RECURRENT LINES IN TWO-PARAMETER ISOTROPIC STABLE LÉVY SHEETS

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Abstract. It is well-known that an $\mathbb{R}^d$-valued isotropic α-stable Lévy process is (neighborhood-) recurrent if and only if $d \le \alpha$. Given an $\mathbb{R}^d$-valued two-parameter isotropic α-stable Lévy sheet $\{X(s,t)\}_{s,t\ge 0}$, this is equivalent to saying that for any fixed $s \in [1,2]$, $\mathrm{P}\{t \mapsto X(s,t) \text{ is recurrent}\} = 0$ if $d > \alpha$ and $= 1$ otherwise. We prove here that $\mathrm{P}\{\exists s \in [1,2] : t \mapsto X(s,t) \text{ is recurrent}\} = 0$ if $d > 2\alpha$ and $= 1$ otherwise. Moreover, for $d \in (\alpha, 2\alpha]$, the collection of all times s at which $t \mapsto X(s,t)$ is recurrent is a random set of Hausdorff dimension $2-d/\alpha$ that is dense in $\mathbb{R}_+$, a.s. When $\alpha = 2$, X is the two-parameter Brownian sheet, and our results extend those of M. Fukushima and N. Kôno.

1. Introduction

It is well-known that d-dimensional Brownian motion is (neighborhood-) recurrent if and only if $d \le 2$; cf. Kakutani [Kak44]. Now consider the process $s^{-1/2}B(s,t)$, where B denotes a d-dimensional two-parameter Brownian sheet. It is clear that for each fixed $s > 0$, $t \mapsto s^{-1/2}B(s,t)$ is a Brownian motion in $\mathbb{R}^d$, and it has been shown that in contrast to the theorem of [Kak44]: (i) If $d > 4$, then with probability one, $t \mapsto s^{-1/2}B(s,t)$ is transient simultaneously for all $s > 0$; and (ii) if $d \le 4$, then there a.s. exists $s > 0$ such that $t \mapsto s^{-1/2}B(s,t)$ is recurrent; cf. Fukushima [Fuk84] for the $d \ne 4$ case, and Kôno [Kôn84] for a proof in the critical case $d = 4$. The goal of this article is to present quantitative estimates that, in particular, imply these results in the more general setting of two-parameter stable sheets.

Henceforth, $X := \{X(s,t)\}_{s,t\ge 0}$ denotes a two-parameter isotropic α-stable Lévy sheet in $\mathbb{R}^d$ with index $\alpha \in (0,2]$; cf. Proposition A.1 below. In particular, note that $t \mapsto s^{-1/\alpha}X(s,t)$ is an ordinary (isotropic) α-stable Lévy process in $\mathbb{R}^d$.

According to Theorem 16.2 of Port and Stone [PoS71, p. 181], an isotropic Lévy process in $\mathbb{R}^d$ is recurrent if and only if $d \le \alpha$. Motivated by this, we will be concerned only with the following transience-type condition that we tacitly assume from now on: Unless the contrary is stated explicitly,

$$d > \alpha. \tag{1.1}$$

Our goal is to find when, under the above condition, $t \mapsto s^{-1/\alpha}X(s,t)$ is recurrent *for some* $s > 0$. That is we ask, "*when are there recurrent lines in the sheet* X"?

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Thus, the set of lines of interest is

$$\mathcal{L}_{d,\alpha} := \bigcap_{\varepsilon>0} \bigcap_{n\geq 1} \left\{ s > 0 : \ \exists t \geq n \text{ such that } X(s,t) \in (-\varepsilon,\varepsilon)^d \right\}. \tag{1.2}$$

One of our main results is the following.

Theorem 1.1. *(a) If $d > 2\alpha$, then $\mathcal{L}_{d,\alpha} = \varnothing$, a.s.*

(b) If $d \in (\alpha, 2\alpha]$, then with probability one, $\mathcal{L}_{d,\alpha}$ is everywhere dense and

$$\dim_{\mathcal{H}}(\mathcal{L}_{d,\alpha}) = 2 - \frac{d}{\alpha}, \qquad \textit{almost surely}, \tag{1.3}$$

where $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension.

Remark 1.2. If X is not the Brownian sheet, then $\alpha \in (0,2)$, and the condition "$d \in (\alpha, 2\alpha]$" is nonvacuous if and only if $d = 2$ and $\alpha \in [1,2)$ or $d = 1$ and $\alpha \in [\frac{1}{2}, 1)$.

Remark 1.3. If X denotes the Brownian sheet, then $\alpha = 2$. In addition, Theorem 1.1 implies that $\dim_{\mathcal{H}}(\mathcal{L}_{3,2}) = \frac{1}{2}$. When $d = 2$, since a.s., $t \mapsto X(s,t)$ is recurrent for almost all s, and since one-dimensional Hausdorff measure is also one-dimensional Lebesgue measure, $\dim_{\mathcal{H}}(\mathcal{L}_{2,2}) = 1$. On the other hand, one-dimensional Brownian motion hits all points, and this means that $\dim_{\mathcal{H}}(\mathcal{L}_{1,2}) = 1$. In fact, Theorem 3.2 of Khoshnevisan et al. [KRS03] shows that $\mathcal{L}_{1,2} = [0,\infty)$. Is $\mathcal{L}_{2,2} = [0,\infty)$? Theorem 2.3 of Adelman et al. [ABP98] suggests a negative answer, although we do not have a completely rigorous proof. In the case $\alpha \in (0,2)$, things are more delicate still, and we pose the following *conjecture: If $\alpha > d = 1$, then almost surely, $\mathcal{L}_{1,\alpha} = [0,\infty)$, whereas $\mathcal{L}_{1,1} \neq [0,\infty)$, a.s.*

Remark 1.4. It would be nice to know more about the critical case $d = 2\alpha$. There are only three possibilities here: (i) $\alpha = \frac{1}{2}$ and $d = 1$; (ii) $\alpha = 1$ and $d = 2$; and (iii) the critical Gaussian case, $\alpha = 2$ and $d = 4$. Theorem 1.1 states that in these cases, $\mathcal{L}_{d,\alpha}$ is everywhere dense but has zero Hausdorff dimension.

This paper is organized as follows. In Section 2, we establish first and second moment estimates of certain functionals of the process X. We use these to estimate the probability that the sample paths of the process hit a ball (see Section 3 for the case $d \geq 2\alpha$ and Section 4 for the case $d \in (\alpha, 2\alpha)$). With these results in hand, we give the proof of Theorem 1.1 in Section 5. This proof also uses the Baire category theorem. In Appendix A, we provide basic information regarding isotropic stable sheets and stable noise, and in Appendix B, some simulations of these processes.

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2. Moment estimates

Throughout, $\mathcal{B}_\varepsilon := (-\varepsilon,\varepsilon)^d$, $|x| := \max_{1\leq j\leq d} |x_j|$, $\|x\| := (x_1^2 + \cdots + x_d^2)^{1/2}$, and $\mathcal{P}(F)$ denotes the collection of all probability measures on any given compact set F in any Euclidean space.

Fix $0 < a < b$, $\varepsilon > 0$, and for all $n \geq 1$ and all $\nu \in \mathcal{P}([a,b])$, define

$$
\begin{aligned}
J_n &:= J_n(a,b;\varepsilon;\nu) := \int_a^b \nu(ds) \int_n^\infty dt\, \mathbf{1}_{\mathcal{B}_\varepsilon}(X(s,t)),\\
\bar{J}_n &:= \bar{J}_n(a,b;\varepsilon;\nu) := \int_a^b \nu(ds) \int_n^{2n} dt\, \mathbf{1}_{\mathcal{B}_\varepsilon}(X(s,t)).
\end{aligned}
\tag{2.1}
$$

The above notations also make sense for any finite measure ν on $[a,b]$.

Lemma 2.1. *Given $\eta > 0$ and $\eta < a < b < \eta^{-1}$, there is a positive and finite constant $A_{2.1} = A_{2.1}(\eta,d,\alpha)$ such that for all $s \in [a,b]$, all $t > 0$, and all $\varepsilon \in (0,1)$,*

$$
A_{2.1}^{-1}(\varepsilon(st)^{-1/\alpha} \wedge 1)^d \leq \mathrm{P}\{|X(s,t)| \leq \varepsilon\} \leq A_{2.1}\varepsilon^d t^{-d/\alpha}. \tag{2.2}
$$

Proof. Set

$$
\phi_\alpha(\lambda) := \mathrm{P}\{|X(1,1)| \leq \lambda\}. \tag{2.3}
$$

Recall that the standard symmetric stable density is bounded above thanks to the inversion theorem for Fourier transforms; it is also bounded below on compacts because of Bochner's subordination ([Kho02, Th. 3.2.2, p. 379]). Thus, there exists a constant $C_\star := C_\star(d,\alpha)$ such that for all $\lambda > 0$,

$$
C_\star^{-1}(\lambda \wedge 1)^d \leq \phi_\alpha(\lambda) \leq C_\star(\lambda \wedge 1)^d. \tag{2.4}
$$

It follows that there is $c < \infty$ depending only on d such that

$$
\mathrm{P}\{|X(s,t)| \leq \varepsilon\} = \phi_\alpha(\varepsilon(st)^{-1/\alpha}) \leq C\varepsilon^d(st)^{-d/\alpha} \leq C\eta^{-d/\alpha}\varepsilon^d t^{-d/\alpha}, \tag{2.5}
$$

and the lower bound follows in the same way. □

Lemma 2.2. *If $d > \alpha$, and if $0 < a < b$ are fixed, then there exists a finite constant $A_{2.2} := A_{2.2}(a,b,d,\alpha) > 1$ such that for all $\varepsilon \in (0,1)$, all $\nu \in \mathcal{P}([a,b])$, and for all $n \geq 1/a$,*

$$
A_{2.2}^{-1}\varepsilon^d n^{-(d-\alpha)/\alpha} \leq \mathrm{E}\left[\bar{J}_n\right] \leq \mathrm{E}\left[J_n\right] \leq A_{2.2}\varepsilon^d n^{-(d-\alpha)/\alpha}. \tag{2.6}
$$

Proof. By scaling,

$$
\mathrm{E}\left[J_n\right] = \int_a^b \nu(ds) \int_n^\infty dt\, \phi_\alpha\left(\varepsilon(st)^{-1/\alpha}\right), \tag{2.7}
$$

where ϕ_α is defined in (2.3). The lemma follows readily from this, its analogue for $\bar{J}_n$, and Lemma 2.1. □

Lemma 2.3. *There exists a positive and finite constant $A_{2.3} := A_{2.3}(d,\alpha)$ such that for all $0 < s < s'$, $0 < t < t'$, and all $\varepsilon \in (0,1)$,*

$$
\begin{aligned}
&\sup_{z \in \mathbb{R}^d} \mathrm{P}\left\{|X(s',t') + z| \leq \varepsilon \,\middle|\, |X(s,t) + z| \leq \varepsilon\right\}\\
&\qquad \leq A_{2.3}\left[\frac{\varepsilon^\alpha}{s|t'-t| + t|s'-s|} \wedge 1\right]^{d/\alpha}.
\end{aligned}
\tag{2.8}
$$

Proof. Consider the decomposition $X(s',t') = V_1 + V_2$, where

$$
V_1 = X(s',t') - X(s,t), \qquad V_2 = X(s,t). \tag{2.9}
$$

Equivalently, in terms of the isostable noise $\mathfrak{X}$ introduced in the Appendix, we can write $V_2 = \mathfrak{X}([0,s]\times[0,t])$ and $V_1 = \mathfrak{X}([0,s']\times[0,t']\setminus[0,s]\times[0,t])$. From this, it is clear that V_1 and V_2 are independent, and so we can write

$$\begin{aligned}&\mathrm{P}\{|X(s,t)+z|\le\varepsilon,\ |X(s',t')+z|\le\varepsilon\}\\&\quad=\mathrm{P}\{|V_2+z|\le\varepsilon,\ |V_1+V_2+z|\le\varepsilon\}\\&\quad\le\mathrm{P}\{|V_2+z|\le\varepsilon\}\sup_{w\in\mathbb{R}^d}\mathrm{P}\{|V_1+w|\le\varepsilon\}.\end{aligned}\tag{2.10}$$

Now V_1 is a symmetric stable random vector in $\mathbb{R}^d$. Thus, its distribution is unimodal: indeed, since the characteristic function of V_1 is a non-negative function, f_{V_1} is positive-definite, and therefore $f_{V_1}(0)\ge f_{V_1}(x)$, for all $x\in\mathbb{R}^d$. In other words, we have $\sup_{w\in\mathbb{R}^d}\mathrm{P}\{|V_1+w|\le\varepsilon\}\le C\varepsilon^d f_{V_1}(0)$, where f_{V_1} denotes the probability density function of V_1. Consequently,

$$\sup_{z\in\mathbb{R}^d}\mathrm{P}\left\{|X(s',t')+z|\le\varepsilon\,\middle|\,|X(s,t)+z|\le\varepsilon\right\}\le C\varepsilon^d f_{V_1}(0).\tag{2.11}$$

Thanks to the Fourier inversion formula, the density function of $V_1=\mathfrak{X}([s,s']\times[t,t'])$ can be estimated as follows:

$$f_{V_1}(x)\le f_{V_1}(0)\le(2\pi)^{-d}\int_{\mathbb{R}^d}e^{-\frac12\|\theta\|^\alpha\lambda}\,d\theta=C\lambda^{-d/\alpha},\qquad\text{for all }x\in\mathbb{R}^d,\tag{2.12}$$

where λ is the area of the ℓ^∞-annulus $([0,s']\times[0,t'])\setminus([0,s]\times[0,t])$, and $C:=C(d,\alpha)$ is some nontrivial constant that does not depend on (s,s',t,t',x). It is easy to see that

$$\begin{aligned}\lambda&=s(t'-t)+t(s'-s)+(s'-s)(t'-t)\\&\ge s(t'-t)+t(s'-s).\end{aligned}\tag{2.13}$$

Thus, for all $0<s<s'$, $0<t<t'$, and all $x\in\mathbb{R}^d$,

$$f_{V_1}(0)\le C\big[s(t'-t)+t(s'-s)\big]^{-d/\alpha}.\tag{2.14}$$

Consequently, the lemma follows from (2.11). □

Lemma 2.4. *There exists a positive and finite constant $A_{2.4}:=A_{2.4}(d,\alpha)$ such that for all $0<s'<s$, $0<t<t'$, and all $\varepsilon\in(0,1)$,*

$$\begin{aligned}&\sup_{z\in\mathbb{R}^d}\mathrm{P}\left\{|X(s',t')+z|\le\varepsilon\,\middle|\,|X(s,t)+z|\le\varepsilon\right\}\\&\quad\le A_{2.4}\left(\frac{s}{s'}\right)^{d/\alpha}\left[\frac{\varepsilon^\alpha}{s'|t'-t|+t|s'-s|}\wedge1\right]^{d/\alpha}.\end{aligned}\tag{2.15}$$

Proof. As in our proof of Lemma 2.3, we begin by a decomposition. Namely, write

$$X(s',t')=V_3+V_4,\qquad X(s,t)=V_4+V_5,\tag{2.16}$$

where $V_4=\mathfrak{X}([0,s']\times[0,t])$, $V_3=\mathfrak{X}([0,s']\times[t,t'])$, $V_5=\mathfrak{X}([s',s]\times[0,t])$, and $\mathfrak{X}$ denotes the isotropic noise defined in the appendix. Note that V_3, V_4 and V_5 are mutually independent, and

$$
\begin{aligned}
&\mathrm{P}\{|X(s,t)+z|\le\varepsilon,\ |X(s',t')+z|\le\varepsilon\}\\
&\quad=\mathrm{P}\{|V_3+V_4+z|\le\varepsilon,\ |V_4+V_5+z|\le\varepsilon\}\\
&\quad\le\mathrm{P}\{|V_3-V_5|\le 2\varepsilon,\ |V_3+V_4+z|\le\varepsilon\}\\
&\quad\le\mathrm{P}\{|V_3-V_5|\le 2\varepsilon\}\ \sup_{w\in\mathbb{R}^d}\mathrm{P}\{|w+V_4|\le\varepsilon\}\\
&\quad\le\mathrm{P}\{|V_3-V_5|\le 2\varepsilon\}\cdot(C\varepsilon^d f_{V_4}(0)\wedge 1).
\end{aligned}
\tag{2.17}
$$

Now we proceed to estimate the probability densities of the stable random vectors V_4 and V_3-V_5, respectively. By Fourier inversion, and arguing as we did for (2.12), we can find a nontrivial constant $C:=C(d,\alpha)$ such that for all $s',t>0$,

$$
f_{V_4}(0)\le C(s't)^{-d/\alpha}. \tag{2.18}
$$

Thus, there exists a nontrivial constant $C:=C(d,\alpha)$ such that for all $s',t>0$ and all $\varepsilon\in(0,1)$,

$$
C\varepsilon^d f_{V_4}(0)\wedge 1\le C\left[\frac{\varepsilon^\alpha}{s't}\wedge 1\right]^{d/\alpha}\le\frac{C}{C_\star}\left(\frac{s}{s'}\right)^{d/\alpha}\mathrm{P}\{|X(s,t)|\le\varepsilon\} \tag{2.19}
$$

[the second inequality uses the lower bound in Lemma 2.1].

Similarly,

$$
f_{V_3-V_5}(0)\le C\lambda^{-d/\alpha}, \tag{2.20}
$$

where λ denotes the area of $([s',s]\times[0,t])\cup([0,s']\times[t,t'])$, that is,

$$
\lambda=t(s-s')+s'(t'-t). \tag{2.21}
$$

Using the last three displays in conjunction yields an upper bound on $P\{|V_3-V_5|\le 2\varepsilon\}$ which establishes (2.15). □

The next technical lemma will be used in Lemma 2.6 below and in the next sections.

Lemma 2.5. *Set*

$$
K_\varepsilon^t(v):=\int_0^1\left(\frac{\varepsilon}{t^{1/\alpha}(u+v)^{1/\alpha}}\wedge 1\right)^d du. \tag{2.22}
$$

(a) If $d\in(\alpha,2\alpha]$, then there a is $A_{2.23}:=A_{2.23}(d,\alpha)\in(0,\infty)$ such that for all $\varepsilon>0$, $t>0$ and $v>0$,

$$
K_\varepsilon^t(v)\le A_{2.23}\frac{\varepsilon^d}{t^{d/\alpha}}\,v^{-(d-\alpha)/\alpha}. \tag{2.23}
$$

(b) If $d\in(\alpha,2\alpha]$ and $M\ge 1$, then there is a constant $A_{2.24}=A_{2.24}(d,\alpha,M)\in(0,1]$ such that for all $v\in(0,M]$, $\varepsilon\in(0,1)$, and $t\ge 3$,

$$
K_\varepsilon^t(v)\ge A_{2.24}\frac{\varepsilon^d}{t^{d/\alpha}}v^{-(d-\alpha)/\alpha}\ 1_{[\varepsilon^\alpha/t,\infty)}(v). \tag{2.24}
$$

(c) If $d\ge 2\alpha$ and $M\ge 1$, then there is a $A_{2.25}:=A_{2.25}(d,\alpha,M)\in(0,\infty)$ such that for all $\varepsilon\in(0,1)$, $t\ge 1$ sufficiently large and $b\le M$,

$$
\int_0^b dv\,K_\varepsilon^t(v)\le A_{2.25}\times\begin{cases}\varepsilon^{2\alpha}t^{-2}, & \text{if } d>2\alpha,\\ \varepsilon^{2\alpha}t^{-2}\log\left(t/\varepsilon^\alpha\right), & \text{if } d=2\alpha.\end{cases} \tag{2.25}
$$

(d) If $d > 2\alpha$, then there is a $A_{2.26} := A_{2.26} \in (0, \infty)$ such that for all $\varepsilon \in (0,1)$, $t \geq 1$ and $a > \varepsilon^\alpha/t$,

$$\int_0^a dv\, K_\varepsilon^t(v) \geq A_{2.26}\varepsilon^{2\alpha} t^{-2}. \tag{2.26}$$

Proof. Throughout this proof, we write C for a generic positive and finite constant. Its dependence on the various parameters $d, \alpha, M, \ldots$ is apparent from the context. Otherwise, C may change from line to line.

(a) Observe that

$$\begin{aligned} K_\varepsilon^t(v) &\leq \int_0^1 du\, \frac{\varepsilon^d}{t^{d/\alpha}(u+v)^{d/\alpha}} = C\frac{\varepsilon^d}{t^{d/\alpha}}(u+v)^{1-d/\alpha}\Big|_1^0 \\ &\leq C\frac{\varepsilon^d}{t^{d/\alpha}}\, v^{-(d-\alpha)/\alpha}. \end{aligned} \tag{2.27}$$

(b) If $v \geq \varepsilon^\alpha/t$, then

$$\begin{aligned} K_\varepsilon^t(v) &= \int_v^{v+1} \left(\frac{\varepsilon}{(tu)^{1/\alpha}} \wedge 1\right)^d du = \frac{\varepsilon^d}{t^{d/\alpha}} \int_v^{v+1} u^{-d/\alpha}\, du \\ &= \left(\frac{d-\alpha}{\alpha}\right) \frac{\varepsilon^d}{t^{d/\alpha}} [v^{-(d-\alpha)/\alpha} - (v+1)^{-(d-\alpha)/\alpha}] \\ &= \left(\frac{d-\alpha}{\alpha}\right) \frac{\varepsilon^d}{t^{d/\alpha}}\, v^{-(d-\alpha)/\alpha} \left[1 - (1+1/v)^{-(d-\alpha)/\alpha}\right]. \end{aligned} \tag{2.28}$$

Since $v \leq M$, the expression in brackets is at least $[1 - (1+1/M)^{-(d-\alpha)/\alpha}] > 0$.

(c) Clearly, since $b \leq M$ and $M \geq 1$,

$$\begin{aligned} \int_0^b dv\, K_\varepsilon^t(v) &\leq \int_{\{x\in\mathbb{R}^2:\, \|x\|\leq M\}} dx \left(\frac{\varepsilon}{(ct\|x\|)^{1/\alpha}} \wedge 1\right)^d \\ &= C\int_0^M dr \left(\frac{\varepsilon}{(ctr)^{1/\alpha}} \wedge 1\right)^d \cdot r \\ &\leq C\left(\frac{\varepsilon^{2\alpha}}{c^2t^2} + \frac{\varepsilon^d}{t^{d/\alpha}} \int_{\varepsilon^\alpha/(ct)}^M dr\, r^{1-d/\alpha}\right). \end{aligned} \tag{2.29}$$

If $d/\alpha > 2$, then this is bounded above by

$$C\left(\frac{\varepsilon^{2\alpha}}{t^2} + \frac{\varepsilon^d}{t^{d/\alpha}} \left(\frac{\varepsilon^\alpha}{ct}\right)^{2-(d/\alpha)}\right) = C\frac{\varepsilon^{2\alpha}}{t^2}, \tag{2.30}$$

while if $d/\alpha = 2$, then this is bounded above by

$$C\left(\frac{\varepsilon^{2\alpha}}{t^2} + \frac{\varepsilon^d}{t^{d/\alpha}} (\log M + \log\left(\frac{ct}{\varepsilon^\alpha}\right)\right) = C\frac{\varepsilon^{2\alpha}}{t^2}\left(1 + \log\left(\frac{Mct}{\varepsilon^\alpha}\right)\right). \tag{2.31}$$

(d) Observe that $\varepsilon t^{-1/\alpha}(u+v)^{-1/\alpha} \geq 1$ if and only if $u + v \leq \varepsilon^\alpha/t$, so for $a > \varepsilon^\alpha/t$,

$$\int_0^a dv\, K_\varepsilon^t(v) \geq \int_0^{\varepsilon^\alpha/t} dv \int_0^{(\varepsilon^\alpha/t)-v} du \cdot 1 = \frac{1}{2}\left(\frac{\varepsilon^\alpha}{t}\right)^2. \tag{2.32}$$

This proves the lemma. □

For $\beta > 0$, define the energy $\mathcal{E}_\beta(\nu)$ of a finite measure ν by

$$\mathcal{E}_\beta(\nu) = \iint |x-y|^{-\beta}\,\nu(dx)\,\nu(dy). \tag{2.33}$$

Lemma 2.6. *If $\alpha < d$, then for any $\eta > 0$, and all $\eta < a < b < \eta^{-1}$, there exists a constant $A_{2.6} := A_{2.6}(\eta, d, \alpha)$ such that for any $\varepsilon \in (0,1)$, all $n \geq 1$, and for all $\nu \in \mathcal{P}([a,b])$,*

$$\mathrm{E}[\bar{J}_n^2] \leq \mathrm{E}[J_n^2] \leq A_{2.6}\varepsilon^{2d} n^{-2(d-\alpha)/\alpha} \mathcal{E}_{\frac{d-\alpha}{\alpha}}(\nu), \tag{2.34}$$

where $\bar{J}_n$ and J_n are defined in (2.1).

Proof. Owing to Taylor's theorem [Kho02, Cor. 2.3.1, p. 525], the conclusion of this lemma is nontrivial if and only if $(d-\alpha)/\alpha < 1$, for otherwise, $\mathcal{E}_{(d-\alpha)/\alpha}(\nu) = +\infty$ for all $\nu \in \mathcal{P}([a,b])$. So we assume that $d \in (\alpha, 2\alpha)$.

Since $\bar{J}_n \leq J_n$, we only have to prove one inequality. Write

$$\mathrm{E}[J_n^2] = 2\mathcal{T}_1 + 2\mathcal{T}_2, \tag{2.35}$$

where

$$\begin{aligned} \mathcal{T}_1 &= \int_a^b \nu(ds) \int_n^\infty dt \int_a^b \nu(ds') \int_t^{2t} dt'\, \mathrm{P}\big\{|X(s,t)| \leq \varepsilon,\ |X(s',t')| \leq \varepsilon\big\}, \\ \mathcal{T}_2 &= \int_a^b \nu(ds) \int_n^\infty dt \int_a^b \nu(ds') \int_{2t}^{\infty} dt'\, \mathrm{P}\big\{|X(s,t)| \leq \varepsilon,\ |X(s',t')| \leq \varepsilon\big\}. \end{aligned} \tag{2.36}$$

One might guess that $\mathcal{T}_1$ dominates $\mathcal{T}_2$, since most self-interactions, along the sheet, are local. We shall see that this is indeed so. We begin by first estimating $\mathcal{T}_2$.

Thanks to Lemmas 2.1, 2.3 and 2.4, there exists a positive and finite constant $C := C(\eta, d, \alpha)$ such that for all $\varepsilon \in (0,1)$, for all $n > a$, for any $s, s' \in [a,b]$, and for all $n < t < t'$,

$$\begin{aligned} \mathrm{P}\{|X(s,t)| \leq \varepsilon,\ |X(s',t')| \leq \varepsilon\} &\leq C\varepsilon^d t^{-d/\alpha} \cdot \left[\frac{\varepsilon^\alpha}{a|t'-t| + t|s'-s|} \wedge 1\right]^{d/\alpha} \\ &\leq C a^{-d/\alpha} t^{-d/\alpha} \varepsilon^{2d} \cdot (t'-t)^{-d/\alpha}. \end{aligned} \tag{2.37}$$

Consequently, there exists a positive and finite $C := C(\eta, d, \alpha)$ such that for all $\varepsilon \in (0,1)$, $n \geq 1$, and all $\nu \in \mathcal{P}(a,b])$,

$$\begin{aligned} \mathcal{T}_2 &\leq C\varepsilon^{2d} \int_a^b \nu(ds) \int_n^\infty dt \int_a^b \nu(ds') \int_{2t}^\infty dt'\, t^{-d/\alpha} \cdot (t'-t)^{-d/\alpha} \\ &= C\varepsilon^{2d} \int_n^\infty dt \int_t^\infty dv\, t^{-d/\alpha} v^{-d/\alpha}. \end{aligned} \tag{2.38}$$

In this way, we obtain the existence of a positive and finite constant $C := C(\eta, d, \alpha)$ such that for all $\varepsilon \in (0,1)$, and all $\nu \in \mathcal{P}([a,b])$,

$$\mathcal{T}_2 \leq C\varepsilon^{2d} n^{-2(d-\alpha)/\alpha}. \tag{2.39}$$

In order to estimate $\mathcal{T}_1$, we still use (2.37), but this time things are slightly more delicate. Indeed, equation (2.37) yields a constant $C := C(\eta, d, \alpha)$ such that for all

$\varepsilon \in (0,1)$ and all $\nu \in \mathcal{P}([a,b])$,

$$\begin{aligned} \mathcal{T}_1 \le C\varepsilon^d \int_a^b \nu(ds) \int_n^\infty dt \int_a^b \nu(ds') \int_t^{2t} dt' \\ \times t^{-d/\alpha} \left[\frac{\varepsilon^\alpha}{a|t'-t| + t|s'-s|} \wedge 1 \right]^{d/\alpha}. \end{aligned} \tag{2.40}$$

Do the change of variables $t' - t = tu$ (t fixed) to see that the right-hand side is equal to

$$C\varepsilon^d \int_a^b \nu(ds) \int_a^b \nu(ds') \int_n^\infty dt\, t^{1-(d/\alpha)} K_\varepsilon^{at}(|s'-s|). \tag{2.41}$$

Use Lemma 2.5(a) and evaluate the dt-integral to get the inequality

$$\mathcal{T}_1 \le C\varepsilon^{2d} n^{-2(d-\alpha)/\alpha} \mathcal{E}_{\frac{d-\alpha}{\alpha}}(\nu). \tag{2.42}$$

In light of (2.39), it remains to get a universal lower bound on $\mathcal{E}_{\frac{d-\alpha}{\alpha}}(\nu)$. But this is easy to do: For any $\beta > 0$ and for all $\nu \in \mathcal{P}([a,b])$,

$$\mathcal{E}_\beta(\nu) = \iint |x-y|^{-\beta}\, \nu(dx)\, \nu(dy) \ge b^{-\beta} \ge \eta^\beta. \tag{2.43}$$

We have used the inequality $|x-y|^\beta \le b^\beta \le \eta^{-\beta}$, valid for all $x, y \in [a,b] \subseteq [\eta, \eta^{-1}]$. □

We now address the analogous problem when $d \ge 2\alpha$ in the *special case where ν is uniform measure on $[a,b]$.*

Lemma 2.7. *(Case $d \ge 2\alpha$). Fix any $\eta > 0$, let $\eta < a < b < \eta^{-1}$ and define J_n as in (2.1) where ν denotes the uniform probability measure on $[a,b]$. Then there exists a constant $A_{2.7} := A_{2.7}(\eta, d, \alpha, a, b)$ such that for any $\varepsilon \in (0,1)$ and for all $n \ge 2$,*

$$\mathrm{E}[J_n^2] \le A_{2.7} \times \begin{cases} \varepsilon^{d+2\alpha} n^{-d/\alpha}, & \text{if } d > 2\alpha, \\ \dfrac{\varepsilon^{4\alpha}}{n^2} \log\left(\dfrac{n}{\varepsilon^\alpha}\right), & \text{if } d = 2\alpha. \end{cases} \tag{2.44}$$

Proof. Recall (2.36), and notice that (2.39) holds for all $d > \alpha$. Thus, it suffices to show that the lemma holds with $\mathrm{E}[J_n^2]$ replaced by $\mathcal{T}_1$. By appealing to (2.40)—with $\nu(dx)$ being the restriction to $[a,b]$ of $(b-a)^{-1}dx$—we can deduce the following for a sequence of positive constants $C := C(\eta, d, \alpha, a, b)$ and $C' := C'(\eta, d, \alpha, a, b)$ that may change from line to line, but never depend on $\varepsilon \in (0,1)$ nor on $n \ge 2$:

$$\mathcal{T}_1 \le C\varepsilon^d \int_a^b ds \int_n^\infty dt \int_a^b ds' \int_t^{2t} dt'\, t^{-d/\alpha} \left[\frac{\varepsilon^\alpha}{a|t'-t| + t|s'-s|} \wedge 1 \right]^{d/\alpha}. \tag{2.45}$$

Use the change of variables $v = s' - s$ (s fixed) and $t' - t = tu$ (t fixed) to see that the right-hand side is bounded above by

$$C\varepsilon^d \int_n^\infty dt \int_0^{b-a} dv\, t^{1-(d/\alpha)} K_\varepsilon^{at}(v/a). \tag{2.46}$$

Apply Lemma 2.5(c) to see that when $d > 2\alpha$, this is not greater than

$$C\varepsilon^{d+2\alpha} \int_n^\infty dt\, t^{-(d/\alpha)-1} = C\varepsilon^{d+2\alpha} n^{-d/\alpha}, \tag{2.47}$$

while when $d = 2\alpha$, this bound becomes $C\varepsilon^{4\alpha} n^{-2} \log(n/\varepsilon^\alpha)$. □

3. The probability of hitting a ball (case $d \geq 2\alpha$)

The following two are the main results of this section. The first treats the case $d > 2\alpha$.

Theorem 3.1 (Case $d > 2\alpha$)**.** *If $\eta > 0$ and $\eta < a < b < \eta^{-1}$ are held fixed, then there exists a constant $A_{3.1} := A_{3.1}(\eta, d, \alpha, a, b) > 1$ such that for all $n \geq 2$ and all $\varepsilon \in (0,1)$,*

$$\frac{\varepsilon^{d-2\alpha}}{A_{3.1}} n^{2-(d/\alpha)} \leq \mathrm{P}\left\{\overline{X\left([a,b] \times [n,2n]\right)} \cap \mathcal{B}_\varepsilon \neq \varnothing\right\} \leq A_{3.1} \varepsilon^{d-2\alpha} n^{2-(d/\alpha)}. \tag{3.1}$$

The case $d = 2\alpha$ is "critical," and the hitting probability of the previous theorem now has logarithmic decay.

Theorem 3.2 (Case $d = 2\alpha$)**.** *If $\eta > 0$ and $\eta < a < b < \eta^{-1}$ are held fixed, then there exists a constant $A_{3.2} := A_{3.2}(\eta, \alpha, a, b) > 1$ such that for all $n \geq 2$ and all $\varepsilon \in (0,1)$,*

$$\frac{A_{3.2}^{-1}}{\log\left(n/\varepsilon^\alpha\right)} \leq \mathrm{P}\left\{\overline{X\left([a,b] \times [n,2n]\right)} \cap \mathcal{B}_\varepsilon \neq \varnothing\right\} \leq \frac{A_{3.2}}{\log\left(n/\varepsilon^\alpha\right)}. \tag{3.2}$$

The case $d = 2\alpha$ looks different in form from the case $d > 2\alpha$, but is proved by similar means; so we omit the details of the proof of Theorem 3.2, and content ourselves with providing the following.

Proof of Theorem 3.1. We begin by deriving the (easier) lower bound. Note that

$$\mathrm{P}\left\{\overline{X\left([a,b] \times [n,2n]\right)} \cap \mathcal{B}_\varepsilon \neq \varnothing\right\} \geq \mathrm{P}\left\{\bar{J}_n > 0\right\}, \tag{3.3}$$

where $\bar{J}_n := \bar{J}_n(a, a+b; \varepsilon, \nu)$, and ν is normalized Lebesgue measure on $[a, a+b]$. By the Paley–Zygmund inequality ([Kho02, Lemma 1.4.1, p. 72]), and Lemmas 2.2 and 2.7,

$$\mathrm{P}\left\{\bar{J}_n > 0\right\} \geq \frac{\left(\mathrm{E}\left[\bar{J}_n\right]\right)^2}{\mathrm{E}\left[\bar{J}_n^2\right]} \geq \frac{\varepsilon^{2d} n^{-2(d-\alpha)/\alpha}}{A_{2.2}^2 A_{2.7} \varepsilon^{d+2\alpha} n^{-d/\alpha}}, \tag{3.4}$$

whence the asserted lower bound. Next we proceed with deriving the corresponding upper bound.

Let $\mathfrak{F}_{u,v}$ denote the σ-algebra generated by $X(s,t)$ for all $s \in [0,u]$ and $t \in [0,v]$, and consider the two-parameter martingale,

$$M(u,v) := \mathrm{E}\left[\bar{J}_n \,\middle|\, \mathfrak{F}_{u,v}\right], \qquad \text{for all } u \in [a,b],\ v \in \left[n, \frac{3n}{2}\right]. \tag{3.5}$$

Clearly,

$$M(u,v) \geq \int_u^{b+a} ds \int_v^{2n} dt\, \mathrm{P}\left\{|X(s,t)| \leq \varepsilon \,\middle|\, \mathfrak{F}_{u,v}\right\} \cdot \mathbf{1}_{\mathbf{G}_\varepsilon(u,v)}, \tag{3.6}$$

where

$$\mathbf{G}_\varepsilon(u,v) := \left\{\omega \in \Omega :\ |X(u,v)|(\omega) < \varepsilon/2\right\}. \tag{3.7}$$

Whenever $s \geq u$ and $t \geq v$, $X(s,t) - X(u,v)$ is independent of $\mathfrak{F}_{u,v}$. Therefore, by this and the triangle inequality, almost surely on $\mathbf{G}_\varepsilon(u,v)$,

$$\begin{aligned} M(u,v) &\geq \int_u^{b+a} ds \int_v^{2n} dt \, \mathrm{P}\left\{|X(s,t) - X(u,v)| \leq \varepsilon/2\right\} \\ &= \int_u^{b+a} ds \int_v^{2n} dt \, \phi_\alpha\left(\frac{\varepsilon/2}{[s(t-v)+v(s-u)]^{1/\alpha}}\right), \end{aligned} \tag{3.8}$$

where ϕ_α is defined in (2.3). By (2.4), on $\mathbf{G}_\varepsilon(u,v)$, for all $u \in [a,b]$ and $v \in [n, \frac{3}{2}n]$,

$$\begin{aligned} M(u,v) &\geq \frac{1}{C_\star} \int_u^{b+a} ds \int_v^{2n} dt \left(\frac{\varepsilon/2}{[s(t-v)+v(s-u)]^{1/\alpha}} \wedge 1\right)^d \\ &\geq \frac{1}{C} \int_u^{b+a} ds \int_v^{2n} dt \left(\frac{\varepsilon/2}{[(t-v)+n(s-u)]^{1/\alpha}} \wedge 1\right)^d. \end{aligned} \tag{3.9}$$

Do the changes of variables $s - u = s'$ and $t - v = \frac{n}{2} w$ to see, noting that $v \leq 3n/2$, that this is bounded below by

$$\int_0^a ds' \, CnK_{\varepsilon/2}^{n/2}(s'). \tag{3.10}$$

By Lemma 2.5(d), this is $\geq C\varepsilon^{2\alpha}/n$. Therefore, with probability one,

$$\sup_{u \in [a,b] \cap \mathbb{Q}} \sup_{v \in [n, \frac{3}{2}n] \cap \mathbb{Q}} \mathbf{1}_{\mathbf{G}_\varepsilon(u,v)} \leq \frac{n^2 C_+^2}{\varepsilon^{4\alpha}} \sup_{u \in [a,b] \cap \mathbb{Q}} \sup_{v \in [n, \frac{3}{2}n] \cap \mathbb{Q}} M^2(u,v). \tag{3.11}$$

Note that the left-hand side is a.s. equal to the indicator of the event $\{\inf |X(u,v)| \leq \varepsilon/2\}$, where the infimum is taken over all $u \in [a,b]$ and $v \in [n, \frac{3}{2}n]$. In particular,

$$\begin{aligned} &\mathrm{P}\left\{\overline{X\left([a,b] \times \left[n, \frac{3n}{2}\right]\right)} \cap \mathcal{B}_{\varepsilon/2} \neq \varnothing\right\} \\ &\leq \frac{n^2 C_+^2}{\varepsilon^{4\alpha}} \mathrm{E}\left\{\sup_{u \in [a,b] \cap \mathbb{Q}} \sup_{v \in [n, \frac{3}{2}n] \cap \mathbb{Q}} M^2(u,v)\right\} \leq \frac{16 n^2 C_+^2}{\varepsilon^{4\alpha}} \mathrm{E}\left\{\bar{J}_n^2\right\}. \end{aligned} \tag{3.12}$$

We have used the maximal L^2-inequality of Cairoli [Kho02, Theorem 1.3.1(ii), p. 222] to derive the last inequality; Cairoli's inequality applies since the two-parameter filtration $(\mathfrak{F}_{u,v})$ is commuting; for a definition, see [Kho02, p. 233]. The proof of this statement, in the Gaussian $\alpha = 2$ case, appears in [Kho02, Theorem 2.4.1, p. 237], and the general case is proved by similar considerations. Thus,

$$\mathrm{P}\left\{\overline{X([a,b] \times [n, 3n/2])} \cap \mathcal{B}_{\varepsilon/2} \neq \varnothing\right\} \leq \frac{32 C_+^2 n^2}{\varepsilon^{4\alpha}} \mathrm{E}\left\{\bar{J}_n^2\right\}. \tag{3.13}$$

Together with Lemma 2.7, this proves the asserted upper bound of the theorem. □

4. The probability of hitting a ball (case $d \in (\alpha, 2\alpha]$)

Recall that for any fixed $r > 0$, the r-dimensional Bessel–Riesz capacity of a compact set $S \subseteq \mathbb{R}_+$ is defined as

$$\mathcal{C}_r(S) := \sup_{\nu \in \mathcal{P}(S)} [\mathcal{E}_r(\nu)]^{-1} \qquad \text{with the convention } 1/\infty := 0. \tag{4.1}$$

The first result of this section is the following.

Theorem 4.1. Case $d \in (\alpha, 2\alpha]$. *If $0 < a < b$ are held fixed, then there exists a constant $A_{4.1} := A_{4.1}(a, b, d, \alpha) > 1$ such that for all compact sets $S \subseteq [a, b]$, all $n \geq 3$, and $\varepsilon \in (0, 1)$,*

$$\mathrm{P}\left\{\overline{X(S \times [n, 2n])} \cap \mathcal{B}_\varepsilon \neq \varnothing\right\} \geq A_{4.1}^{-1} \mathcal{C}_{\frac{d-\alpha}{\alpha}}(S). \tag{4.2}$$

Proof of Theorem 4.1. For any $\nu \in \mathcal{P}(S)$, $0 < a < b$, for all compact $S \subseteq [a, b]$, and $\varepsilon > 0$, consider $\bar{J}_n := \bar{J}_n(a, b; \varepsilon, \nu)$ as defined in (2.1). By the Paley–Zygmund inequality [Kho02, Lemma 1.4.1, p. 72], and Lemmas 2.2 and 2.6

$$\mathrm{P}\left\{\overline{X(S \times [n, 2n])} \cap \mathcal{B}_\varepsilon \neq \varnothing\right\} \geq \frac{\left(\mathrm{E}\{\bar{J}_n\}\right)^2}{\mathrm{E}\left\{\bar{J}_n^2\right\}} \geq \left[C_{2.2}^2 A_{2.6} \mathcal{E}_{\frac{d-\alpha}{\alpha}}(\nu)\right]^{-1}, \tag{4.3}$$

and this makes sense whether or not $\mathcal{E}_{(d-\alpha)/\alpha}(\nu)$ is finite. Optimize over $\nu \in \mathcal{P}(S)$ to deduce (4.2). □

As for an analogous upper bound, we shall prove the following:

Theorem 4.2. Case $d \in (\alpha, 2\alpha]$. *If $M \geq 1$ is fixed, then there exists a constant $A_{4.2} := A_{4.2}(d, \alpha, M)$ such that for all $\varepsilon \in (0, 1)$, $n \geq 3$, and all $[a, b] \subseteq [M^{-1}, M]$ that satisfies $b - a \geq M\varepsilon^\alpha n^{-1}$,*

$$\mathrm{P}\left\{\overline{X([a, b] \times [n, 2n])} \cap \mathcal{B}_\varepsilon \neq \varnothing\right\} \leq A_{4.2}(b - a)^{(d-\alpha)/\alpha}. \tag{4.4}$$

It is not difficult to show that $\mathcal{C}_{(d-\alpha)/\alpha}([a, b]) = c(b-a)^{(d-\alpha)/\alpha}$ for some constant $c := c(d, \alpha)$. Therefore, Theorem 4.2 shows that Theorem 4.1 is best possible. On the other hand, Theorem 4.1 does not have a corresponding capacity upper bound as can be seen by considering $S = \{1\}$. In fact, this shows that even the condition $b - a \geq 2\varepsilon^\alpha n^{-1}$ of Theorem 4.2 cannot, in a sense, be improved.

Proof of Theorem 4.2. Throughout, let $\bar{J}_n := \bar{J}_n(a, 2b - a; \varepsilon, \lambda)$, where λ denotes the Lebesgue measure on $[0, 2b - a]$. Although λ is not a probability measure, it is easy to see as in Lemma 2.6 that

$$\begin{aligned} \mathrm{E}\left\{\bar{J}_n^2\right\} &\leq 4^d A_{2.6} \varepsilon^{2d} n^{-2(d-\alpha)/\alpha} \mathcal{E}_{\frac{d-\alpha}{\alpha}}(\lambda) \\ &= C\varepsilon^{2d} n^{-2(d-\alpha)/\alpha}(b - a)^{3-(d/\alpha)}, \end{aligned} \tag{4.5}$$

where $C := 2^{1+2d} A_{2.6} \alpha^2 (3\alpha - d)^{-1}(2\alpha - d)^{-1}$.

Next define the two-parameter martingale

$$M(u, v) := \mathrm{E}\left[\bar{J}_n \,\middle|\, \mathfrak{F}_{u,v}\right\}, \qquad \text{for all } u \in [a, b],\ v \in \left[n, \frac{3}{2}n\right]. \tag{4.6}$$

By Cairoli's L^2-maximal inequality and (4.5),

$$\mathrm{E}\left\{\sup_{u,v \in \mathbb{Q}_+} M^2(u, v)\right\} \leq 16C\varepsilon^{2d} n^{-2(d-\alpha)/\alpha}(b - a)^{3-(d/\alpha)}. \tag{4.7}$$

Evidently,

$$M(u, v) \geq \int_u^{2b-a} \int_v^{2n} \mathrm{P}\left\{|X(s, t)| \leq \varepsilon \,\middle|\, \mathfrak{F}_{u,v}\right\} dt\, ds \cdot \mathbf{1}_{\mathbf{G}_\varepsilon(u,v)}, \tag{4.8}$$

where $\mathbf{G}_\varepsilon(u, v)$ is defined in (3.7). Whenever $s \geq u$ and $t \geq v$, the random variable $X(s, t) - X(u, v)$ is independent of $\mathfrak{F}_{u,v}$, and has the same distribution as

$\rho^{1/\alpha}X(1,1)$, where ρ denotes the area of $([0,s]\times[0,t])\setminus([0,u]\times[0,v])$. Hence, almost surely on $\mathbf{G}_\varepsilon(u,v)$,

$$\begin{aligned} M(u,v) &\geq \int_u^{2b-a} ds \int_v^{2n} dt\, \mathrm{P}\left\{|X(s,t)-X(u,v)|\leq \varepsilon/2\right\} \\ &= \int_u^{2b-a} ds \int_v^{2n} dt\, \phi_\alpha\left(\frac{\varepsilon/2}{\rho^{1/\alpha}}\right). \end{aligned} \tag{4.9}$$

But for any $s\in[u,2b-a]$ and $t\in[v,2n]$, $\rho\leq 2b(t-v)+2n(s-u)$, and so from (2.4), we have the following a.s. on $\mathbf{G}_\varepsilon(u,v)$:

$$M(u,v)\geq \int_u^{2b-a} ds\int_v^{2n} dt\left(\frac{\varepsilon/2}{[2b(t-v)+2n(s-u)]^{1/\alpha}}\wedge 1\right)^d. \tag{4.10}$$

Do the changes of variables $s-u=s'$ and $t-v=\frac{n}{2}t'$ to see that this is bounded below by

$$\begin{aligned} &\frac{n}{2}\int_0^{b-a} ds' \int_0^1 dt'\left(\frac{\varepsilon/2}{(bn)^{1/\alpha}(t'+2s'/b)^{1/\alpha}}\wedge 1\right)^d \\ &\quad = \frac{n}{2}\int_0^{b-a} ds'\, K_{\varepsilon/2}^{2bn}(2s'/b). \end{aligned} \tag{4.11}$$

By Lemma 2.5(b), for $b-a\geq M\varepsilon^\alpha/(2n)$, this is not less than

$$\begin{aligned} &C\frac{n}{2}\frac{\varepsilon^d}{n^{d/\alpha}}\int_{\frac{M\varepsilon^\alpha}{2n}}^{b-a} ds\, s^{-(d-\alpha)/\alpha} \\ &\quad \geq C\varepsilon^d n^{-(d-\alpha)/\alpha}(b-a)^{-(d-\alpha)/\alpha}\left(b-a-\frac{M\varepsilon^\alpha}{2n}\right). \end{aligned} \tag{4.12}$$

For $b-a\geq M\varepsilon^\alpha/n$, this is not less than

$$\frac{C}{2}\varepsilon^d n^{-(d-\alpha)/\alpha}(b-a)^{2-(d/\alpha)}. \tag{4.13}$$

This shows that a.s.,

$$M(u,v)\geq A\varepsilon^d n^{-(d-\alpha)/\alpha}(b-a)^{2-(d/\alpha)}\cdot \mathbf{1}_{\mathbf{G}_\varepsilon(u,v)}. \tag{4.14}$$

In particular, with probability one,

$$\sup M^2(u,v)\geq A^2\varepsilon^{2d}n^{-2(d-\alpha)/\alpha}(b-a)^{4-2(d/\alpha)}\cdot \sup \mathbf{1}_{\mathbf{G}_\varepsilon(u,v)}, \tag{4.15}$$

where both suprema are taken over $\{(u,v)\in([a,b]\times[n,\frac{3}{2}n])\cap\mathbb{Q}\}$. The path-regularity of the random field X (Proposition A.2) ensures that $\mathrm{E}\{\sup \mathbf{1}_{\mathbf{G}_\varepsilon(u,v)}\}$ is the probability that $\overline{X([a,b]\times[n,\frac{3}{2}n])}\cap\mathcal{B}_{\varepsilon/2}$ is nonempty. Therefore, the preceding display together with (4.7) readily prove the theorem. ☐

5. Proof of Theorem 1.1

(a) We shall show that when $d>2\alpha$, $\mathfrak{L}_{d,\alpha}=\varnothing$, a.s. Thanks to Theorem 3.1, for any $[a,b]\subset(0,\infty)$ with $b>a$, we can find a constant $A:=A(a,b,d,\alpha)>1$ such that for all $\varepsilon \in (0,1)$ and $n \geq 2$,

$$\begin{aligned}
&\sum_{n=5}^{\infty} \mathrm{P}\left\{\overline{X\left([a,b]\times[2^n,\infty)\right)} \cap \mathcal{B}_\varepsilon \neq \varnothing\right\} \\
&\qquad \leq \sum_{n=5}^{\infty}\sum_{j=n}^{\infty} \mathrm{P}\left\{\overline{X\left([a,b]\times[2^j,2^{j+1})\right)} \cap \mathcal{B}_\varepsilon \neq \varnothing\right\} \qquad (5.1)\\
&\qquad \leq A\varepsilon^{d-2\alpha} \sum_{n=5}^{\infty}\sum_{j=n}^{\infty} (2^j)^{2-(d/\alpha)} < +\infty.
\end{aligned}$$

Thus, the Borel–Cantelli lemma guarantees that a.s., for all but a finite number of n's, $X([a,b]\times[n,\infty))\cap\mathcal{B}_\varepsilon = \varnothing$. This yields $\mathcal{L}_{d,\alpha} = \varnothing$, a.s., as asserted.

(b) We divide the proof of (1.3) into two cases: $d \in (\alpha, 2\alpha)$ and $d = 2\alpha$.

5.1. **The case $d \in (\alpha, 2\alpha)$.** We begin our analysis of this case with a weak codimension argument. To do so, we will need the notion of a *upper Minkowski dimension* ([Mat95, p. 76–77]), which is described as follows: Given any bounded set $S \subset \mathbb{R}$ and $k \geq 1$, define

$$\mathcal{N}_S(k) := \#\left\{ j \in \mathbb{Z} : \ \left[\frac{j}{k}, \frac{j+1}{k}\right] \cap S \neq \varnothing \right\}. \qquad (5.2)$$

[As usual, # denotes cardinality.] Note that the boundedness of S ensures that $\mathcal{N}_S(k) < +\infty$. The *upper Minkowski dimension* of S is then defined as

$$\dim_{\mathcal{M}}(S) := \limsup_{k\to\infty} \frac{\log \mathcal{N}_S(k)}{\log k}. \qquad (5.3)$$

It is not hard to see that we always have $\dim_{\mathcal{H}}(S) \leq \dim_{\mathcal{M}}(S)$, and the inequality can be strict; cf. Mattila [Mat95, p. 77].

The following refines half of what is known as the codimension argument. Part (b) is within Taylor [Tay66, Theorem 4], but we provide a brief self-contained proof for the sake of completeness.

Proposition 5.1. *If $\mathbf{X}$ is a random analytic subset of $\mathbb{R}$, then:*

(a) Suppose that there exists a nonrandom number $a \in (0,1)$ such that for all nonrandom bounded sets $T \subset \mathbb{R}$ with $\dim_{\mathcal{M}}(T) < a$ we have $\mathrm{P}\{\mathbf{X}\cap T = \varnothing\} = 1$. Then $\dim_{\mathcal{H}}(\mathbf{X}) \leq 1-a$, a.s.

(b) Suppose that there exists a nonrandom number $a \in (0,1)$ such that for all nonrandom bounded sets $T \subset \mathbb{R}$ such that $\dim_{\mathcal{H}}(T) > a$ we have $\mathrm{P}\{\mathbf{X}\cap T \neq \varnothing\} = 1$. Then $\dim_{\mathcal{H}}(\mathbf{X}) \geq 1-a$, a.s.

Proof. (a) Without loss of generality, we can assume that $\mathbf{X} \subseteq [1,2]$ a.s.

For any $r \in (0,1)$, let us consider a one-dimensional symmetric stable Lévy process $Z_r := \{Z_r(t);\ t \geq 0\}$ with $Z_r(0) = 0$ and index $r \in (0,1)$. If $\mathbf{Z}_r := \overline{Z_r([1,2])}$, then it is well-known that:

(i) $\mathbf{Z}_r$ is a.s. a compact set;
(ii) for all analytic sets $F \subset \mathbb{R}$ with $\dim_{\mathcal{H}}(F) > 1-r$, $P\{\mathbf{Z}_r \cap F \neq \varnothing\} > 0$;
(iii) for all analytic sets $F \subset \mathbb{R}$ with $\dim_{\mathcal{H}}(F) < 1-r$, $\mathbf{Z}_r \cap F = \varnothing$, a.s.; and
(iv) with probability one, $\dim_{\mathcal{H}}(\mathbf{Z}_r) = \dim_{\mathcal{M}}(\mathbf{Z}_r) = r$.

An explanation is in order: Part (i) follows from the cádlág properties of Z_r; parts (ii) and (iii) follows from the connections between probabilistic potential theory and Frostman's lemma [Kho02, Th. 3.5.1, p. 385]; and part (iv) is a direct computation that is essentially contained in McKean [McK55].

Now to prove the proposition, suppose to the contrary that with positive probability, $\dim_{\mathcal{H}}(\mathbf{X}) > 1-a$. This and (ii) together prove that for any $r \in (0,a)$, $\mathbf{X} \cap \mathbf{Z}_r \neq \varnothing$ with positive probability. On the other hand, by (iv), the upper Minkowski dimension of $\mathbf{Z}_r$ is $r < a$, a.s. Therefore, the property of $\mathbf{X}$ mentioned in the statement of the proposition implies that a.s., $\mathbf{X} \cap \mathbf{Z}_r = \varnothing$, which is the desired contradiction, and (a) is proved.

(b) Choose $r \in (a,1)$, and recall $\mathbf{Z}_r$ from (a) above. By item (iv) of the proof of (a), $\dim(\mathbf{Z}_r) = r > a$, a.s. The assumed hitting property of $\mathbf{X}$ implies that $\mathrm{P}\{\mathbf{X} \cap \mathbf{Z}_r \neq \varnothing\} = 1$. On the other hand, if $\dim_{\mathcal{H}}(\mathbf{X}) < 1-r$ with positive probability, then (iii) of the proof of part (a) would imply that $P\{\mathbf{X} \cap \mathbf{Z}_r = \varnothing\} > 0$, which is a contradiction. Thus, we have shown that almost surely, $\dim_{\mathcal{H}}(\mathbf{X}) \geq 1-r$. Let $r \downarrow a$ to finish. □

The property of not hitting sets of small upper Minkowski dimension is shared by $\mathcal{L}_{d,\alpha}$—defined in (1.2)—as we shall see next. Note that Proposition 5.2 and Corollary 5.3 below include the case $d = 2\alpha$.

Proposition 5.2. (Case $d \in (\alpha, 2\alpha]$). *If $S \subset (0,\infty)$ is compact, and if its upper Minkowski dimension is strictly below $(d-\alpha)/\alpha$, then almost surely, $\mathcal{L}_{d,\alpha} \cap S = \varnothing$.*

Proof. Without loss of generality, we assume that $S \subset [1,2)$. Now by Theorem 4.2, for all $\ell \geq 3$, $\varepsilon \in (0,1)$, and all closed intervals $I \subset [1,2)$ with $|I| \geq 2\varepsilon^\alpha/\ell$,

$$\mathrm{P}\left\{\overline{X\left(I \times [\ell, 2\ell]\right)} \cap \mathcal{B}_\varepsilon \neq \varnothing\right\} \leq A_{4.2} |I|^{(d-\alpha)/\alpha}, \tag{5.4}$$

where $|I|$ denotes the length of I. Next we define

$$\gamma_{n,\varepsilon} := \left\lfloor \frac{2^{n-1}}{\varepsilon^\alpha} \right\rfloor, \tag{5.5}$$

and cover S with $\mathcal{N}_S(\gamma_{n,\varepsilon})$-many of the intervals $I_1, \ldots, I_{\gamma_{n,\varepsilon}}$ with $I_l := [l\gamma_{n,\varepsilon}^{-1}, (l+1)\gamma_{n,\varepsilon}^{-1}]$ ($l = \gamma_{n,\varepsilon}, \ldots, 2\gamma_{n,\varepsilon} - 1$). We then apply the preceding inequality to deduce the following: Since $\gamma_{n,\varepsilon}^{-1} \geq 2\varepsilon^\alpha/2^n$,

$$\mathrm{P}\left\{\overline{X\left(S \times [2^n, 2^{n+1}]\right)} \cap \mathcal{B}_\varepsilon \neq \varnothing\right\} \leq A_{4.2} \gamma_{n,\varepsilon}^{-(d-\alpha)/\alpha} \mathcal{N}_S(\gamma_{n,\varepsilon}). \tag{5.6}$$

But as $n \to \infty$, $\gamma_{n,\varepsilon} = (1+o(1))\varepsilon^{-\alpha} 2^{n-1}$ and $\mathcal{N}_S(\gamma_{n,\varepsilon}) = O(\gamma_{n,\varepsilon}^{-q+(d-\alpha)/\alpha})$, as long as $-q + (d-\alpha)/\alpha > \dim_{\mathcal{M}}(S)$. This yields the following: as $n \to \infty$,

$$\mathrm{P}\left\{\overline{X\left(S \times [2^n, \infty)\right)} \cap \mathcal{B}_\varepsilon \neq \varnothing\right\} \leq A_{4.2} \sum_{k=n}^{\infty} \gamma_{k,\varepsilon}^{-(d-\alpha)/\alpha} \gamma_{k,\varepsilon}^{-q+(d-\alpha)/\alpha}, \tag{5.7}$$

and this is $O(2^{-nq})$. Owing to the Borel–Cantelli lemma, with probability one,

$$\overline{X\left(S \times [2^n, \infty)\right)} \cap \mathcal{B}_\varepsilon = \varnothing, \tag{5.8}$$

for all but a finite number of n's. In addition, by monotonicity, this statement's null set can be chosen to be independent of $\varepsilon \in (0,1)$. This shows that $\mathcal{L}_{d,\alpha} \cap S = \varnothing$, a.s., as desired. □

An immediate consequence of Propositions 5.1(a) and (5.2) is the following, which proves half of the dimension formula (1.3) in Theorem 1.1.

Corollary 5.3. (Case $d \in (\alpha, 2\alpha]$). *With probability one,*

$$\dim_{\mathcal{H}}(\mathcal{L}_{d,\alpha}) \leq 2 - \frac{d}{\alpha}. \tag{5.9}$$

The remainder of this subsection is devoted to deriving the converse inequality. We need a lemma which is contained in Joyce and Preiss [JoP95].

Lemma 5.4. *Given a number* $a \in (0,1)$ *and a compact set* $F \subset \mathbb{R}$ *with* $\dim_{\mathcal{H}}(F) > a$*, there is a single non-empty compact set* $F_\star \subseteq F$ *with the following property: For any rational open interval* $I \subset \mathbb{R}$*, if* $I \cap F_\star \neq \varnothing$*, then* $\dim_{\mathcal{H}}(I \cap F_\star) > a$*.*

We provide a proof of this simple result for the sake of completeness.

Proof. Define

$$\begin{aligned} &\mathcal{R} := \left\{\text{rational open intervals } I: \ I \cap F \neq \varnothing, \text{ but } \dim_{\mathcal{H}}(I \cap F) \leq a\right\}, \\ &F_\star := F \setminus \bigcup_{I \in \mathcal{R}} I, \qquad G := \bigcup_{I \in \mathcal{R}} (I \cap F). \end{aligned} \tag{5.10}$$

The second equation above defines the set $F_\star$ of our lemma, as we shall see next. Note that $F_\star \neq \varnothing$ since $\dim_{\mathcal{H}}(F) > a$.

Because $\mathcal{R}$ is denumerable, $\dim_{\mathcal{H}}(G) = \sup_{I \in \mathcal{R}} \dim_{\mathcal{H}}(I \cap F) \leq a$. On the other hand, $F_\star \cup G = F$; thus, for any rational interval I, $(F_\star \cap I) \cup (G \cap I) = F \cap I$. By monotonicity, $\dim_{\mathcal{H}}(F_\star \cap I) \leq \dim_{\mathcal{H}}(F \cap I) \leq a$.

Now suppose, to the contrary, that there exists a rational interval I such that $\dim_{\mathcal{H}}(I \cap F_\star) \leq a$, although $I \cap F_\star \neq \varnothing$. This shows that $\dim_{\mathcal{H}}(I \cap F) \leq \max(\dim_{\mathcal{H}}(F_\star \cap I), \dim_{\mathcal{H}}(G \cap I)) \leq a$ and $I \cap F \neq \varnothing$. In other words, such an I is necessarily in $\mathcal{R}$. In light of our definition of $F_\star$, we have $I \cap F_\star = \varnothing$, which is the desired contradiction. □

Proof of Theorem 1.1 in the case $d \in (\alpha, 2\alpha)$. Theorem 4.1 and Frostman's theorem ([Kho02, Th. 2.2.1, p. 521]), used in conjunction, tell us that whenever $S \subseteq [1,2]$ is compact and satisfies $\dim_{\mathcal{H}}(S) \in ((d-\alpha)/\alpha, 1]$ (note that the case $d = 2\alpha$ is *not* included here),

$$\inf_{\varepsilon \in (0,1)} \inf_{n \geq 3} \mathrm{P}\left\{\overline{X\left(S \times [n,\infty)\right)} \cap \mathcal{B}_\varepsilon \neq \varnothing\right\} > 0. \tag{5.11}$$

Consequently, by monotonicity and the Hewitt–Savage 0-1 law,

$$\mathrm{P}\left\{\overline{X\left(S \times [n,\infty)\right)} \cap \mathcal{B}_\varepsilon \neq \varnothing \text{ infinitely often for each } \varepsilon \in (0,1)\right\} = 1. \tag{5.12}$$

By path regularity (Proposition A.2), and since $\varepsilon \in (0,1)$ can be adjusted up a little, we have

$$\mathrm{P}\left\{X\left(S \times [n,\infty)\right) \cap \mathcal{B}_\varepsilon \neq \varnothing \text{ infinitely often for each } \varepsilon \in (0,1)\right\} = 1. \tag{5.13}$$

Now for each $\varepsilon \in (0,1)$ and $n \geq 3$ consider the sets

$$\begin{aligned} &\tilde{\Gamma}_\varepsilon^n := \left\{s \in [1,2]: \ \exists t \geq n \text{ such that } X(s,t) \in \mathcal{B}_\varepsilon\right\}, \\ &\Gamma_\varepsilon^n := \left\{s \in [1,2]: \ \exists t \geq n \text{ such that } X(s,t) \in \mathcal{B}_\varepsilon \text{ and } X(s-,t) \in \mathcal{B}_\varepsilon\right\}. \end{aligned} \tag{5.14}$$

By the path-regularity of X (Proposition A.2), Γ_ε^n is (a.s.) an open subset of $[1,2]$ no matter the value of α, whereas $\tilde{\Gamma}_\varepsilon^n$ is an open set only in the case $\alpha = 2$ (and in this case, $\tilde{\Gamma}^n_\varepsilon = \Gamma^n_\varepsilon$). On the other hand, by (5.13), as long as $\dim_{\mathcal{H}}(S) > (d-\alpha)/\alpha$, we have

$$\mathrm{P}\left\{{}^\forall n \geq 3,\ {}^\forall \varepsilon \in \mathbb{Q}_+ : S \cap \tilde{\Gamma}^n_\varepsilon \neq \varnothing\right\} = 1. \tag{5.15}$$

Now we appeal to Lemma 5.4 to extract a compact set $S_\star \subseteq S$ such that if $I \subseteq [1,2]$ is any rational open interval such that $I \cap S_\star \neq \varnothing$, then $\dim_{\mathcal{H}}(S_\star \cap I) > (d-\alpha)/\alpha$. In particular, by (5.15), for all such rational open intervals I,

$$\mathrm{P}\left\{{}^\forall n \geq 3,\ {}^\forall \varepsilon \in \mathbb{Q}_+ : S_\star \cap \bar{I} \cap \tilde{\Gamma}^n_\varepsilon \neq \varnothing\right\} = 1. \tag{5.16}$$

We would like to have the same statement with $\tilde{\Gamma}^n_\varepsilon$ replaced by Γ^n_ε. If $\alpha = 2$, this is clear; thus, one can go directly to (5.18). Assuming that $\alpha \in (0,2)$, observe that the set S_q of elements s of $S_\star \cap \bar{I}$ which are isolated on the right (i.e., there is $\eta > 0$ such that $S_\star \cap \bar{I} \cap [s, s+\eta) = \{s\}$) is countable. By Dalang and Walsh [DW92b, Corollary 2.8], with probability one, there is no point (s_n, t_n) with the properties that $\square X(s_n, t_n) \neq 0$ and $s_n \in S_q$; see also (A.10) below.

Now set

$$\begin{aligned} F &:= \left\{\omega \in \Omega :\ {}^\forall n \geq 3,\ {}^\forall \varepsilon \in \mathbb{Q}_+,\ S_\star \cap \bar{I} \cap \tilde{\Gamma}^n_\varepsilon \neq \varnothing\right\}, \\ G &:= \left\{\omega \in \Omega :\ {}^\forall n \geq 3,\ {}^\forall \varepsilon \in \mathbb{Q}_+,\ S_\star \cap \bar{I} \cap \Gamma^n_\varepsilon \neq \varnothing\right\}. \end{aligned} \tag{5.17}$$

Fix $\omega \in F$, and suppose that $\square X(s,t)(\omega) \neq 0$ for all $s \in S_q$ and $t \geq 0$. We shall show that $\omega \in G$. Indeed, fix $n \geq 3$ and $\varepsilon \in \mathbb{Q}_+$. If there is some $s \in S_q \cap \bar{I} \cap \tilde{\Gamma}^n_\varepsilon$, then there is a $t \geq n$ such that $X(s,t)(\omega) \in \mathcal{B}_\varepsilon$. Because $X(s-,t)(\omega) = X(s,t)(\omega) \in \mathcal{B}_\varepsilon$, we see that $\omega \in G$. If $S_q \cap \bar{I} \cap \tilde{\Gamma}^n_\varepsilon = \varnothing$, then there is an $s \in (S_\star \setminus S_q) \cap \bar{I} \cap \tilde{\Gamma}^n_\varepsilon$ and a $t \geq n$ such that $X(s,t)(\omega) \in \mathcal{B}_\varepsilon$. Since $s \notin S_q$, by the path regularity of X, there is an $r \in S$ such that $r > s$, $X(r,t)(\omega) \in \mathcal{B}_\varepsilon$ and $X(r-,t)(\omega) \in \mathcal{B}_\varepsilon$, so $\omega \in G$.

We have shown that $F \subset G$ a.s., and therefore,

$$\mathrm{P}\left\{{}^\forall n \geq 3,\ {}^\forall \varepsilon \in \mathbb{Q}_+ : S_\star \cap \bar{I} \cap \Gamma^n_\varepsilon \neq \varnothing\right\} = 1. \tag{5.18}$$

It follows that $S_\star \cap \Gamma^n_\varepsilon$ is a relatively open subset of $S_\star$ that is everywhere dense (in $S_\star$). By the Baire category theorem, with probability one, $S_\star \cap \cap_{\varepsilon \in \mathbb{Q}_+} \cap_{n \geq 3} \Gamma^n_\varepsilon$ is an uncountable dense subset of $S_\star$. In particular, with probability one, we can find uncountably-many $s \in S$ such that for all $\varepsilon > 0$ and for infinitely-many integers $n \geq 1$, there exists $t \geq n$ such that $X(s,t) \in \mathcal{B}_\varepsilon$.

In other words, we have shown that whenever $S \subset [1,2]$ is compact (and hence analytic) with $\dim_{\mathcal{H}}(S) > (d-\alpha)/\alpha$, then almost surely, $\mathcal{L}_{d,\alpha} \cap S \neq \varnothing$. In particular, $\mathcal{L}_{d,\alpha}$ is dense in $\mathbb{R}_+$ and Proposition 5.1(b) shows that with probability one, $\dim_{\mathcal{H}}(\mathcal{L}_{d,\alpha}) \geq 1 - (d-\alpha)/\alpha = 2 - (d/\alpha)$. In conjunction with Corollary 5.3, this proves Theorem 1.1(b) in the case $d \in (\alpha, 2\alpha)$. $\square$

5.2. **The Case $d = 2\alpha$.** According to Corollary 5.3, $\dim_{\mathcal{H}}(\mathcal{L}_{2\alpha,\alpha}) = 0$, so it remains to prove that $\mathcal{L}_{2\alpha,\alpha}$ is a.s. everywhere-dense. We do this in successive steps.

The first step is the classical reflection principle (the discrete-time analogue is for instance in [CaD96, Lemma p. 34]).

Lemma 5.5 (The Maximal Inequality). *If $\{L(t)\}_{t\geq 0}$ denotes a symmetric Lévy process with values in a separable Banach space $(\mathbb{B}, \|\cdot\|)$, then for all $t, \lambda > 0$,*

$$\mathrm{P}\left\{\sup_{s\in[0,t]} \|L(s)\| \geq \lambda\right\} \leq 2\mathrm{P}\left\{\|L(t)\| \geq \lambda\right\}. \tag{5.19}$$

Proof. Consider the stopping time,

$$T := \inf\{s > 0 : \; \|L(s)\| \geq \lambda\}, \tag{5.20}$$

with the convention $\inf \varnothing := +\infty$. Clearly,

$$\begin{aligned} &\mathrm{P}\left\{\sup_{s\in[0,t]} \|L(s)\| \geq \lambda\right\} \\ &= \mathrm{P}\{T < t, \|L(t)\| \geq \lambda\} + \mathrm{P}\{T < t, \|L(t)\| < \lambda\} \\ &\leq \mathrm{P}\{\|L(t)\| \geq \lambda\} + \mathrm{P}\{T < t, \|L(t) - L(T) + L(T)\| < \lambda\}. \end{aligned} \tag{5.21}$$

By symmetry and the strong Markov property, the conditional distributions of $L(t) - L(T)$ and $L(T) - L(t)$ given $L(T)$ are identical on $\{T < t\}$. Therefore, the preceding becomes

$$\mathrm{P}\{\|L(t)\| \geq \lambda\} + \mathrm{P}\{T < t, \| - L(t) + 2L(T)\| < \lambda\}. \tag{5.22}$$

Because Lévy processes are right-continuous, on the set $\{T < t\}$, we have $\|L(T)\| \geq \lambda$. Therefore, the triangle inequality implies that, on the set $\{T < t\}$, we always have $\| - L(t) + 2L(T)\| \geq 2\lambda - \|L(t)\|$. This proves the result. □

We return to the proof of the fact that $\mathfrak{L}_{2\alpha,\alpha}$ is everywhere-dense. Fix $0 < a < b$, $\theta > 0$, $\varepsilon \in (0,1)$, and define

$$\begin{aligned} &\boldsymbol{\chi}_N := \sum_{j=1}^N \mathbf{1}_{\mathbf{G}_j\cap\mathbf{H}_j}, \text{ where} \\ &\mathbf{G}_j := \left\{\omega \in \Omega : \; \overline{X([a,b] \times [2^j, 2^{j+1}])}(\omega) \cap \mathcal{B}_\varepsilon \neq \varnothing\right\}, \text{ and} \\ &\mathbf{H}_j := \left\{\omega \in \Omega : \; \sup_{s\in[a,b]} \left|X\left(s, 2^{j+1}\right)(\omega)\right|^\alpha \leq \theta j 2^j\right\}. \end{aligned} \tag{5.23}$$

Thanks to Theorem 3.2, there exists a constant $A_{5.24} := A_{5.24}(d, \alpha, a, b, \varepsilon) \in (0,1)$ such that for all $j \geq 3$,

$$\frac{A_{5.24}}{j} \leq \mathrm{P}(\mathbf{G}_j) \leq \frac{A_{5.24}^{-1}}{j}. \tag{5.24}$$

We now improve this slightly by proving the following:

Lemma 5.6. *There exists a constant $\theta_0 = \theta_0(\alpha, d) \in (0,1)$ such that whenever $\theta \geq \theta_0$,*

$$\mathrm{P}\left(\mathbf{G}_j \cap \mathbf{H}_j\right) \geq \frac{A_{5.24}}{2j}, \qquad \textit{for all } j \geq 1. \tag{5.25}$$

Proof. Thanks to (5.24), Lemma 5.5, and scaling,

$$\begin{aligned} \mathrm{P}\left(\mathbf{G}_j^{\complement} \cup \mathbf{H}_j^{\complement}\right) &\leq 1 - \frac{A_{5.24}}{j} + \mathrm{P}\left(\mathbf{H}_j^{\complement}\right) \\ &\leq 1 - \frac{A_{5.24}}{j} + 2\mathrm{P}\left(\left|X\left(b, 2^{j+1}\right)\right|^\alpha \geq \theta j 2^j\right) \\ &= 1 - \frac{A_{5.24}}{j} + 2\mathrm{P}\left(|\mathcal{S}_\alpha| \geq \frac{1}{2b}\left(\theta j\right)^{1/\alpha}\right), \end{aligned} \tag{5.26}$$

where $\mathcal{S}_\alpha$ is an isotropic stable random variable in $\mathbb{R}^d$; see (A.1). Now, we recall that there exists a constant $C := C(d, \alpha) > 1$ such that for all $x \geq 1$, $\mathrm{P}\{|\mathcal{S}_\alpha| \geq$

$x\} \le Cx^{-\alpha}$; see, for instance, [Kho02, Prop. 3.3.1, p. 380]. In particular, whenever $\theta > (2b)^\alpha$, we have, for all $j \ge 1$,

$$\mathrm{P}\left(\mathbf{G}_j^{\complement} \cup \mathbf{H}_j^{\complement}\right) \le 1 - \frac{A_{5.24}}{j} + 2C\frac{(2b)^\alpha}{\theta j}. \tag{5.27}$$

Because $A_{5.24} \in (0,1)$ and $C > 1$, we can choose $\theta_0 := 4C(2b)^\alpha A_{5.24}^{-1}$ to finish. $\square$

Henceforth, we fix $\theta > \theta_0$ so that, by Lemma 5.6, there exists a constant $A_{5.28} := A_{5.28}(d,\alpha,a,b,\varepsilon) > 0$ with the property that

$$\mathrm{E}\left[\boldsymbol{\chi}_N\right] \ge A_{5.28} \log N, \qquad \text{for all } N \ge 3. \tag{5.28}$$

Next we show that

$$\mathrm{E}\left[\boldsymbol{\chi}_N^2\right] = O\left(\log^2 N\right), \qquad (N \to \infty). \tag{5.29}$$

To prove this, note that whenever $k \ge j+2$,

$$\begin{aligned} &\mathrm{P}\left(\mathbf{G}_k \cap \mathbf{H}_k \,\middle|\, \mathbf{G}_j \cap \mathbf{H}_j\right) \\ &\le \mathrm{P}\left\{ \inf_{s\in[a,b]} \inf_{t\in[2^k,2^{k+1}]} \left|X(s,t) - X(s,2^{j+1})\right| \le \varepsilon + \left(\theta j 2^j\right)^{\frac{1}{\alpha}} \,\middle|\, \mathbf{G}_j \cap \mathbf{H}_j \right\}. \end{aligned} \tag{5.30}$$

Because X has stationary and independent increments, this is equal to

$$\begin{aligned} &\mathrm{P}\left\{ \inf_{s\in[a,b]} \inf_{t\in[2^k,2^{k+1}]} \left|X(s,t) - X(s,2^{j+1})\right| \le \varepsilon + \left(\theta j 2^j\right)^{1/\alpha} \right\} \\ &\le \mathrm{P}\left\{ \inf_{s\in[a,b]} \inf_{t\in[2^k - 2^{j+1}, 2^{k+1} - 2^{j+1}]} |X(s,t)| \le (1+\theta)^{1/\alpha} \left(j2^j\right)^{1/\alpha} \right\} \\ &= \mathrm{P}\left\{ \inf_{s\in[a,b]} \inf_{t\in[2,5]} |X(s,t)| \le (1+\theta)^{1/\alpha} \left(\frac{j}{2^{k-j-1}-1}\right)^{1/\alpha} \right\} \end{aligned} \tag{5.31}$$

For the last equality, we have used the scaling property of X. For $k \ge j+2$, the ratio on the right-hand side is $\le 4j2^{j-k}$, and there are $c > 0$, $\gamma > 0$ and $C < \infty$ such that for $k > c + j + \gamma\log(j)$, $4(1+\theta)j2^{j-k} \le C(2/3)^{k-j} \le 1$. By (5.30), (5.31) and Theorem 3.2, we conclude that there is $A_{5.32} < \infty$ not depending on N such that for such j and k,

$$\mathrm{P}\left(\mathbf{G}_k \cap \mathbf{H}_k \,\middle|\, \mathbf{G}_j \cap \mathbf{H}_j\right) \le \frac{A_{5.32}}{k-j}. \tag{5.32}$$

Next we use (5.24) to estimate $\mathrm{E}[\boldsymbol{\chi}_N^2]$ as follows:

$$\begin{aligned} \mathrm{E}\left[\boldsymbol{\chi}_N^2\right] &\le 2 \sum\sum_{1\le j\le k\le N} \mathrm{P}\left(\mathbf{G}_j\right) \mathrm{P}\left(\mathbf{G}_k \cap \mathbf{H}_k \mid \mathbf{G}_j \cap \mathbf{H}_j\right) \\ &\le 2A_{5.24}^{-1} \sum\sum_{1\le j\le k\le N} \frac{\mathrm{P}\left(\mathbf{G}_k \cap \mathbf{H}_k \mid \mathbf{G}_j \cap \mathbf{H}_j\right)}{j}. \end{aligned} \tag{5.33}$$

We split this double-sum into two parts according to the value of the variable k: Where $j \le k \le c + j + \gamma\log(j)$ and where $c + j + \gamma\log(j) \le k \le N$. For the first part, we estimate the conditional probability by one, and for the second part by

(5.32). This yields

$$\begin{aligned}\mathrm{E}\left[\boldsymbol{\chi}_N^2\right] &\le 2A_{5.24}^{-1}\sum_{1\le j\le N}\frac{c+j+\gamma\log(j)}{j}\\ &\quad + 2A_{5.24}^{-1}A_{5.32}\sum_{1\le j\le N}\ \sum_{c+j+\gamma\log(j)\le k\le N}\frac{1}{j(k-j)}\\ &= O\left(\log^2 N\right)\quad (N\to\infty).\end{aligned}\tag{5.34}$$

This establishes (5.29).

Now by the Paley–Zygmund inequality [Kho02, Lemma 1.4.1, p. 72], (5.28) and (5.29),

$$\mathrm{P}\left\{\boldsymbol{\chi}_N \ge \frac{A_{5.24}^{-1}}{2}\log N\right\} \ge \mathrm{P}\left\{\boldsymbol{\chi}_N \ge \frac{1}{2}\mathrm{E}\left[\boldsymbol{\chi}_N\right]\right\} \ge \frac{1}{4}\frac{(\mathrm{E}\left[\boldsymbol{\chi}_N\right])^2}{\mathrm{E}(\boldsymbol{\chi}_N^2)},\tag{5.35}$$

and this is bounded away from zero, uniformly for all large N. Therefore, $\mathrm{P}\{\boldsymbol{\chi}_\infty = +\infty\}$ is positive, and hence is one by the Hewitt–Savage zero-one law. That is, for each fixed $\varepsilon \in (0,1)$ and $0<a<b$, with probability one there are infinitely many n's such that

$$\overline{X([a,b]\times[n,\infty))}\cap\mathcal{B}_\varepsilon \ne \varnothing.\tag{5.36}$$

Let $\tilde{\Gamma}_\varepsilon^n$ and Γ_ε^n be as in (5.14). By (5.36),

$$\mathrm{P}\left\{^\forall n\ge 1,\ ^\forall\varepsilon\in\mathbb{Q}_+,\ [a,b]\cap\tilde{\Gamma}_\varepsilon^n\ne\varnothing\right\}=1,\tag{5.37}$$

which is analogous to (5.15). We now use the Baire Category argument that follows (5.15) to conclude that with probability one, there are uncountably many $s\in[a,b]$ such that for all $\varepsilon\in(0,1)$ and for infinitely-many n's, there exists $t\ge n$ such that $X(s,t)\in\mathcal{B}_\varepsilon$. Because with probability one this holds simultaneously for all rational intervals $[a,b]\subset(0,\infty)$, $\mathfrak{L}_{2\alpha,\alpha}$ is everywhere-dense and Theorem 1.1 is proved.

Appendix A. Isotropic Stable Sheets and the Stable Noise

Throughout this appendix, $\alpha\in(0,2]$ is held fixed, and $\mathcal{S}_\alpha$ denotes an isotropic stable random variable in $\mathbb{R}^d$; i.e., $\mathcal{S}_\alpha$ is infinitely-divisible, and

$$\mathrm{E}\left[e^{it\cdot\mathcal{S}_\alpha}\right]=e^{-\frac{1}{2}\|t\|^\alpha},\qquad\text{for all } t\in\mathbb{R}^d,\tag{A.1}$$

where $\|t\|^2:=t_1^2+\cdots+t_d^2$.

Here, we collect (and outline the proofs of) some of the basic facts about stable sheets of index $\alpha\in(0,2)$. More details can be found within Adler and Feigin [AdF84], Bass and Pyke [BaP84, BaP87], Dalang and Walsh [DW92b, Sections 2.2–2.4], Dalang and Hou [DaH97, §2]. Related facts can be found in Bertoin [Ber96, pp. 11–16], Dalang and Walsh [DW92a], and Dudley [Dud69].

Let us parametrize $x\in\mathbb{R}^d$ as $x:=r\varphi$ where $r:=\|x\|>0$ and $\varphi\in\mathbb{S}^{d-1}:=\{y\in\mathbb{R}^d:\ \|y\|=1\}$. Then given any $\alpha\in(0,2)$, let $\nu_\alpha(dx)$ be the measure on $\mathbb{R}^d$ such that

$$\int f(x)\,\nu_\alpha(dx):=\int_0^\infty dr\int_{\mathbb{S}^{d-1}}\sigma_d(d\varphi)\,\mathbf{c}r^{-\alpha-1}\,f(r,\varphi),\tag{A.2}$$

where σ_d denotes the uniform probability measure on $\mathbb{S}^{d-1}$, and $\mathbf{c} := \mathbf{c}(d, \alpha) > 0$ is the following normalizing constant:

$$\mathbf{c} := \left[2 \int_0^\infty \int_{\mathbb{S}^{d-1}} \left(\frac{1 - e^{ir\varphi_1}}{r^{1+\alpha}} \right) \sigma_d(d\varphi)\, dr \right]^{-1}. \tag{A.3}$$

It is easy to see that $\mathbf{c} \in \mathbb{R}$, and hence,

$$\begin{aligned} \mathbf{c} &:= \left[2 \int_{\mathbb{S}^{d-1}} \int_0^\infty \left(\frac{1 - \cos(r\,|\varphi_1|)}{r^{1+\alpha}} \right) dr\, \sigma_d(d\varphi) \right]^{-1} \\ &= -\frac{1}{\pi} \Gamma(1+\alpha) \cos\left(\frac{\pi(1+\alpha)}{2} \right) \left[\int_{\mathbb{S}^{d-1}} |\varphi_1|^{1+\alpha}\ \sigma_d(d\varphi) \right]^{-1}. \end{aligned} \tag{A.4}$$

This choice of $\mathbf{c}$ makes ν_α out to be the Lévy measure of $\mathcal{S}_\alpha$ with normalization given by (A.1); cf. also [Ber96, p. 11–16].

Next consider the Poisson point process $\Pi := \{(s, t, \mathsf{e}(s,t));\ s, t \geq 0\}$ whose characteristic measure is defined as $ds \times dt \times \nu_\alpha(dx)$ $(s, t \geq 0,\ x \in \mathbb{R}^d)$. Since this characteristic measure is locally finite on $\mathbb{R}_+ \times \mathbb{R}_+ \times (\mathbb{R}^d \setminus \{0\})$, Π can be identified with a purely atomic Poisson random measure,

$$\Pi_{s,t}(G) := \#\left\{ (u,v) \in \mathbb{R}_+^2 :\ u \leq s,\ v \leq t,\ \mathsf{e}(u,v) \in G \right\}, \tag{A.5}$$

where $(s,t) \in \mathbb{R}_+^2$ and $G \subset \mathbb{R}^d$ is a Borel set. We note that $\Pi_{s,t}(G)$ is finite for all G such that $\nu_\alpha(G) < +\infty$, which is equivalent to the condition that the distance between G and $0 \in \mathbb{R}^d$ is strictly positive.

Next define

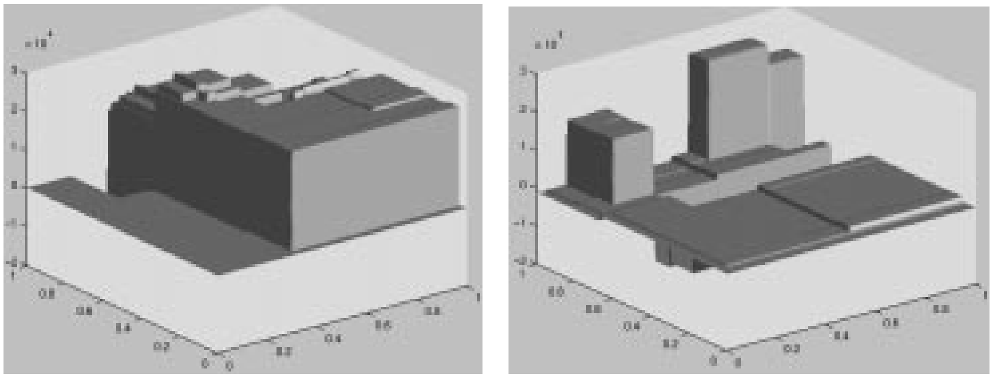
$$\begin{aligned} Y(s,t) &:= \sum\sum_{(u,v) \in [0,s] \times [0,t]} \mathsf{e}(u,v) \mathbf{1}_{\{\|\mathsf{e}(u,v)\| \geq 1\}}, \\ Z^\delta(s,t) &:= \sum\sum_{(u,v) \in [0,s] \times [0,t]} \mathsf{e}(u,v) \mathbf{1}_{\{\delta \leq \|\mathsf{e}(u,v)\| < 1\}}, \\ W^\delta(s,t) &:= Z^\delta(s,t) - \mathrm{E}\left\{ Z^\delta(s,t) \right\}, \end{aligned} \tag{A.6}$$

for all $s, t \geq 0$ and $\delta \in (0,1)$. Since $s \mapsto \Pi_{s,\bullet}(\bullet)$ is an ordinary one-parameter Poisson process, and because the (infinite-dimensional) compound Poisson processes $s \mapsto Y(s, \bullet)$ and $s \mapsto W^\delta(s, \bullet)$ do not jump simultaneously, they are independent; cf. [Ber96, Proposition 1, p. 5].

For any $\eta \in (0, \delta)$, consider

$$\begin{aligned} &\mathrm{E}\left\{ \sup_{(u,v) \in [0,s] \times [0,t]} \left\| W^\delta(u,v) - W^\eta(u,v) \right\|^2 \right\} \\ &\leq 16\mathrm{E}\left\{ \left\| W^\delta(s,t) - W^\eta(s,t) \right\|^2 \right\} \\ &= 16st \int_{\eta \leq \|x\| < \delta} \|x\|^2\, \nu_\alpha(dx). \end{aligned} \tag{A.7}$$

The inequality follows from Cairoli's maximal L^2-inequality [Kho02, Th. 1.3.1(ii), p. 222], and the readily-checkable fact that $(s,t) \mapsto W^\delta(s,t)$ is a two-parameter martingale with respect to the commuting filtration generated by the process e. The equality is a straight-forward about the variance of the sum of mean-zero $L^2(\mathrm{P})$-random variables. Since $\int (1 \wedge \|x\|^2)\, \nu_\alpha(dx) < +\infty$, we have shown that $\eta \mapsto W^\eta(s,t)$ is a Cauchy sequence in $L^2(\mathrm{P})$, uniformly over (s,t) in a compact set.



(a) $(d, \alpha) = (1, 0.8)$ (b) $(d, \alpha) = (1, 0.6)$

FIGURE 1. The $d > \alpha$ Case

Now we compute characteristic functions directly to deduce the following [DW92b, Th. 2.3]:

Proposition A.1. *If $\alpha \in (0, 2)$, then the process $X := \{X(s,t);\ s,t \geq 0\}$ defined by*

$$X(s,t) = Y(s,t) + \lim_{\delta \downarrow 0} W^{\delta}(s,t) \tag{A.8}$$

is well-defined. Here, the limit exists uniformly over (s,t) in compact subsets of $\mathbb{R}^2_+$, a.s., and:

(1) *For all $s,t,r,h \geq 0$, $\Delta_{r,h}(s,t)$ is independent of $\{X(u,v);\ (u,v) \in [0,s] \times [0,t]\}$, where*

$$\Delta_{r,h}(s,t) := X(s+r, t+h) - X(s+r, t) - X(s, t+h) + X(s,t). \tag{A.9}$$

(2) *For all $s,t,r,h, \geq 0$, $\Delta_{r,h}(s,t)$ has the same distribution as $(rh)^{1/\alpha}\mathcal{S}_\alpha$.*

The process X is termed a *two-parameter isotropic α-stable Lévy sheet*. Note that the case $\alpha = 2$ is substantially different: The process X is continuous and is the classical *Brownian sheet* [DW92b, Prop. 2.4]. For various α, a simulation of the sample paths of X is shown in Figures 1 and 2. These simulations are explained in Appendix B.

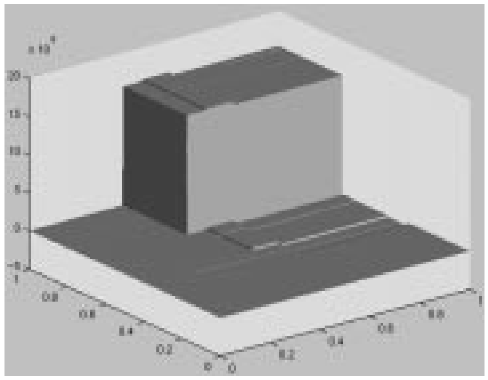
Many of the regularity features of the samples of Y and W^δ automatically get passed onto the sample functions of X, as can be seen from the construction of X. In particular, we have the following [DW92b, §2.4]:

Proposition A.2. *The process X a.s. has the following regularity properties:*

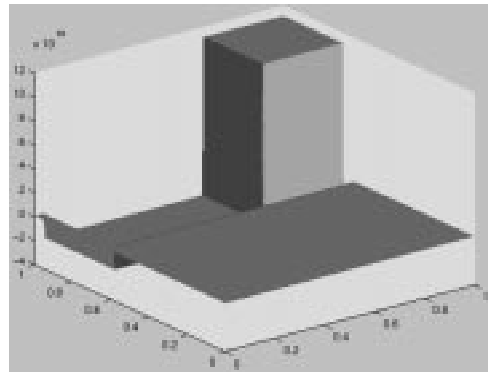
(1) *X is right-continuous with limits in the other three quadrants.*
(2) *$\square X(s,t) = 0$ except for a countable set of (random) points $(s_n, t_n) \in \mathbb{R}^2_+$, where*

$$\square X(s,t) = X(s,t) - X(s-,t) - X(s,t-) + X(s-,t-). \tag{A.10}$$

(3) *If $\square X(s_n, t_n) = x$, then $X(s_n, t) - x(s_n-, t) = x$ for all $t \geq t_n$, and $X(s, t_n) - x(s, t_n-) = x$ for all $s \geq s_n$.*
(4) *The sample paths of X have no other discontinuities than those in 2. and 3. In particular:*



(a) $(d, \alpha) = (1, 0.5)$



(b) $(d, \alpha) = (1, 0.4)$

FIGURE 2. The $d \leq \alpha$ Case

(5) *the set* $\{s \geq 0 : \exists t \geq 0 : X(s,t) \neq X(s-,t)\}$ *is countable;*
(6) *the set* $\{t \geq 0 : \exists s \geq 0 : X(s,t) \neq X(s,t-)\}$ *is countable.*

Finally, we mention a few facts about the isotropic stable noise. Fix $\alpha \in (0,2]$ fixed (with $\alpha = 2$ allowed), and define

$$\mathfrak{X}\left([s, s+r] \times [t, t+h]\right) := \Delta_{r,h}(s,t), \qquad \text{for all } s,t,r,h \geq 0. \tag{A.11}$$

This can easily be extended, by linearity, to construct a finitely-additive random measure on the algebra generated by rectangles of the form $[s, s+r] \times [t, t+h]$. The extension to Borel subsets of $\mathbb{R}^2_+$—that we continue to write as $\mathfrak{X}$—is the socalled *isotropic stable noise* of index α (in $\mathbb{R}^d$). It is a.s. a genuine random measure on the Borel subsets of $\mathbb{R}^d$ if and only if $\alpha \in (0,1)$.

Appendix B. Simulating Stable Processes

B.1. **Some Distribution Theory.** One simulates one-dimensional symmetric stable sheets of Figures 1 and 2 by first simulating positive stable random variables; these generate the law of stable subordinators. The basic idea is to use a representation of Kanter [Kan75], which relies on the so-called *Ibragimov–Chernin* function [IbC59],

$$\mathrm{IC}_\alpha(v) := \frac{\sin(\pi(1-\alpha)v)(\sin(\pi\alpha v))^{\alpha/(1-\alpha)}}{(\sin(\pi v))^{1/(1-\alpha)}}, \qquad \text{for all } \alpha, v \in (0,1). \tag{B.1}$$

We then have:

Proposition B.1 (Kanter [Kan75])**.** *If* $\alpha \in (0,1)$ *and* U *and* V *are independent, and uniformly distributed on* $[0,1]$*, then* $W := |\mathrm{IC}_\alpha(U)/\ln(V)|^{(1-\alpha)/\alpha}$ *has a positive* α*-stable distribution with characteristic function*

$$\phi_W(t) = \exp\left(-|t|^\alpha e^{-\frac{1}{2}i\pi\alpha \mathrm{sign}(t)}\right), \qquad \textit{for all } \alpha \in (0,1). \tag{B.2}$$

One then uses Bochner's subordination [Kho02, Th. 3.2.2, p. 379] to simulate symmetric α-stable random variables for any $\alpha \in (0,2]$. Formally, this is:

Proposition B.2 (Bochner's subordination)**.** *Suppose X and Y are independent, Y is a positive α-stable variable whose characteristic function is in (B.2), and X is a centered normal variate with variance* 2*. Then the characteristic function of $Z := X\sqrt{e^{-\pi}Y}$ is $\phi_Z(t) = \exp(-|t|^\alpha)$, and Z is symmetric. That is, Z is symmetric α-stable.*

Note that the simulations here generate variates with characteristic function $\phi(t) = \exp(-|t|^\alpha)$ instead of $\exp(-\frac{1}{2}|t|^\alpha)$. The adjustment is simple, though unnecessary for us, and we will not bother with this issue.

B.2. **Simulating Symmetric Stable Sheets.** In order to simulate the sheet, we run a two-parameter random walk with symmetric α-stable increments. That is, let $\{\xi_{i,j}\}_{i,j\geq 1}$ denote i.i.d. symmetric α-stable random variables, and approximate the symmetric α-stable sheet $X(s,t)$, in law, by $n^{-2/\alpha}S^n_{\lfloor ns\rfloor,\lfloor nt\rfloor}$, where

$$S^n_{k,\ell} := \sum_{1\leq i\leq k}\ \sum_{1\leq j\leq \ell} \xi_{i,j}$$

is a two-parameter random walk. It is easy to see that as $n\to\infty$,

$$n^{-2/\alpha}S^n_{\lfloor ns\rfloor,\lfloor nt\rfloor} \xrightarrow{(d)} X(s,t) \tag{B.3}$$

in the sense of finite-dimensional distributions. By this weak approximation result, for large n, the two-parameter random walk yields a good approximation of the stable sheet. A simulation of the random walk produces the pictures in Figures 1 and 2.

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