expectation.

Example 1. *Pass line bet at craps with β-times odds.* Here β is a positive integer specified by the casino, typically 1, 2, 3, 5, 10, 20, or 100. (1-, 2-, and 3-times odds are called single, double, and triple odds.)

First, the gambler makes a one-unit *pass line* bet. A pair of dice is rolled. This is called the *come-out* roll. If a 7 or 11 appears, the bet is won. If a 2, 3, or 12 appears, the bet is lost. If any other number appears (4,5,6,8,9,10), that number becomes the gambler's *point*. The dice are then rolled repeatedly until either the point is repeated (the point is "made"), in which case the bet is won, or a 7 appears (the point is "missed"), in which case the bet is lost. The pass line bet pays even money.

Second, if a point is established on the come-out roll, the gambler makes an additional β-unit *odds* bet (so-named because it pays fair odds). This is in effect a side bet that the as-yet-unresolved pass line bet will be won. The odds bet pays 2-to-1 if the point is 4 or 10; 3-to-2 if the point is 5 or 9; 6-to-5 if the point is 6 or 8.

The case $\beta = 1$ is summarized in Table 1.

With (Y, Z) as in Table 1, the result of a one-unit pass line bet with β-times odds can be represented by

$$X := Y + \beta Z, \tag{4.1}$$

which has moments

$$m_1 = -\frac{7}{495} \tag{4.2i}$$

$$m_2 = 1 + \frac{784}{495}\beta + \beta^2 \tag{4.2ii}$$

$$m_3 = -\frac{7}{495} + \frac{103}{165}\beta^2 + \frac{17}{30}\beta^3 \tag{4.2iii}$$

$$m_4 = 1 + \frac{1568}{495}\beta + 6\beta^2 + \frac{538}{99}\beta^3 + \frac{599}{300}\beta^4. \tag{4.2iv}$$

Table 1

The joint distribution of profit Y from a one-unit pass line bet and profit Z from an associated one-unit odds bet

result	Y	Z	probability	probability $\times$ 990
7 or 11	1	0	$\frac{8}{36}$	220
2, 3, or 12	-1	0	$\frac{4}{36}$	110
point 4 or 10; make point	1	2	$\frac{6}{36}\frac{3}{9}$	55
point 5 or 9; make point	1	$\frac{3}{2}$	$\frac{8}{36}\frac{4}{10}$	88
point 6 or 8; make point	1	$\frac{6}{5}$	$\frac{10}{36}\frac{5}{11}$	125
point 4, 5, 6, 8, 9, or 10; miss point	-1	-1	$\frac{6}{36}\frac{6}{9} + \frac{8}{36}\frac{6}{10} + \frac{10}{36}\frac{6}{11}$	392

We would now like to illustrate our results with some numerical computations. Consider three gamblers, A, B, and C, who each start with \$50,000 and decide to play until they either double it or lose it. Gambler A bets strictly on the pass line without taking odds. Gambler B bets the pass line and takes double odds. Gambler C bets the pass line and takes 100-times odds. Suppose C bets \$1 on each come-out roll and backs up each point with \$100 in odds. Since a point is established 2/3 of the time, the expected amount bet by C per decision is \$1 + (2/3)\$100 = \$67.67. Keeping in mind the fact that the larger one's bets are in a subfair game, the better one's chances are of reaching a goal, we ensure a fair comparison by arranging a similar average bet size for the other two players. We assume that A bets \$70 on each come-out while B bets \$30 on each come-out with \$60 in odds on each point (\$30 + (2/3)\$60 = \$70). What is the probability of success for each player?

For gambler A it can be evaluated exactly using (1.8). Here $p = 244/495$, $q = 1 - p$, and 1 unit is \$70, so $W = L = 715$. (We allow overshoot, meaning in particular that gambler A is prepared to risk \$50,050.) We can also evaluate the moment bounds, taking $W^* = L^* = 715$ as well (overshoot is nonrandom). We find that

$$1.64600 \times 10^{-9} < P(\text{A succeeds}) = 1.64822 \times 10^{-9} < 1.65044 \times 10^{-9}, \tag{4.3}$$

the bounds having been obtained from (3.6), so the relative error is at most 0.135 percent. (Of course, in this case the bounds would not ordinarily be evaluated.)

For gambler B we take 1 unit to be \$30, so X is as in (4.1) with $\beta = 2$. Notice the possibility of a fractional payoff. With $W = L = 5000/3$, $\mu = 5$, $\nu = 3$, and $\varepsilon = \delta = 1/15$ in (3.5), we deduce from (3.6) that

$$0.00307093 < 0.00307116 \leq P(\text{B succeeds}) \leq 0.00312412 < 0.00312435. \tag{4.4}$$

(We used Newton's method to evaluate ρ_0.) The outer probabilities are our moment bounds (relative error at most 1.7397 percent) and the inner ones are essentially Feller's bounds (relative error at most 1.7245 percent). Most of the inaccuracy in our moment bounds is already found in Feller's bounds and is due to overshoot. Of course, we could have taken 1 unit to be \$1, in which case X is 30 times that in (4.1) with $\beta = 2$. Here $W = L = 50{,}000$, $\mu = 150$, $\nu = 90$, and $\varepsilon = \delta = 2$ in (3.5). The results are identical to (4.4) by virtue of the scale invariance.

For gambler C we take 1 unit to be \$1, so X is as in (4.1) with $\beta = 100$. With $W = L = 50{,}000$, $\mu = 201$, $\nu = 101$, and $\varepsilon = \delta = 1$ in (3.5), we deduce from (3.6) that

$$0.464195 \leq P(\text{C succeeds}) \leq 0.465722. \tag{4.5}$$

Our moment bounds differ from Feller's bounds only in the tenth significant digit and beyond. The relative error is at most 0.329 percent and is due almost entirely to overshoot.

The results (4.3)–(4.5), which first appeared informally in the *Las Vegas Advisor* (October 1996), tell us something remarkable about the effect of odds bets in craps. We leave it as an exercise for the interested reader to determine which gambler finishes earliest on average (see the corollary).

We mention in passing that we could have just as well considered the don't pass bet with β-times odds. This bet is exactly the opposite of the pass line bet with β-times odds (i.e., the criteria for winning and losing are reversed), except that a 12 on the come-out roll results in a tie instead of a win for the gambler. Also, the odds bet can be made not just for β units, but for an amount *sufficient to win* β units. An unusual feature of this bet is that $m_3 < 0$ for all $\beta \geq 0$, that is, its distribution is skewed to the left. Gamblers do not usually like to make bets in which the potential loss exceeds the potential win, which may help to explain why this bet is relatively unpopular.

Example 2. *Video poker, "Deuces Wild."* In video poker, the player inserts several coins into the machine and receives 5 cards, with each of the $\binom{52}{5}$ hands equally likely. He then has the option of holding or discarding each of his 5 cards (2^5 ways to play the hand). If he discards k of the 5 initial cards, he draws k additional cards, with each of the $\binom{47}{k}$ possibilities equally likely. He then receives a certain specified return based on the number of coins played and the rank of his final hand. Surprisingly, several versions of this game offer a positive expectation to the player with a nearly optimal drawing strategy.

Once a drawing strategy is fully specified, the probabilities in such a game can in theory be computed exactly, with $\binom{52}{5}\binom{47}{5}$ being a common denominator. In practice, however, they are usually presented as rounded decimal expansions. Consequently, it is important to determine whether the ruin probabilities are robust under small perturbations in the probabilities with which the values of X are assumed.

With this in mind, suppose that $x_0 < x_1 < \cdots < x_l$ are the values of X and that

$$p_i := P\{X = x_i\}, \qquad i = 0, 1, \ldots, l, \tag{4.6}$$

are the exact but unknown probabilities determining the distribution of X. Suppose further that we can bound these probabilities (except p_0) above and below by known probabilities:

$$0 \leq p_i^- \leq p_i \leq p_i^+, \qquad i = 1, \ldots, l, \tag{4.7}$$

where $\sum_{i=1}^l p_i^+ \leq 1$. We define $p_0^- = 1 - \sum_{i=1}^l p_i^-$ and $p_0^+ = 1 - \sum_{i=1}^l p_i^+$, and let X^- and X^+ satisfy $P\{X^- = x_i\} = p_i^-$ and $P\{X^+ = x_i\} = p_i^+$ for $i = 0, 1, \ldots, l$. Let $\{X_j^-\}$ and $\{X_j^+\}$ be i.i.d. sequences distributed as X^- and X^+, respectively. Let S_n^- and S_n^+ be as in (1.5) and $N^-(-L, W)$ and $N^+(-L, W)$ be as in (1.6), with W and L being positive numbers. The following lemma shows that the success probability based on X lies between the success probability based on X^- and the one based on X^+.

Lemma 3. Under the assumptions of the preceding paragraph,

$$P\{S^-_{N^-(-L,W)} \geq W\} \leq P\{S_{N(-L,W)} \geq W\} \leq P\{S^+_{N^+(-L,W)} \geq W\}. \tag{4.8}$$

Proof. The result is intuitively clear, but a proof requires a coupling argument. Define (X^-, X, X^+) by

$$(X^-, X, X^+) = \begin{cases} (x_i, x_i, x_i) & \text{with probability } p_i^-, \quad i = 1, \ldots, l, \\ (x_0, x_i, x_i) & \text{with probability } p_i - p_i^-, \quad i = 1, \ldots, l, \\ (x_0, x_0, x_i) & \text{with probability } p_i^+ - p_i, \quad i = 1, \ldots, l, \\ (x_0, x_0, x_0) & \text{with probability } p_0^+. \end{cases} \tag{4.9}$$

Then $P\{X^- = x_i\} = p_i^-$, $P\{X = x_i\} = p_i$, and $P\{X^+ = x_i\} = p_i^+$ for $i = 1, \ldots, l$, hence also for $i = 0$, and

$$P\{X^- \leq X \leq X^+\} = 1. \tag{4.10}$$

This leads to

$$P\{S_n^- \leq S_n \leq S_n^+ \text{ for all } n \geq 1\} = 1, \tag{4.11}$$

from which (4.8) follows. $\diamondsuit$

Now let us consider a specific video poker game known as "Deuces Wild." Ruin probabilities for this game were studied by Dunbar and B. (1999) and by Canjar (2000). As the name of the game suggests, each of the four 2's is a wild card. The payoff structure is listed in Table 2, as are the probabilities of the various payoffs, assuming the optimal drawing strategy, which is too complicated to describe here. The probabilities, or actually their reciprocals, come from Panamint Software's *Video Poker Tutor*, and it appears that they have been rounded to five significant digits. We assume that to be the case.

The player's profit from a one-unit bet is

$$X := Y - 1 \tag{4.12}$$

with Y as in Table 2. (Typically, to qualify for the 800-for-1 payoff listed in Table 2, one must bet five coins, so 1 unit is 5 times the value of one coin.) Let us apply Lemma 3 to this game. Here $l = 9$ (the straight and flush probabilities are combined) and the bounding probabilities in (4.7) can be determined from Table 2 as follows:

$$p_1^- = \frac{1}{3.51275}, \quad p_1^+ = \frac{1}{3.51265}, \quad \ldots, \quad p_9^- = \frac{1}{45281.5}, \quad p_9^+ = \frac{1}{45280.5}. \tag{4.13}$$

We observe first that this is a positive-expectation game:

$$0.00760848 = E[X^-] \leq E[X] \leq E[X^+] = 0.00765100. \tag{4.14}$$

¿From (3.10) and Lemma 3 we find that

$$(\rho_1^+)^L < (\rho_0^+)^L \leq P\{S_{N(-L,\infty)} = -L\} \leq (\rho_0^-)^L < (\rho_2^-)^L \tag{4.15}$$

for all positive integers L, where the plus and minus superscripts refer to the use of X^+ and X^- as the underlying random variables. In particular, with $L = 1$, this reduces to

$$\rho_1^+ < \rho_0^+ \leq \rho_0 \leq \rho_0^- < \rho_2^-, \tag{4.16}$$

which in the case of our example becomes

$$0.999334094 < 0.999343595 \leq \rho_0 \leq 0.999347772 < 0.999359827. \tag{4.17}$$

The maximum relative error from the inner bounds (i.e., the error due to roundoff in Table 2) is 0.640 percent. The maximum relative error from the outer bounds (i.e., the error from both roundoff in Table 2 and the use of the moment bounds of Lemma 2) is 4.02

percent. Evidently, the highly skewed nature of this example makes the moment bounds less accurate than they are in Example 1.

Table 2

The distribution of return Y from a one-unit bet on the video poker game "Deuces Wild." Assumes optimal drawing strategy.

result	Y	rounded reciprocal of probability
natural royal flush	800	45281.
four deuces	200	4909.1
wild royal flush	25	556.84
five of a kind	15	312.34
straight flush	9	240.01
four of a kind	5	15.399
full house	3	47.105
flush	2	59.516
straight	2	17.836
three of a kind	1	3.5127
other	0	1.8285

If we use (3.11) to determine the necessary bankroll to achieve a 5 percent probability of ruin, the inner bounds in (4.17) tell us that we would need between 4562.3 and 4591.6 betting units. The outer bounds in (4.17) put the figure between 4497.2 and 4678.1 betting units.

Finally, as first pointed out by Dunbar and B. (1999), we can also take into account "cash back," an incentive offered by many casinos to the video poker player. Specifically, a certain percentage of the amount bet (typically, less than 1 percent) is returned to the player once it has reached a certain threshhold. Let β denote the fraction of each bet returned to the player. Then the player's profit from a one-unit bet on "Deuces Wild" becomes

$$X := Y - 1 + \beta \tag{4.18}$$

with Y being the instantaneous return from such a bet as in Table 2. This example illustrates why we do not want to restrict X to be an integer-valued random variable (although we could achieve this if necessary by rescaling, since β is rational). The first four moments of X can be expressed in terms of β and the first four moments of $Y - 1$, so the outer bounds in the β-dependent version of (4.15) provide bounds on the ruin probability as explicit functions of β. Note that, even if the exact probabilities in Table 2 were known, the equality in (3.10) would not apply.

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