ANNALES DE L'INSTITUTE HENRI POINCARÉ, **29**(3), 419–451. (1993)

Intersection Local Times and Tanaka Formulas

Richard F. Bass [1] Davar Khoshnevisan

Department of Mathematics
University of Washington
Seattle, Washington 98195

Summary. A new approach to intersection local times of Brownian motion is given, using additive functionals of a single Markov process and stochastic calculus. New results include the Tanaka formula for the k–multiple points of self–intersection local time and the joint Hölder continuity in all variables of renormalized self–intersection local time for k–multiple points, $k \geq 4$.

Résumé Nous donnons une nouvelle approche à l'étude des temps locaux d'intersection du mouvement brownien. Elle se sert de la théorie de fonctionelles additives d'un seul processus de Markov et de calcul stochastique. Parmi les resultats nouveaux sont la formule de Tanaka pour les points de multiplicité k de temps locaux d' intersection et la continuité dans toutes les variables du temps locaux d'intersection renormalizés pour les points de multiplicité k, $k \geq 4$.

Short title: Intersection local times

French title: Temps locaux d'intersection et formules de Tanaka

[1] Research partially supported by NSF grant

ANNALES DE L'INSTITUTE HENRI POINCARÉ, **29**(3), 419–451. (1993)

Key words: Intersection local times, Tanaka formula, renormalization, Brownian motion, diffusions, multiple points

1. Introduction. It has been known for quite some time that 3–dimensional Brownian motion has double points and that 2–dimensional Brownian motion has k–multiple points for every positive integer k. It has been known for not quite as long a time that one can construct a local time for these multiple points, that is, a functional that increases only at the times when Brownian motion has a multiple point and that measures in some sense how many of these times there are. These intersection local times (ILTs) have been constructed by means of Fourier analysis, by means of stochastic calculus, and by the study of additive functionals of several Markov processes. Through the work of Dynkin, LeGall, Rosen, Yor, and others, a great deal is now known about ILTs; see [D1], [L], [R3], [RY], and the references therein.

One of the main purposes of this paper is to introduce a new method of approaching the study of ILTs, via a combination of the theory of additive functionals for a single Markov process and stochastic calculus. This new method allows us to obtain, if not easily, at least systematically, many of the known results about ILTs. We concentrate primarily on Brownian motion in this paper, but the method also works for other diffusions and, to some extent, stable processes (see Section 10).

In addition to discussing our method, we obtain some new results as well. For example, we obtain the Tanaka formulas for self-intersections of 2–dimensional Brownian motion of order k for any k (Sections 7, 8). These had previously been known only when $k = 2, 3$; see [Y], [RY]. (While we were writing up this paper, we learned of the preprint of Shieh [Sh] who had also obtained the Tanaka formulas for any k by using white noise analysis.)

Using these Tanaka formulas we prove that one can renormalize ILT for k-multiple points in terms of lower order ILTs in such a way that the renormalized ILT is jointly Hölder continuous in every variable almost surely. This had been previously known only when $k = 2, 3$ (see [D2, R1]). For other k various sorts of renormalizations were known, but the almost sure joint continuity of any of these renormalizations had been an open problem.

We also can obtain both weak and strong invariance principles that are uniform over all levels x for the convergence of ILTs of lattice valued random walks satisfying suitable moment conditions; these can be found in [BK1] and [BK2]. Finally, we mention that one can use our method to construct a local time for intersections that occur in certain random sets (see Section 10).

The basic idea is simple. Let us first consider the intersection of two independent Brownian motions X_t, Y_t. Fix u and define the (random) measure

$$\mu(A) = \int_0^u 1_A(X_s)\, ds.$$

Note μ is supported on the path of X_s. Elementary estimates show that a.s. the measure μ is sufficiently regular so that there is an additive functional of Brownian motion associated to it. That additive functional (for Y_t) is ILT for intersections of X_t and Y_t. Slightly more complicated measures give rise to ILTs for 3 or more Brownian motions. To get self–ILT for a single Brownian motion, we partition $[0, u]$ by points $s_0, s_1, \ldots, s_n$, we look at the intersections of $X_t, s_i \leq t < s_{i+1}$, with $X_r, 0 \leq r < s_i$, we sum over i, and we then prove we get convergence as we let the partition become finer.

In Section 2, we construct ILTs for the intersection of 2 independent Brownian motions, while in Section 3 we do the same for the intersection of k independent Brownian motions. In Section 4 we obtain the Tanaka formula for these ILTs. Section 5 has some estimates on certain potentials, and Section 6 contains some preliminaries on the Hölder continuity of processes. Section 7 has the construction of ILTs for double points of a single Brownian motion and also the derivation of the Tanaka formula; Section 8 considers multiple points of a single Brownian motion. The proof of the a.s. joint continuity of renormalized ILTs is in Section 9. Finally, in Section 10, we discuss other diffusions, stable processes, and ILTs in random sets.

The letter c, with or without subscripts, will denote constants whose exact value is unimportant and may change from line to line. The open ball of radius s about the point y is denoted $B(y, s)$.

2. Intersections of 2 Brownian Motions. Let X_t, Y_t be two independent Brownian motions in $\mathbb{R}^d, d = 2$ or 3. If $d = 3$, let $g(x, y)$ denote the Green function of Brownian motion. If $d = 2$, let $g_R(x, y)$ denote the Green function of Brownian motion killed on exiting the ball $B(0, R)$.

Let

$$T_R = T_R(X) = \inf\{t\colon | X_t | \geqslant R\}.$$

For each $x \in \mathbb{R}^d$ and $u \leqslant 1$, define the random measure $\mu_{x,u}$ by

$$\mu_{x,u}(A) = \int_0^u 1_A(X_r + x)\, dr. \tag{2.1}$$

Lemma 2.1. *For each $\varepsilon \in (0, 1]$, for almost all ω there exists $K_\varepsilon(\omega)$ such that*

$$\mu_{x,u}\left(B(y, s)\right)(\omega) \leqslant K_\varepsilon(\omega)\left(s^{2-\varepsilon} \wedge 1\right) \tag{2.2}$$

for all $y \in \mathbb{R}^d$.

Proof. Since $\mu_{x,u}(\mathbb{R}^d) \leqslant u$, we may assume $s \leqslant 1/2$. Let $R \geqslant 2 + 2 \mid x \mid$ and let

$$A_t = \int_0^{t \wedge T_R} 1_{B(y,s)}(X_r + x) dr.$$

If $d=2, g_R(w,z) \leqslant c(1 \vee \log(1/\mid w-z \mid))\, dz$, and so if $w \in B(0,R)$,

$$E^w A_{T_R} \leqslant c \int_{B(y-x,s)} (1 \vee \log(1/\mid w-z \mid)) dz \leqslant c \int_{B(0,s)} \log(1/\mid z \mid) dx \leqslant cs^2 \log(1/s).$$

A similar calculation for $d=3$ gives $E^w A_{T_R} \leqslant cs^2$.

Since A_t is an additive functional, the above implies

$$E^0[A_{T_R} - A_t | \mathcal{F}_t] \le E^{X_t} A_{T_R} \leqslant \sup_w E^w A_{T_R} \le cs^{2-\varepsilon/2}.$$

By [DM, p. 193], $E^0 \exp(\lambda A_{T_R}) \leqslant 2$ if $\lambda \leqslant 1/8 \sup_w E^w A_{T_R}$. Using Chebyshev, we get

$$P^0\left(A_{T_R} > c_1 s^{2-\varepsilon}\right) \leqslant 2 \exp\left(-c_2 s^{-\varepsilon/2}\right). \tag{2.3}$$

Now $B(0,3R)$ can be covered by $N = cs^{-d}$ balls of radius $2s$, say $B_1, \cdots, B_N$, so that every ball $B(y,s), y \in B(0,2R)$, is contained in one of the B_i's. Writing

$$D_R = \{\sup_{t \leqslant 1} \mid X_t \mid \leqslant R\},$$

(2.3) yields

$$\begin{aligned} P^0(\mu_{x,u}(B(y,s)) \geqslant c_1 s^{2-\varepsilon} &\text{ for some } y \in B(0,2R); D_R) \\ &\leqslant P^0(\mu_{x,u}(B_i) \geqslant c_1 s^{2-\varepsilon} \text{ for some } i = 1, \cdots, N; D_R) \\ &\leqslant c_2 s^{-d} \exp(-c_3 s^{-\varepsilon/2}). \end{aligned}$$

By a straightforward Borel-Cantelli argument with $s = 2^{-i}, i = 0, 1, 2, \cdots$,

$$P^0(\text{for some } y \in B(0,2R),\ \mu_{x,u}(B(y,2^{-i}))/(2^{-i})^{2-\varepsilon} > c, \text{ i.o. }; D_R) = 0.$$

Hence, if $\omega \in D_R$, then for some $K_{\varepsilon R}(\omega)$,

$$\mu_{x,u}(B(y,2^{-i})) \leqslant K_{\varepsilon R}(\omega)(2^{-i})^{2-\varepsilon}$$

for all $y \in B(0,2R), i = 0,1,2,\cdots$. If $s \in (0,1]$, then $s \in (2^{-(i+1)}, 2^{-i}]$ for some i. So, provided $\omega \in D_R$,

$$\mu_{x,u}(B(y,s)) \leqslant K_{\varepsilon R}(\omega)(2^{-i})^{2-\varepsilon} \leqslant cK_{\varepsilon R}(\omega) s^{2-\varepsilon}. \tag{2.4}$$

for all $y \in B(0,2R)$, all $s \in (0,1]$. If $\omega \in D_R, \mu_{x,u}(B(y,s)) = 0$ if $y \notin B(0,2R)$.

Finally, each $\omega \in D_R$ for some R sufficiently large (except for a null set). This observation with (2.4) yields (2.2). □

Define $\mathcal{L} = \{\psi\colon \psi$ maps $\mathbb{R}^d$ to $[-1,1]$, $\|\psi\|_\infty \le 1$, and ψ is Lipschitz with Lipschitz constant 1$\}$. Define $d_L(\mu,\nu) = \sup\{|\int \psi d\mu - \int \psi d\nu| \colon \psi \in \mathcal{L}\}$.

Lemma 2.2.

(a) $d_L(\mu_{x,u}, \mu_{x,v}) \leqslant |\, u - v \,|$;

(b) $d_L(\mu_{x,u}, \mu_{y,u}) \leqslant u \,|\, x - y \,|$.

Proof. (a) is obvious. For (b), notice

$$|\int \psi \, d(\mu_{x,u} - \mu_{y,u})| = |\int_0^u [\psi(X_t + x) - \psi(X_t + y)]dt| \leqslant u \,|\, x - y \,|$$

since $\psi \in \mathcal{L}$. □

Let $\alpha_2(x, \cdot, u)$ be the continuous additive functional of Y_t associated with $\mu_{x,u}$, that is, the continuous additive functional such that $E^z \alpha_2(x, T_R(Y), u) = g_R \mu_{x,u}(z)$ for all z and R (see [BG]). In stochastic calculus terms, $\alpha_2(x, \cdot, u)$ is the increasing part of the supermartingale $g_R \mu_{x,u}(Y_{t \wedge T_R(Y)})$.

We will show that α_2 is jointly Hölder continuous in each variable. Before doing so, we need

Proposition 2.3. *Suppose* $c, \gamma > 0$ *and* μ *is a positive measure satisfying* $\mu(B(y,s)) \leqslant c(s^{d-2+\gamma} \wedge 1)$ *for all* $s \in (0, \infty), y \in \mathbb{R}^d$. *Let* L_t^μ *be the associated continuous additive functional. Then* L_t^μ *is Hölder continuous in* t*, a.s.*

Remark. See [BK1, Section 2] for the contruction of L_t^μ.

Proof. That L_t^μ is nondecreasing and continuous follows from its construction. So we only need the Hölder continuity. Let g be the Green function if $d \geqslant 3$, the Green function killed at an independent exponential if $d = 2$. By [BK1, Proposition 2.7], $g\mu$ is Hölder continuous. Hence,

$$E^x \,|\, g\mu(X_t) - g\mu(X_0) \,| \leqslant cE^x \,|\, X_t - X_0 \,|^\alpha \leqslant ct^{\alpha/2}$$

for some $\alpha > 0$, using the Burkholder–Davis–Gundy inequalities [ReY, p. 151] Since $g\mu(X_t) - g\mu(X_0) + L_t^\mu$ is a mean 0 martingale, $E^x L_h^\mu \leqslant ch^{\alpha/2}$, independent of x. By the argument of the first part of Lemma 2.1,

$$P^x(L_h^\mu \geqslant c_1 h^{\alpha/2-\varepsilon}) \leqslant c_2 \exp(-c_3 h^{-\varepsilon}).$$

Using the Markov property,

$$P^x(L_{t+h}^\mu - L_t^\mu \geqslant c_1 h^{\alpha/2-\varepsilon}) \leqslant c_2 \exp(-c_3 h^{-\varepsilon}).$$

Our result now follows by standard metric entropy (i.e., chaining) arguments. □

Theorem 2.4. *There is a version of $\alpha_2(x,r,u)$ that is jointly Hölder continuous in x,r,u.*

Proof. It is enough to let $R \geqslant 1$ be arbitrary and to show Hölder continuity for $|\,x\,| \leqslant R$. In view of Lemma 2.2, the d_L–metric entropy $\mathcal{H}(\delta)$ of $\{\mu_{x,u}\colon x \in B(0,R), u \in (0,1]\}$ satisfies $\mathcal{H}(\delta) \leqslant c\log(1/\delta)$. By applying Theorem 2.9 of [BK1], there exists a version of $\alpha_2(x,r,u)$ that is jointly Hölder continuous in x and u. The Hölder continuity in r follows by Proposition 2.3. □

The question that remains is whether $\alpha_2(x,r,u)$ is actually what one means by ILT.

Theorem 2.5. *There exists a null set N such that if $\omega \notin N$, then*

$$\int_{\mathbb{R}^d} f(x)\alpha_2(x,r,u)(\omega)dx = \int_0^u \int_0^r f(Y_s(\omega) - X_t(\omega))ds\,dt \tag{2.5}$$

for all bounded measurable f.

Proof. Suppose $d = 2$ and suppose f, h are continuous with compact support. Let

$$B_u^{x,h} = \int_0^u h(X_t - x)\,dt.$$

The potential of $B_u^{x,h}$ on the ball of radius R is

$$E^z B_{T_R}^{x,h} = \int g_R(z,y)h(y-x)\,dy.$$

So the potential of $\int f(x)B_u^{x,h}dx$ is

$$\int\int g_R(z,y)f(x)h(y-x)\,dy\,dx = \int\int g_R(z,y)h(x)f(y-x)\,dy\,dx,$$

which is the potential of $\int h(x)B_u^{x,f}\,dx$. If two additive functionals of Brownian motion have the same potential, they are equal ([BG]). Hence

$$\int f(x)B_u^{x,h}dx = \int h(x)B_u^{x,f}\,dx, \qquad \text{a.s.},$$

or

$$\int f(x)\left(\int h(y)\mu_{-x,u}(dy)\right)dx = \int h(x)\left(\int f(y)\mu_{-x,u}(dy)\right)dx. \tag{2.6}$$

Now the right-hand side of (2.5) equals $\int_0^r \left(\int f(-y)\mu_{-Y_s,u}(dy)\right)ds$. So its potential in $B(0,R)$, considered as a continuous additive functional of Y, is

$$\int g_R(z,y)\left(\int f(-w)\mu_{-y,u}(dw)\right)dy.$$

By (2.6), this equals

$$\int f(-x)\int g_R(z,y)\mu_{-x,u}(dy)dx = \int f(x)g_R\mu_{x,u}(z)dx,$$

which is the potential of the left-hand side of (2.5). Since R is arbitrary, this proves (2.5) when $d=2$ for this particular f. The case $d=3$ is similar but easier. Let N_f be the null set.

Let $\{f_i\}$ be a countable dense subset of the bounded continuous functions on $\mathbb{R}^d$ and let $N=\bigcup_i N_{f_i}$. If $\omega\notin N$, then by taking limits, (2.5) holds for bounded continuous f. It then holds for all bounded measurable f by a monotone class argument. □

3. Intersections of k Brownian Motions. In this section, we require $d=2$. We construct ILTs for k Brownian motions by induction. Denote the measures $\mu_{x,u}$ of Section 2 by $\mu^2_{x,u}$. Suppose $k\geqslant 3$. Let $X^1_t,\cdots,X^{k-1}_t$ be $k-1$ independent Brownian motions and let Y_t be an additional independent Brownian motion. Suppose we have measures $\mu^{k-1}_{x_{k-1},r_1,\cdots,r_{k-2}}$ (denoted μ^{k-1} when no confusion results) and associated continuous additive functionals $\alpha_{k-1}(x_1,\cdots,x_{k-2},r_1,\cdots,r_{k-2},r_{k-1})$ satisfying

(3.1) for each ε there exists $K_\varepsilon(\omega)$ such that

$$\mu^{k-1}(B(y,s))\leqslant K_\varepsilon(\omega)(s^{2-\varepsilon}\wedge 1)$$

for all $y\in\mathbb{R}^2$, $s\in(0,\infty)$, and

(3.2)$\alpha_{k-1}(x_1,\cdots,x_{k-2},r_1,\cdots,r_{k-2})$ is Hölder continuous in each variable.

Define the random measure $\mu^k=\mu^k_{x_{k-1},r_1,\cdots,r_{k-1}}$ by

$$\mu^k(A)=\int_0^{r_{k-1}} 1_A(X^{k-1}_t+x_{k-1})\alpha_{k-1}(x_1,\cdots,x_{k-2},r_1,\cdots,r_{k-2},dt).$$

We need the analog of Lemma 2.1.

Lemma 3.1. *Suppose (3.1) and (3.2) hold. If $\varepsilon>0$, there exists $K_1(\omega)$ such that*

$$\mu^k(B(y,s))\leqslant K_1(\omega)(s^{2-\varepsilon}\wedge 1)$$

for all $s\in(0,\infty),y\in\mathbb{R}^2$.

Proof. Define the additive functional A_t of X^{k-1}_t by

$$A_t=\int_0^t 1_{B(y,s)}\left(X^{k-1}_r+x_{k-1}\right)\alpha_{k-1}\left(x_1,\cdots,x_{k-2},r_1,\cdots,r_{k-2},dr\right). \tag{3.3}$$

Since the potential of α_{k-1} on $B(0,R)$ (considered as an additive functional of X_t^{k-1}) is $g_R\mu^{k-1}$, then the potential of A_t (conditional on the processes $X^1,\dots,X^{k-1}$) is

$$\int g_R(w,z)1_{B(y,s)}(z)\mu^{k-1}(dz).$$

By Hölder's inequality with $p^{-1}+q^{-1}=1$, this is less than or equal to

$$\left(\int g_R(w,z)^p\mu^{k-1}(dz)\right)^{1/p}\left(\int 1_{B(y,s)}(z)\mu^{k-1}(dz)\right)^{1/q}. \tag{3.4}$$

The second term in the product is bounded by $(K_{\varepsilon/3}(\omega)(s^{2-\varepsilon/3}\wedge 1))^{1/q}$, using (3.1). For the first term in the product, we write

$$\begin{aligned}\int g_R(w,z)^p\mu^{k-1}(dz) &\leqslant c\int(1\vee\log\mid w-z\mid)^p\mu^{k-1}(dz)\\ &\leqslant c\sum_{j=0}^{\infty}\int_{2^{-j}\leqslant|w-z|\leqslant 2^{-j+1}}(1\vee\log\mid w-z\mid)^p\mu^{k-1}(dz)+c\mu^{k-1}(\mathbb{R}^2)\\ &\leqslant c\sum_{j=0}^{\infty}(j+1)^p\mu^{k-1}(B(w,2^{-j}))+c\mu^{k-1}(\mathbb{R}^2)\\ &\leqslant c(\omega),\end{aligned}$$

using (3.1). Taking q sufficiently close to 1, we get that the potential of A, conditional on the processes $X^1,\dots,X^{k-1}$, is bounded by $c(\omega)(s^{2-\varepsilon/2}\wedge 1)$.

Using this estimate, we now proceed in a fashion very similar to Lemma 2.1. □

Theorem 3.2. *For each k, a version of α_k exists that is jointly Hölder continuous in each variable.*

Proof. The proof is by induction. Suppose (3.1) and (3.2) hold. The measures μ^k are Hölder continuous with respect to d_L as a function of $x_1,\cdots,x_{k-2}$ and $r_1,\cdots,r_{k-1}$ by the Hölder continuity of α_{k-1}. The Hölder continuity in x_{k-1} follows as in the proof of Lemma 2.2.

Let $\alpha_k(x_1,\cdots,x_{k-1},r_1,\cdots,r_{k-1})$ be the continuous additive functional of Y_t corresponding to the measure μ^k. The metric entropy of the set $\{\mu^k\colon x_1,\cdots,x_{k-1},\in B(0,R),$ $r_1,\cdots,r_{k-1}\in[0,1]\}$ still is bounded by $c\log(1/\delta)$. So as in the proof of Theorem 2.4, there is a version of α_k that is jointly Hölder continuous in each variable. This establishes (3.2) with $k-1$ replaced by k. Lemma 3.1 establishes (3.1) with $k-1$ replaced by k. So by induction, (3.1) and (3.2) hold for all k. □

Theorem 3.3. *Except for a null set independent of f,*

$$\int \cdots \int f(x_1, \cdots, x_{k-1}) \alpha_k(x_1, \cdots, x_{k-1}, r_1, \cdots, r_k) dx_1 \cdots dx_{k-1}$$
$$= \int_0^{r_k} \cdots \int_0^{r_1} f(X_{t_2}^2 - X_{t_1}^1, \cdots, Y_{t_k} - X_{t_{k-1}}^{k-1}) dt_1 \cdots dt_k,$$

for f bounded and measurable on $(\mathbb{R}^2)^{k-1}$, a.s.

The proof of Theorem 3.3 is very similar to that of Theorem 2.5 and is left to the reader.

4. Tanaka Formulas. The Tanaka formulas for ILTs of independent Brownian motions are actually quite simple. We do the case $d = 2$. Let us suppose $k = 2$ first. Define

(4.1) $$G(x) \equiv \frac{1}{\pi} \log(1/\mid x \mid).$$

Note $G(-x) = G(x)$.

By a formula of Brosamler [Br]

(4.2). $$g_R \mu_{x,u}(Y_{t \wedge T_R}) - g_R \mu_{x,u}(Y_0) = \int_0^{t \wedge T_R} \nabla g_R \mu_{x,u}(Y_s) \cdot dY_s - \alpha_2(x, t \wedge T_R, u).$$

Since $G(\cdot - y) - g_R(\cdot, y)$ is harmonic in $B(0, R)$ for each y, so is $G\mu_{x,u}(\cdot) - g_R\mu_{x,u}(\cdot)$, and we also have by [Br]

(4.3)
$$(G\mu_{x,u} - g_R\mu_{x,u})(Y_{t \wedge T_R}) - (G\mu_{x,u} - g_R\mu_{x,\mu})(Y_0) = \int_0^{t \wedge T_R} \nabla (G\mu_{x,u} - g_R\mu_{x,u})(Y_s) \cdot dY_s.$$

Here

(4.4) $$G\mu_{x,u}(y) = \int G(y - z) \mu_{x,u}(dz).$$

Adding (4.2) and (4.3) and letting $R \to \infty$,

$$G\mu_{x,u}(Y_t) - G\mu_{x,u}(Y_0) = \int_0^t \nabla G\mu_{x,u}(Y_s) \cdot dY_s - \alpha_2(x, t, u).$$

Finally, recalling the definition of $\mu_{x,u}$, this and (4.4) yield

(4.5) $$\int_0^u G(Y_t - X_r - x) dr - \int_0^u G(Y_0 - X_r - x) dr$$
$$= \int_0^t [\int_0^u \nabla G(Y_s - X_r - x) dr] \cdot dY_s - \alpha_2(x, t, u).$$

The argument for ILTs of k Brownian motions is the same, and we get

Theorem 4.1.

(4.6)

$$\int_0^{r_k}[G(Y_t-X_r^{k-1}-x_{k-1})-G(Y_0-X_r^{k-1}-x_{k-1})]\alpha_{k-1}(x_1,\cdots,x_{k-2},r_1,\cdots,r_{k-2},dr)$$
$$=\int_0^{r_{k-1}}[\int_0^{r_k}\nabla G(Y_s-X_r^{k-1}-x_{k-1})\alpha_{k-1}(x_1,\cdots,x_{k-1},r_1,\cdots,r_{k-2},dr)]\cdot dY_s$$
$$-\alpha_k(x_1,\cdots,x_{k-1},r_1,\cdots,r_k).$$

Remark. Recall that the way Brosamler's formulas are proved is by using Ito's formula and taking limits (see also [B1]). Therefore, provided μ is a sufficiently nice measure, we have

$$G\mu(Y_t)-G\mu(Y_0)=\int_0^t\nabla G\mu(Y_s)\cdot dY_s-L_t^\mu$$

whenever $Y_0\in\mathcal{F}_0(Y)$, that is, if Y_0 is independent of $\sigma(Y_s-Y_0\colon s\geqslant 0)$. We will apply this fact in Sections 7 and 8 with μ taken to be μ^k.

5. Some Estimates. Before proceeding to the construction of ILT of double and multiple points of a single Brownian motion, we need some preliminary estimates.

Proposition 5.1. *Suppose $a>0$. Suppose $\beta(t)$ is a nondecreasing continuous process with $\beta(0)\equiv 0$. Suppose for each $p\geqslant 1$ there exists $c(p)$ such that*

$$E[\beta(t)-\beta(s)]^p\leqslant c(p)\mid t-s\mid^{ap},\quad s,t\leqslant 1. \tag{5.1}$$

Let Y_r be 2-dimensional Brownian motion. Then there exists $b>0$ (not depending on p) and constants $c(p)$ such that if $p\geqslant 1, x\in\mathbb{R}^2$, and $\sigma<1$, then

$$P\Big[\int_0^1 1_{B(x,\sigma)}(Y_r)\beta(dr)>\lambda\Big]\leqslant c(p)\frac{\sigma^{bp}}{\lambda^{bp}}. \tag{5.2}$$

Proof. Let us assume $\lambda>2\sigma$, for otherwise the result is trivial. Fix x and define $R_t=\mid Y_t-x\mid$. Let $\varepsilon=1/16$. Let $S_1=\inf\{t\colon R_t\leqslant\sigma\}, T_1=\inf\{t>S_1\colon R_t\geqslant\sigma^{1-\varepsilon}\}, S_{i+1}=\inf\{t>T_i\colon R_t\leqslant\sigma\}$, and $T_{i+1}=\inf\{t>S_{i+1}\colon R_t\geqslant\sigma^{1-\varepsilon}\}$. Let $D_u=\inf\{i\colon S_i>u\}$. So D_u is greater than or equal to the number of upcrossings of $[\sigma,\sigma^{1-\varepsilon}]$ by R_t up to time u.

Since $\log R_t$ is a martingale, by the upcrossing inequality (see, e.g., [Ch, p. 332])

$$\sup_z E^zD_1=E^\sigma D_1\leqslant\frac{E^\sigma\mid\log R_1\mid+\mid\log\sigma\mid}{\mid\log\sigma^{1-\varepsilon}-\log\sigma\mid}\leqslant c_1.$$

By Chebyshev,

$$\sup_z P^z(D_1\geqslant 2c_1)\leqslant 1/2.$$

So by the strong Markov property applied at $\inf\{t: D_t \geqslant 2nc_1\}$,

$$\sup_z P^z\left(D_1 \geqslant 2c_1(n+1)\right) \leqslant \frac{1}{2}\sup_z P^z\left(D_1 \geqslant 2c_1 n\right),$$

which leads to

$$P(D_1 \geqslant n) \leqslant c_2 \exp(-c_3 n), \qquad n \geqslant 1. \tag{5.3}$$

By the strong Markov property applied at S_i and standard estimates on Brownian motion,

$$P\left(T_i - S_i > K\sigma^{2-3\varepsilon}\right) \leqslant P^0\left(T_1 > K\sigma^{2-3\varepsilon}\right) \leqslant c_4 \exp\left(-c_5 K\right). \tag{5.4}$$

Let $h \in [0,1]$. If $\beta((t+h)\wedge 1) - \beta(t) \geqslant Lh^{a/2}$ for some $t \in [0,1]$, then $\beta((j+2)h \wedge 1) - \beta(jh) \geqslant Lh^{a/2}$ for some $j \leqslant [1/h]+1$. Equation (5.1) implies

$$P\left(\beta(t) - \beta(s) \geqslant L \mid t-s \mid^{a/2}\right) \leqslant c(p) \mid t-s \mid^{ap/2} / L^p,$$

and so if $p > p_0 = 8/a$,

$$P\left(\sup_{t\leqslant 1}[\beta((t+h)\wedge 1) - \beta(t)] \geqslant Lh^{a/2}\right) \leqslant c(p)\frac{2}{h}\frac{h^{ap/2}}{L^p} \leqslant c(p)\frac{2h^{ap/4}}{L^p}. \tag{5.5}$$

Note $R_{T_i} \geqslant \sigma^{1-\varepsilon}$ and R_t does not return to the interval $[0,\sigma]$ until time S_{i+1}. So if $Y_r \in B(x,\sigma)$, then $r \in [S_i, T_i]$ for some i. Hence

$$\int_0^1 1_{B(x,\sigma)}(Y_r)\beta(dr) \leqslant \sum_{i=1}^{\infty}[\beta(T_i \wedge 1) - \beta(S_i \wedge 1)]. \tag{5.6}$$

Let $n = [\lambda^d/\sigma^d], K = n^d, h = K\sigma^{2-5\varepsilon}, L = \lambda/2h^{a/2}n$, where d will be chosen in a moment. If the sum on the right-hand side of (5.6) is bigger than λ, then either (a) $D_1 > n$ or (b) $T_i - S_i \geqslant K\sigma^{2-3\varepsilon}$ for some $i \leqslant n$ or (c) $\beta(T_i\wedge 1) - \beta(S_i \wedge 1) > \lambda/2n$ for some $i \leqslant n$. So

$$\begin{aligned} P\Big(\int_0^1 1_{B(x,\sigma)}(Y_r)\beta(dr) > \lambda\Big) &\leqslant P(D_1 > n) + n\sup_i P(T_i - S_i \geqslant K\sigma^{2-3\varepsilon}) \\ &\quad + P\big(\sup_{t\leqslant 1}[\beta((t+h)\wedge 1) - \beta(t)] > \lambda/2n\big) \\ &\leqslant c_2 e^{-c_3 n} + nc_4 e^{-c_5 K} + 2h^{ap/4}/L^p. \end{aligned}$$

If we substitute for n, K, h, and L, recall that $\lambda > 2\sigma$ and $\sigma < 1$, and take d sufficiently small, we obtain our result for $p \geqslant p_0$. The result (with the same b) for $p \in [1, p_0)$ follows since $\sigma < \lambda$. □

Define, for $\zeta \in (0,1)$,

$$G_\zeta(x) = G(x) \wedge \frac{1}{\pi}\log(1/\zeta), \qquad H_\zeta(x) = G(x) - G_\zeta(x). \tag{5.7}$$

A consequence of Proposition 5.1 is

Proposition 5.2. *Suppose $a > 0$ and β satisfies the hypotheses of Proposition 5.1. There exists $d > 0$ and $\zeta_0 < 1$ such that if $p \geqslant 1, q \geqslant 1$, then*

$$E\Big[\int_0^u |H_\zeta(X_u - X_r - x)|^q \beta(dr)\Big]^p \leqslant c(p,q)\zeta^{dp}$$

if $u \in [0,1]$, and $\zeta \leqslant \zeta_0$.

Proof. Write $V = \int_0^u \mid H_\zeta(X_u - X_r - x) \mid^q \beta(dr)$. Let $Y_r = X_u - X_r$. This is again 2-dimensional Brownian motion. Let $n = [4/b] + 4$, where b is the constant in the conclusion of Proposition 5.1.

Note

$$H_\zeta^q(z+x) \leqslant c_1 \sum_{\{j:2^{-j} \le \zeta\}}^\infty j^q 1_{B(x,2^{-j})}(z).$$

Note also that if $\lambda \geqslant \zeta^{1/4}, \zeta$ is sufficiently small, and $2^{-j} \le \zeta$, then $\lambda/40c_1 j^{2+q} \geqslant 2^{-j/2}$. So, using Proposition 5.1,

$$\begin{aligned}
P[V > \lambda] &\leqslant \sum_{\{j:2^{-j} \le \zeta\}}^\infty P\Big(c_1 j^q \int_0^u 1_{B(x,2^{-j})}(Y_r)\beta(dr) \geqslant \lambda/20j^2\Big) \\
&\leqslant c(np) \sum_{\{j:2^{-j} \le \zeta\}}^\infty \frac{(2^{-j})^{bnp}}{(\lambda/20j^{q+2})^{bnp}} \\
&= c(p,q) \sum_{\{j:2^{-j} \le \zeta\}}^\infty \frac{2^{-jp/2}}{\lambda^{bnp}} \\
&\leqslant c(p,q)\zeta^{d_1 p}/\lambda^{p+2}
\end{aligned}$$

if ζ is sufficiently small.

Multiplying by $p\lambda^{p-1}$ and integrating from $\zeta^{1/4}$ to ∞ gives

$$E[V^p; V \geqslant \zeta^{1/4}] \leqslant c(p,q)\zeta^{d'p}.$$

Since $E[V^p; V \leqslant \zeta^{1/4}] \leqslant \zeta^{p/2}$, adding gives our result. □

6. Stochastic Calculus. When we get to double points and multiple points of a single Brownian motion, the joint Hölder continuity will take some work. In preparation for this, we derive some stochastic calculus results.

Suppose $U_t = M_t - B_t$, where M_t is mean zero martingale, B_t is a nondecreasing process, $B_0 \equiv 0$, and U, M, and B have right continuous paths with left limits and are adapted to a filtration satisfying the usual conditions.

Proposition 6.1. *Let $a > 0$. Suppose for each $p \geq 1$ there exists $K(p)$ such that*

$$E \mid U_t \mid^p \leqslant K(p), \qquad t \leqslant 1 \tag{6.1}$$

and

$$E \mid U_t - U_s \mid^p \leqslant K(p) \mid t-s \mid^{ap}, \qquad s,t \leqslant 1. \tag{6.2}$$

Let $K'(p) = K(p) \vee K(p+1)$. Then there exists $b > 0$ (independent of p) and constants $c(p)$ such that if $p \geq 1$, then

$$EB_1^p \leqslant c(p)K'(p) \tag{6.3}$$

and

$$E(B_t - B_s)^p \leqslant c(p)K'(p) \mid t-s \mid^{bp}, \qquad s,t \leqslant 1. \tag{6.4}$$

Remark. Applying (6.4) with $p > 1/b$ implies that there is a dense subset of $[0,1]$ on which B_t is Hölder continuous, a.s. Since B_t is increasing, this implies B_t is continuous, a.s.

Proof. It suffices to prove the result for $p \geqslant p_0 = 2/a$, since we can get the result for $p < p_0$ by using Jensen's inequality.

By a standard chaining argument as in the proof of Kolmogorov's theorem (see the remark following the proof of Theorem 9.3), (6.1) and (6.2) imply that we can find a version of U_t such that $E\sup_{t \leqslant 1} \mid U_t \mid^p \leqslant c(p)K'(p)$. Since U_t and $-B_t$ differ by a martingale, for all $t \leq 1$

$$E(B_1 - B_t \mid \mathcal{F}_t) = E(U_t - U_1 \mid \mathcal{F}_t) \leqslant 2E(\sup_s \mid U_s \mid \ \mid \mathcal{F}_t).$$

By a standard inequality (see, for example, [B2, Lemma 2.3]),

$$EB_1^p \leqslant c(p)E\sup_t \mid U_t \mid^p .$$

This and (6.1) proves (6.3).

Similarly, $E\sup_{s \leqslant r \leqslant t} \mid U_r - U_s \mid^p \leqslant c(p)K'(p) \mid t-s \mid^{ap}$. To get (6.4), apply the above argument to $B'_r = B_{s+r} - B_s, U'_r = U_{s+r} - U_s, M'_r = M_{s+r} - M_s, r \leqslant t-s$. □

Now suppose $U_t^i = M_t^i - B_t^i, i = 1,2$, with $B_0^i \equiv 0$, B_t^i nondecreasing, and M_t^i a martingale. Let $B_t = B_t^1 - B_t^2$, and similarly for M_t, U_t.

Proposition 6.2. *Let $a, b, \delta \in (0,1)$. Suppose for each p there exists $K(p)$ such that*

$$E \mid U_t^i \mid^p \leqslant K(p), \qquad t \leqslant 1, i = 1, 2,$$
$$E \mid U_t^i - U_s^i \mid^p \leqslant K(p) \mid t - s \mid^{ap}, \qquad s, t \leqslant 1, i = 1, 2,$$

and

$$E \mid U_t \mid^p \leqslant K(p)\delta^{bp}, \qquad s, t \leqslant 1, i = 1, 2 \tag{6.5}$$

Let $K'(p) = K(p) \vee K(p+1)$. Then there exists $d > 0$ such that

$$E \mid B_t \mid^p \leqslant c(p)K'(p)\delta^{dp}, \qquad t \leqslant 1. \tag{6.6}$$

Proof. We may again suppose $p > 2/a + 2$. As in the proof of Proposition 6.1,

$$E \sup_{s \leqslant t \leqslant s+h} \mid U_t^i - U_s^i \mid^p \leqslant c(p)K'(p)h^{ap}.$$

If $n \geqslant 1$,

$$\sup_{t \leqslant 1} \mid U_t \mid \leqslant \sup_{j \leqslant n} \mid U_{j/n} \mid + \sum_{i=1}^{2} \sup_{j \leqslant n} \sup_{j/n \leqslant t \leqslant (j+1)/n} \mid U_t^i - U_{j/n}^i \mid .$$

Hence

$$\begin{aligned} E \sup_t \mid U_t \mid^p &\leqslant c(p)n \sup_{j \leqslant n} E \mid U_{j/n} \mid^p + 2c(p)n \max_{1 \le i \le 2} \sup_{j \leqslant n} E(\sup_{j/n \leqslant t \leqslant (j+1)/n} \mid U_t^i - U_{j/n}^i \mid^p) \\ &\leqslant c(p)nK'(p)\delta^{bp} + 2nc(p)K'(p)(1/n)^{ap}. \end{aligned}$$

Since $ap > 2$, take $n = [\delta^{-b/2}] + 1$ to get

$$E \sup_t \mid U_t \mid^p \leqslant c(p)K'(p)\delta^{abp/2}. \tag{6.7}$$

Let $Z = \sup_t \mid U_t \mid$ and $W = 1 + B_1^1 + B_1^2$. By Proposition 6.1, $W \in L^p$ for all p. Observe that if $t \leqslant 1$,

$$|E\left(B_1 - B_t|\mathcal{F}_t\right)| = |E\left(U_t - U_1|\mathcal{F}_t\right)| \leqslant 2E\left(Z|\mathcal{F}_t\right).$$

So as in the proof of [B2, Lemma 2.3],

$$\begin{aligned} E[(B_1 - B_t)^2|\mathcal{F}_t] &= 2E[\int_t^1 (B_1 - B_s)dB_s|\mathcal{F}_t] \\ &= 2E[\int_t^1 E(B_1 - B_s|\mathcal{F}_s)dB_s|\mathcal{F}_t] \\ &\leqslant 2E[\int_t^1 E(Z|\mathcal{F}_s)d(B_s^1 + B_s^2)|\mathcal{F}_t] \\ &\leqslant 2E[Z(B_1^1 + B_1^2)|\mathcal{F}_t] \leqslant 2E[ZW|\mathcal{F}_t]. \end{aligned} \tag{6.8}$$

Next, let $V_t = E(B_1 - B_t|\mathcal{F}_t)$, and $N_t = E(B_1|\mathcal{F}_t)$, so that $V_t = N_t - B_t$ (take the right continuous version of V and N). By Jensen's inequality,

$$V_t^2 = (E(B_1 - B_t|\mathcal{F}_t))^2 \leqslant E[(B_1 - B_t)^2|\mathcal{F}_t] \leqslant 2E[ZW|\mathcal{F}_t].$$

Also, by Ito's lemma,

$$V_1^2 - V_t^2 = 2\int_t^1 V_s dV_s + < N >_1 - < N >_t$$

Therefore,

$$\begin{aligned} E[< N >_1 - < N >_t |\mathcal{F}_t] &\leqslant |E[V_1^2 - V_t^2|\mathcal{F}_t]| + 2|E[\int_t^1 V_s dV_s|\mathcal{F}_t]| \\ &\leqslant 4E[ZW|\mathcal{F}_t] + 2|E[\int_t^1 V_s dB_s|\mathcal{F}_t]| \\ &\leqslant 4E[ZW|\mathcal{F}_t] + 2E[\int_t^1 2E(Z|\mathcal{F}_s) d(B_s^1 + B_s^2)|\mathcal{F}_t] \\ &\leqslant 8E[ZW|\mathcal{F}_t]. \end{aligned}$$

Finally, by [B2, Lemma 2.3] and Proposition 6.1,

$$\begin{aligned} E < N >_1^p \leqslant c(p)E(ZW)^p &\leqslant (EZ^{2p})^{1/2}(EW^{2p})^{1/2} \\ &\leqslant c(2p)K'(2p)\delta^{2abp/2})^{1/2}(K'(2p))^{1/2} \\ &\leqslant c(p)K'(2p)\delta^{abp/2}. \end{aligned}$$

By Jensen again,

$$E \mid V_t \mid^{2p} \leqslant E[(2E[ZW|\mathcal{F}_t]^p)] \leqslant c(p)E[(ZW)^p] \leqslant c(p)K'(2p)\delta^{abp/2}.$$

Therefore,

$$\begin{aligned} E \mid B_t \mid^{2p} &\leqslant c(p)E \mid N_t \mid^{2p} + c(p)E \mid V_t \mid^{2p} \leqslant c(p)E < N >_1^p + c(p)E \mid V_t \mid^{2p} \\ &\leqslant c(p)K'(2p)\delta^{abp/2}. \end{aligned}$$

Letting $d = ab/4$ completes the proof. □

7. Double Points. We now want to construct self-ILT for double points for a single Brownian motion X_t and derive the associated Tanaka formula. These results were first obtained by Yor [Y]. For concreteness, we restrict ourselves to 2-dimensional Brownian motion. Write $\beta(s) = s$ so that $\beta(ds) = ds$.

Fix t, let $\Delta_n = 2^{-n}$, and let $s_i = ti\Delta_n, i = 0, \cdots, 2^n$. We want to apply the results of Sections 2 and 4 with $u = s_i$ and $Y_r = (X_{s_i+r} - X_{s_i}) + X_{s_i} = X_{s_i+r}, 0 \leqslant r \leqslant \Delta$. For $x \in \mathbb{R}^2$, let $\mu_{x,u}(A) = \int_0^{s_i} 1_A(X_r)dr$. As in Section 2, there exist continuous additive functionals of Y_r, say $\alpha_2^{ni}(x, \cdot)$, that if $A_{n,i,x} = \alpha_2^{ni}(x, \Delta_n)$, then

$$\begin{aligned} &\int_0^{s_i} [G(X_{s_{i+1}} - X_r - x) - G(X_{s_i} - X_r - x)]\beta(dr) \\ &= \int_{s_i}^{s_{i+1}} [\int_0^{s_i} \nabla G(X_s - X_r - x)\beta(dr)] \cdot dX_s - A_{n,i,x}, \end{aligned} \tag{7.1}$$

$A_{n,i,x} \geqslant 0, A_{n,i,x}$ is continuous in x, and

$$\int f(x) A_{n,i,x} dx = \int_{s_i}^{s_{i+1}} \int_0^{s_i} f(X_r - X_s)\, ds\, dr. \tag{7.2}$$

Note that $X_{s_i+r} - X_{s_i}$ is independent of $Y_0 = X_{s_i}$ and recall the remark following Theorem 4.1.

Let

$$U_t^n = U_t^n(x) = \sum_{i=0}^{2^n-1} \int_0^{s_i} [G(X_{s_{i+1}} - X_r - x) - G(X_{s_i} - X_r - x)]\beta(dr),$$

$$M_t^n = M_t^n(x) = \sum_{i=0}^{2^n-1} \int_{s_i}^{s_{i+1}} [\int_0^{s_i} \nabla G(X_s - X_r - x)\beta(dr)] \cdot dX_s,$$

$$\beta_t^n(x) = \sum_{i=0}^{2^n-1} A_{n,i,x},$$

$$U_t = U_t(x) = \int_0^t [G(X_t - X_r - x) - G(-x)]\beta(dr),$$

and

$$M_t = M_t(x) = \int_0^t [\int_0^s \nabla G(X_s - X_r - x)\beta(dr)] \cdot dX_s,$$

Summing (7.1) over i, we get

$$U_t^n = M_t^n - \beta_t^n(x). \tag{7.3}$$

Proposition 7.1. *Suppose $x \neq 0$. Then $U_t^n \to U_t$ in $L^p, p > 1$.*

Proof. We have

$$\begin{aligned} U_t^n &= \sum_{i=0}^{2^n-1} \sum_{j=0}^{i-1} \int_{s_j}^{s_{j+1}} [G(X_{s_{i+1}} - X_r - x) - G(X_{s_i} - X_r - x)]\beta(dr) \\ &= \sum_{j=0}^{2^n-1} \sum_{i=j+1}^{2^n-1} \int_{s_j}^{s_{j+1}} [G(X_{s_{i+1}} - X_r - x) - G(X_{s_i} - X_r - x)]\beta(dr) \\ &= \sum_{j=0}^{2^n-1} \int_{s_j}^{s_{j+1}} [G(X_t - X_r - x) - G(X_{s_{j+1}} - X_r - x)]\beta(dr). \end{aligned}$$

So to prove the proposition, it suffices to prove

$$\int_0^t h_r^n \beta(dr) \to 0 \text{ in } L^p,$$

where

$$h_r^n = \sum_{j=0}^{2^{n-1}} [G(-x) - G(X_{s_{j+1}} - X_r - x)]1_{(s_j, s_{j+1})}(r).$$

By Hölder's inequality and then Cauchy–Schwarz,

$$E|\int_0^t h_r^n \, \beta(dr)|^p \leqslant \left(E \int_0^t \mid h_r^n \mid^{2p} \beta(dr) \right)^{1/2} (E\beta(t)^{2p-1})^{1/2},$$

and since $E\beta(t)^{2p-1} = t^{2p-1}$, it suffices to prove

$$E \sum_{j=0}^{2^n-1} \int_{s_j}^{s_{j+1}} |G(-x) - G(X_{s_{j+1}} - X_r - x)|^{2p} \beta(dr) \to 0. \tag{7.4}$$

Choose ζ small enough so that $G_\zeta(z) = G(z)$ for $z \in B(x, \mid x \mid /2)$. Note

$$\begin{aligned} &E \sum_{j=0}^{2^n-1} \int_{s_j}^{s_{j+1}} |G_\zeta(-x) - G_\zeta(X_{s_{j+1}} - X_r - x)|^{2p} \beta(dr) \\ &\leqslant \parallel \nabla G_\zeta \parallel^{2p} E[\sum_j \int_{s_i}^{s_{j+1}} \beta(dr) \sup_{\substack{u,v \leqslant 1 \\ |u-v| \leqslant \Delta_n}} |X_u - X_v|^{2p}] \\ &\leqslant c\zeta^{-2p} (E\beta(1)^2)^{1/2} (E(\sup_{\substack{|u-v| \leqslant \Delta_n \\ u,v \leqslant 1}} \mid X_u - X_v \mid^{4p}))^{1/2} \leqslant c(p)\zeta^{-2p} \Delta_n^p. \end{aligned} \tag{7.5}$$

Let

$$V = \{ \sup_{\substack{|u-v|\leqslant \Delta_n \\ u,v\leqslant 1}} |X_u - X_v| >| x \mid /2\}.$$

By our choice of $\zeta, H_\zeta(-x) = 0$ and

$$E[\sum_{j=0}^{2^n-1} \int_{s_j}^{s_{j+1}} |H_\zeta(-x) - H_\zeta(X_{s_{j+1}} - X_r - x)|^{2p}\beta(dr); V^c] = 0. \tag{7.6}$$

On the other hand, noticing the inequality

$$E(\sum_{j=0}^{2^n-1} Z_j)^2 \leq 2^n E \sum Z_j^2 \leq 2^{2n} \sup_j EZ_j^2,$$

we get

$$\begin{aligned} &E[\sum \int_{s_j}^{s_{j+1}} |H_\zeta\left(X_{s_{j+1}} - X_r - x\right)|^{2p}\beta(dr); V] \\ &\leqslant \left(E\left(\sum \int_{s_j}^{s_{j+1}} |H_\zeta\left(X_{s_{j+1}} - X_r - x\right)|^{2p}\beta(dr)\right)^2\right)^{1/2} (PV)^{1/2} \\ &\leqslant c2^n \left(\sup_j E\left(\int_0^{s_{j+1}} |H_\zeta\left(X_{s_{j+1}} - X_r - x\right)|^{2p}\beta(dr)\right)^2\right)^{1/2} \exp(- \mid x \mid^2 /16\Delta_n) \\ &\leqslant c2^n \exp(- \mid x \mid^2 /16\Delta_n), \end{aligned} \tag{7.7}$$

using Proposition 5.2. If we add (7.5), (7.6), and (7.7), and let $\zeta = \zeta_n \to 0$ as $n \to \infty$ so that $\Delta_n^{\frac{1}{2}} \leqslant \zeta_n^2$, we get our desired result. □

Proposition 7.2. *$\beta_t^n(x)$ increases as $n \to \infty$. If we call the limit $\beta_2(x,t)$, and if f is continuous with compact support, then a.s.*

$$\int f(x)\beta_2(x,t)dx = \int_0^t \int_0^s f(X_r - X_s)\, dr\, ds. \tag{7.8}$$

Proof. If φ_ε is a nonnegative symmetric approximation to the identity with compact support, then by (7.2),

$$\int \varphi_\varepsilon(x - x_0)\beta_t^n(x)dx = \sum_{i=0}^{2^n-1} \int_{s_i}^{s_{i+1}} \int_0^{s_i} \varphi_\varepsilon(X_r - X_s - x_0)\, dr\, ds. \tag{7.9}$$

For each n, the left-hand side converges a.s. to $\beta_t^n(x_0)$ as $\varepsilon \to 0$ since each $A_{n,i,x}$ is continuous in x. And for each fixed ε, the right-hand side of (7.9) is increasing in n. We conclude that for each $x_0 \neq 0, \beta_t^n(x_0)$ increases as $n \to \infty$. Call the limit $\beta_2(x_0, t)$.

By monotone convergence

$$\begin{aligned}\int f(x)\beta_2(x,t)dt &= \lim_{n\to\infty}\int f(x)\beta_t^n(x)dx\\ &= \lim_{n\to\infty}\int_0^t\int_0^s f(X_r-X_s)1_{(r\leqslant s_i \text{ if } s_i\leqslant s<s_{i+1})}\,dr\,ds\\ &= \int_0^t\int_0^s f(X_r-X_s)\,dr\,ds,\end{aligned}$$

and (7.8) is proved. □

We define $\beta_2'(x,t)$ to be the limit of $\beta_t^n(x)$ for each $x\in\mathbb{R}^2-\{0\}, t$ rational. By the argument of Proposition 7.2, it is easy to see that $\beta_2'(x,t)\geqslant\beta_2'(x,s)$, a.s., if $t\geqslant s$. For $t\in[0,1]$, let

$$\beta_2(x,t)=\inf_{u\geqslant t, u\,\text{rational}}\beta_2'(x,u).$$

Recall $G(-x)=G(x)$.

Lemma 7.3. *For each $p\geq 1$, there exists $\nu(p)$ such that*
(a) $E\mid U_t(x)\mid^p\leqslant c(p)(1\vee|G(x)|)^{\nu(p)},\qquad t\leqslant 1;$
(b) There exists $a>0$ such that

$$E\mid U_t(x)-U_s(x)\mid^p\leqslant c(p)(1\vee|G(x)|)^{\nu(p)}\mid t-s\mid^{ap},\qquad s,t\leqslant 1.$$

Proof. $G(x)\beta(t)$ trivially has moments of all orders. Take ζ small but fixed. Note that $\int_0^t H_\zeta(X_t-X_r-x)\beta(dr)$ has p^{th} moments by Proposition 5.2, while

$$|\int_0^t G_\zeta(X_t-X_r-x)\beta(dr)|\leqslant c\log(1/\zeta)\beta(t).$$

This proves (a).

For (b),

$$\begin{aligned}|U_t-U_s|\leqslant{}&|G(x)|[\beta(t)-\beta(s)]+|\int_0^t H_\zeta(X_t-X_r-x)\,\beta(dr)|\\ &+|\int_0^s H_\zeta(X_s-X_r-x)\beta(dr)|+|\int_s^t G_\zeta(X_t-X_r-x)\beta(dr)|\\ &+|\int_0^s[G_\zeta(X_t-X_r-x)-G_\zeta(X_s-X_r-x)]\beta(dr)|\end{aligned}$$

So by Proposition 5.2,

$$
\begin{aligned}
E|U_t-U_s|^p \leqslant{}& c(p)|G(x)|^pE|\beta(t)-\beta(s)|^p\\
&+c(p)\zeta^{dp}+c(p)\zeta^{dp}+c(p)|\log(1/\zeta)|^pE|\beta(t)-\beta(s)|^p\\
&+\|\nabla G_\zeta\|^pE\left(\int_0^s\beta(dr)\mid X_t-X_s\mid\right)^p\\
\leqslant{}&(1\vee|G(x)|)^p\mid t-s\mid^p+c(p)\zeta^{dp}+|\log(1/\zeta)|^p\mid t-s\mid^p\\
&+c\mid t-s\mid^{p/2}/\zeta^p,
\end{aligned}
\tag{7.10}
$$

using Cauchy–Schwarz to get the last term on the right of (7.10). Taking $\zeta=\mid t-s\mid^b$ for suitable b proves (b). □

Proposition 7.4. *For each $p\geq 1$ there exists $\nu(p)$ such that*

(a) $E\beta_2(x,t)^p\leqslant c(p)(1\vee\mid G(x)\mid)^{\nu(p)},\qquad t\leqslant 1,$

(b) There exists $a>0$ such that

$$E|\beta_2(x,t)-\beta_2(x,s)|^p\leqslant c(p)(1\vee|G(x)|)^{\nu(p)}\mid t-s\mid^{ap},\qquad s,t\leqslant 1.$$

Proof. We have

$$E[\beta_1^n(x)-\beta_t^n(x)|\mathcal{F}_t]=E[U_t^n(x)-U_1^n(x)|\mathcal{F}_t].$$

Using the monotone convergence of $\beta_t^n(x)$ to $\beta_2'(x,t)$ for t rational, the monotonicity of $\beta_2(x,t)$ in t, and the L^p convergence of $U_t^n(x)$ to $U_t(x)$, we get $E[\beta_2(x,1)-\beta_2(x,t)|\mathcal{F}_t]=E[U_t(x)-U_1(x)|\mathcal{F}_t]$. So $M_t=U_t(x)+\beta_2(x,t)$ is a martingale. Our result now follows from Proposition 6.1. □

Remark. Since β_2 is increasing, Proposition 7.4(b) implies $\beta_2(x,t)$ is Hölder continuous in t. As a consequence $\beta_t^n(x)\to\beta_2(x,t)$, uniformly for $t\in[0,1]$, a.s., for each x.

Proposition 7.5. *The Tanaka formula*

(7.11)

$$\int_0^t[G(X_t-X_r-x)-G(-x)]\beta(dr)=\int_0^t[\int_0^s\nabla G(X_s-X_r-x)\,\beta(dr)]\cdot dX_s-\beta_2(x,t)$$

holds.

Proof. Since $\beta_t^n(x)\uparrow\beta_2(x,t)$ and $\beta_2(x,t)$ is in $L^p, p\geqslant 1$, then the convergence is in L^p. Since $U_t^n(x)\to U_t(x)$ in L^p, we conclude $M_t^n(x)$ converges in L^p, say to N_t. Since $M_t^n(x)=\int_0^t h_s^n\cdot dX_s$, where

$$h_s^n=\int_0^s\nabla G(X_s-X_r-x)1_{(r\leqslant s_i\text{ if }s_i\leqslant s<s_{i+1})}\beta(dr),$$

then $\int_0^t \mid h_s^n - h_s^m \mid^2 ds =< M^n - M^m >_t \to 0$. Since h_s^n converges for each s to $h_s = \int_0^s \nabla G(X_s - X_r - x)\beta(dr)$, then $\int_0^t \mid h_s^n - h_s \mid^2 ds \to 0$. It follows that N_t must equal $M_t(x)$. We then get (7.11) by taking a limit in (7.3). ☐

Proposition 7.6. *There exists $a > 0$ such that if $x, x' \neq 0$,*

$$E \mid \beta_2(x,t) - \beta(x',t) \mid^p \leqslant c(p)(\mid x \mid \wedge \mid x' \mid)^{-p} \mid x - x' \mid^{ap} .$$

Proof. We can connect x to x' by an arc of length less than $c \mid x - x' \mid$ which never gets closer to the point 0 than $\mid x \mid \wedge \mid x' \mid$. Along this arc, ∇G is bounded by $c(\mid x \mid \wedge \mid x' \mid)^{-1}$. So

$$\mid G(-x) - G(-x') \mid \beta(t) \leqslant \frac{c}{\mid x \mid \wedge \mid x' \mid} \mid x - x' \mid \beta(t),$$

which has p^{th} moments of the desired form.

$E|\int_0^t H_\zeta(X_t - X_r - x)\beta(dr)|^p \leqslant c\zeta^{bp}$ by Proposition 5.2 and similarly with x replaced by x'. And finally,

$$E|\int_0^t [G_\zeta(X_t - X_r - x) - G_\zeta(X_t - X_r - x')]\beta(dr)|^p$$
$$\leqslant c\|\nabla G_\zeta\|^p \mid x - x' \mid^p E\beta(t)^p \leqslant c \mid x - x' \mid^p /\zeta^p.$$

So if we let $\zeta = \mid x - x' \mid^{1/2}$ and sum, we get

$$E|U_t^x - U_t^{x'}|^p \leqslant c(p)\,(\mid x \mid \wedge \mid x' \mid)^{-p} |x - x'|^{ap}. \tag{7.12}$$

Now apply Proposition 6.2, using Lemma 7.3. ☐

Remark. The $G(-x)\beta(t)$ term is what contributes the highly singular $(\mid x \mid \wedge \mid x' \mid)^{-p}$ term.

We finally can prove

Theorem 7.7. *There exists a version of $\beta_2(s,t)$ which is jointly Hölder continuous in $t \in [0,1]$ and $x \in \mathbb{R}^2 - \{0\}$ and that satisfies (7.8) and (7.11). Moreover, outside a single null set, (7.8) holds for all bounded and measurable f.*

Proof. By Propositions 7.4 and 7.6, there is a countable dense subset D of $\mathbb{R}^2$ and a countable dense subset T of $[0,1]$ so that $\beta_2(x,t)$ is uniformly continuous on $(x,t) \in (D \cap B(0,\delta^{-1}) - B(0,\delta)) \times T$ a.s. for each $\delta \in (0,1)$. For $x \neq 0$, define

$$\hat{\beta}_2(x,t) = \lim_{\substack{x_n \in D, t_n \in T \\ x_n \to x, t_n \to t}} \beta_2(x_n, t_n).$$

By the uniform continuity of $\beta_2(x_n,t_n)$, we see that $\hat\beta_2(x,t)$ is jointly continuous in x and t on $(\mathbb{R}^2-\{0\})\times[0,1]$. By Propositions 7.4 and 7.6, in fact $\hat\beta_2(x,t)=\beta_2(x,t)$, a.s., the null set depending on x and t. Since both β_2 and $\hat\beta_2$ are continuous in t, $\hat\beta_2(x,t)=\beta_2(x,t), t\leqslant 1$, a.s., the null set depending on x. Hence (7.11) holds with β_2 replaced by $\hat\beta_2$.

By Fubini, there is a null set N such that if $\omega\notin N, \hat\beta_2(x,t)=\beta_2(x,t)$ for a.e. x. If f is smooth with compact support in $\mathbb{R}^2-\{0\}$ and $\omega\notin N$, then

$$\int f(x)\hat\beta_2(x,t)dx=\int f(x)\beta_2(x,t)=\int_0^t\int_0^s f(X_r-X_s)\,dr\,ds. \tag{7.13}$$

This shows that (7.8) holds for each f with β_2 replaced by $\hat\beta_2$. We now proceed as in the last paragraph of the proof of Theorem 2.5 to obtain the last assertion of our theorem. □

Remark. Using (7.12), it is not hard to show we can find a version of $U_t(x)$ that is jointly continuous in x and t provided $x\neq 0$. Defining $\hat M_t(x)=U_t(x)+\hat\beta(x,t)$, we see that we can find a single null set outside of which (7.11) holds for all $x\neq 0$ and all t.

For the purposes of the next section, we need

Proposition 7.8. *If $x\neq 0$, there exists $K(\omega)$ and $\gamma>0$ such that*

$$\int_0^1 1_{B(y,s)}(X_r)\beta_2(x,dr)\leqslant K(\omega)(s\wedge 1)^\gamma,\qquad y\in\mathbb{R}^2, s\in(0,\infty). \tag{7.14}$$

Proof. By the finiteness of $\beta_2(x,1)$, we may assume $s\leqslant 1/2$. By Proposition 7.4(b) and Proposition 5.1 with $Y_r=X_r, \beta=\beta_2$,

$$P[\int_0^1 1_{B(y,s)}(X_r)\beta_2(x,dr)>\lambda]\leqslant c(p)s^{ap}/\lambda^{ap}$$

for each $p\geqslant 1$. With this estimate for $p\geqslant 8/a$ in place of (2.3), we may proceed very much as in the proof of Lemma 2.1. □

8. Multiple Points.

We now want to construct ILT for k-multiple points of a single Brownian motion. Here $d=2$. The proof is by induction. Recall $G(-x)=G(x)$. We let $G^\vee$ denote the quantity $1\vee|G(x_1)|\vee\cdots\vee|G(x_{k-1})|$.

Theorem 8.1. *Suppose $k\geqslant 2$. Suppose $x_i\neq 0, i=1,\cdots,k-1$. There exist positive reals $a,\gamma,\nu(p)$ for $p\geqslant 1$ and nondecreasing processes $\beta_k(x_1,\cdots,x_{k-1},t)$ such that*

$$E|\beta_k(x_1,\cdots,x_{k-1},t)|^p\leqslant c(p)(G^\vee)^{\nu(p)}; \tag{8.1}$$

(8.2) $$E|\beta_k(x_1,\cdots,x_{k-1},t)-\beta_k(x_1,\cdots,x_{k-1},s)|^p \leqslant c(p)(G^\vee)^{\nu(p)} \mid t-s\mid^{ap};$$

(8.3) $$E|\beta_k(x_1,\cdots,x_{k-1},t)-\beta_k(x_1',\cdots,x_{k-1}',t)|^p \leqslant c(p)(G^\vee)^{\nu(p)}|(x_1,\cdots,x_{k-1})-(x_1',\cdots,x_{k-1}')|^{ap};$$

(8.4) *there exists* $K(\omega)$ *such that*
$$\int_0^1 1_{B(y,s)}(X_r)\beta_k(x_1,\cdots,x_{k-1},dr)\leqslant K(\omega)(s\wedge 1)^\gamma$$
for $y\in\mathbb{R}^2$, $s\in(0,\infty)$;

(8.5) β_k *is jointly Hölder continuous on* $(\mathbb{R}^2-\{0\})^{k-1}\times[0,1]$;

(8.6) $$\int_0^t [G(X_t-X_r-x_1)-G(-x_1)]\beta_{k-1}(x_2,\cdots,x_{k-1},dr) = \int_0^t [\int_0^s \nabla G(X_s-X_r-x_1)\beta_{k-1}(x_2,\cdots,x_{k-1},dr)]\cdot dx_s - \beta_k(x_1,\cdots,x_{k-1},t);$$

(8.7) *except for a null set independent of* f,
$$\int\cdots\int \beta_k(x_1,\cdots,x_{k-1})dx_1\cdots dx_{k-1} = \int_0^t\int_0^{s_1}\cdots\int_0^{s_{k-2}} f(X_{s_1}-X_t,\cdots,X_{s_{k-1}}-X_{s_{k-2}})ds_{k-1}\cdots ds_1$$
for all bounded measurable f.

Remark. (8.6) was independently obtained by Shieh ([Sh]).

Proof. If we write $\beta_1(t)=t$, (8.1) – (8.7) for the case $k=2$ follow by Section 7. We use induction: we suppose we have the result for k and prove it for $k+1$. We write x for x_1, y for $(x_2,\cdots,x_{k-1})$.

Let $\Delta_n=2^{-n}$ and let $s_i=ti\Delta_n$. Fix $u=s_i$ for the moment and set
$$\mu_{x,u}(A)=\int_0^u 1_A(X_r+x)\beta_k(y,dr).$$
If $Y_s=(X_{u+s}-X_u)+X_u=X_{u+s}$, $\quad s\leqslant\Delta_n$, then by Sections 2 and 4 and the remark following Theorem 4.1, there is a continuous additive functional, $A_{n,x,i}(s)$ say, associated to $\mu_{x,u}$. By Section 4,
$$\int_0^{s_i}[G(X_{s_{i+1}}-X_r-x)-G(X_{s_i}-X_r-x)]\beta_k(y,dr) = \int_{s_i}^{s_{i+1}}[\int_0^{s_i}\nabla G(X_{s_{i+1}}-X_r-x)\beta_k(y,dr)]\cdot dX_s - A_{n,x,i}(\Delta_n).$$

If we let $A_{n,x} = \sum_{i=0}^{2^n-1} A_{n,x,i}(\Delta_n)$ and we sum over i, we get

$$\begin{aligned}\sum_{i=0}^{2^n-1}[\int_0^{s_i}[G(X_{s_{i+1}}-X_r-x)-G(X_{s_i}-X_r-x)]\beta_k(y,dr)\\ =\sum_{i=0}^{2^n-1}\int_{s_i}^{s_{i+1}}[\int_0^{s_i}\nabla G(X_{s_{i+1}}-X_r-x)]\cdot dX_s - A_{n,x}.\end{aligned} \tag{8.8}$$

We set $\beta = \beta_k(y,dr)$ and then proceed as in Section 7: using (8.5), $A_{n,x}$ increases as $n\to\infty$. We let $\beta_{k+1}(x,y,t)$ denote the limit. As in Proposition 7.1, the left-hand side of (8.8) converges in L^p to

$$U_t(x,y) = \int_0^t [G(X_t - X_r x) - G(-x)]\beta_k(y,dr). \tag{8.9}$$

Continuing exactly as in Section 7, we obtain (8.1), (8.2), (8.4), (8.6), and (8.7). (8.5) will follow, then, once we obtain (8.3).

We have $E|\beta_{k+1}(x,y,t)-\beta_{k+1}(x',y,t)|^p \leqslant c(p)(G^\vee)^{\nu(p)} \mid x-x' \mid^{ap}$ by arguing as in Proposition 7.6. So it remains to show

$$E|\beta_{k+1}(x,y,t)-\beta_{k+1}(x,y',t)|^p \leqslant c(p)(G^\vee)^{\nu(p)} \mid y-y' \mid^{ap}, \qquad y,y'\in(\mathbb{R}^2-\{0\})^{k-1}. \tag{8.10}$$

By Section 6, this will follow if we show

$$E|U_t(x,y)-U_t(x,y')|^p \leqslant c(p)(G^\vee)^{\nu(p)} \mid y-y' \mid^{bp} \tag{8.11}$$

for some b.

Now $G(-x)[\beta_k(y,t)-\beta_k(y',t)]$ has p^{th} moments of the desired form by the induction hypothesis. By Proposition 5.2,

$$E|\int_0^t H_\zeta(X_t-X_r-x)\beta_k(y,dr)|^p \leqslant c(p)(G^\vee)^{\nu(p)}\zeta^{ap}, \tag{8.12}$$

and similarly with y replaced by y'.

Let

$$V = \{|X_{s+u}-X_u| \geqslant u^{1/4}/\zeta \text{ for some } s\in[0,1], u\in[0,1]\}.$$

By standard estimates on the Brownian path,

$$\begin{aligned}E[\int_0^t |G_\zeta(X_t-X_r-x)|\beta_k(y,dr);V]^p\\ \leqslant (E(\int_0^t |G_\zeta(X_t-X_r-x)|\beta_k(y,dr))^{2p})^{1/2}(PV)^{1/2}\\ \leqslant c(p)(G^\vee)^{\nu(p)}\zeta^{dp}\end{aligned} \tag{8.13}$$

for some $d > 0$ independent of p.

On $V^c, f(r) = G_\zeta(X_t - X_r - x)$ is Hölder continuous of order 1/4:

$$|G_\zeta(X_t - X_r - x) - G_\zeta(X_t - X_s - x)| \leqslant (1/\zeta)|X_r - X_s| \leqslant |\, r - s\,|^{1/4} /\zeta^2.$$

Hence for each $\omega \in V^c$, we can find $f_h(r)$ such that $|\, f - f_h \,| \leqslant ch^{1/4}$ and f_h is Lipschitz with constant $\|f\|_\infty/h \leqslant c\log(1/\zeta)/h \leqslant c/\zeta h$, namely by letting

$$f_h(t) = \frac{1}{h}\int_t^{t+h} f(u)du.$$

Set $h = \zeta$. Then

$$E[\int_0^t |G_\zeta(X_t - X_r - x) - f_h|(r)\beta_k(y, dr)^p; V^c] \leqslant c(p)(G^\vee)^{\nu(p)}\zeta^{p/4}, \tag{8.14}$$

and similarly with y replaced by y'.

Finally, by integration by parts and the induction hypothesis,

$$\begin{aligned} E[|\int_0^t f_h(r)[\beta_k(y, dr) - \beta_k(y', dr)]|^p; V^c] \\ \leqslant c(p)E|f_h(t)|^p|\beta_k(y, t) - \beta_k(y', t)|^p \\ + E|\int_0^t [\beta_k(y, r) - \beta_k(y', r)]f_h(dr)|^p \\ \leqslant c(p)(G^\vee)^{\nu(p)}\zeta^{-2p} \mid y - y' \mid^{ap} . \end{aligned} \tag{8.15}$$

(Since f_h is Lipschitz in $r \in [0, 1]$, it is of bounded variation.)

Adding (8.12) – (8.15) and letting $\zeta =\mid y - y' \mid^{a/4}$ yields (8.11). □

9. Renormalization. Again, $d = 2$. For $x \neq 0$, let

$$\xi_2(x, t) = G(x)t, \qquad \gamma_2(x, t) = \beta_2(x, t) - \xi_2(x, t).$$

Hence, since $G(-x) = G(x)$,

$$\int_0^t G(X_t - X_r - x)dr = \int_0^t [\nabla G(X_s - X_r - x)dr] \cdot dX_s - \gamma_2(x, t).$$

If $y = (x_2, \cdots, x_{k-1})$ with $x_i \neq 0, i = 2, \cdots, k-1$, define by induction

$$\begin{aligned} \xi_{k+1}(x, y, t) = G(x)\beta_k(y, t) - \int_0^t G(X_t - X_r - x)\xi_k(y, dr) \\ + \int_0^t [\int_0^s \nabla G(X_s - X_r - x)\xi_k(y, dr)] \cdot dX_s, \end{aligned} \tag{9.1}$$

and

$$\gamma_{k+1}(x,y,t)=\beta_{k+1}(x,y,t)-\xi_{k+1}(x,y,t). \tag{9.2}$$

By (8.6),

$$\gamma_{k+1}(x,y,t)=\int_0^t[\int_0^s G(X_s-X_r-x)\gamma_k(y,dr)]\cdot dX_s-\int_0^t G(X_t-X_r-x)\gamma_k(y,dr). \tag{9.3}$$

We call γ_{k+1} renormalized ILT.

Define $\varphi_{ki}\colon(\mathbb{R}^2)^k\times\{1,\cdots,k\}^i\to(\mathbb{R}^2)^{k-1}$ by letting $\varphi_{ki}(x_1,\cdots,x_k,j_1,\cdots,j_i)$ be the sequence $x_1,\cdots,x_k$ with the $j_1,j_2,\ldots$, and j_i entries deleted. For example,

$$\varphi_{4,2}(x_1,x_2,x_3,x_4;2,4)=(x_1,x_3).$$

Let $\beta_1(t)=t$.

Proposition 9.1.

$$\begin{aligned}&\xi_{k+1}(x_1,\cdots,x_k,t)\\&\quad=\sum_{i=1}^k(-1)^{i+1}\sum_{j_1<\cdots<j_i}G(x_{j_1})\cdots G(x_{j_i})\beta_{k+1-i}(\varphi_{ki}(x_1,\cdots,x_k,j_1,\cdots,j_i),t).\end{aligned}$$

Remark. The proposition says, for example,

$$\begin{aligned}\xi_3(x,y,t)&=G(x)\beta_2(y,t)+G(y)\beta_2(x,t)-G(x)G(y)t;\\ \xi_4(x,y,z,t)&=G(x)\beta_3(y,z,t)+G(y)\beta_3(x,z,t)+G(z)\beta_3(x,y,t)\\&\quad-G(x)G(y)\beta_2(z,t)-G(x)G(z)\beta_2(y,t)-G(y)G(z)\beta_2(x,t)\\&\quad+G(x)G(y)G(z)t,\end{aligned}$$

and so on. Recall $\gamma_k=\beta_k-\xi_k$.

Proof. The proof is by induction: the $(k+1)^{st}$ formula follows from the k^{th} formula, (8.6), (9.1), (9.2), and some routine calculations. □

Set

$$\begin{aligned}\gamma^+_{k+1}(x_1,\cdots,x_k,t)&=\beta_{k+1}(x_1,\cdots,x_k,t)+\sum_{i\leqslant k,i\,\text{even}}\;\sum_{j_1<\cdots<j_i}G(x_{j_1})\cdots G(x_{j_i})\\&\qquad\times\beta_{k+1-i}(\varphi_{ki}(x_i,\cdots,x_k,j_1,\cdots,j_i),t),\\ \gamma^-_{k+1}(x_1,\cdots,x_k,t)&=-(\gamma_{k+1}-\gamma^+_{k+1}).\end{aligned} \tag{9.4}$$

If $G^\vee = 1 \vee |G(x_1)| \vee \cdots \vee |G(x_{k-1})|$, for each p

$$E|\gamma_k(x_1, \cdots, x_{k-1}, t)|^p \leqslant c(p)(G^\vee)^{\nu(p)}, \qquad t \leqslant 1, \tag{9.5}$$

and

$$E|\gamma_k(x_1, \cdots, x_{k-1}, t) - \gamma_k(x_1, \cdots, x_{k-1}, s)|^p \leqslant c(p)(G^\vee)^{\nu(p)} \mid t - s \mid^{ap}, \qquad s, t \leqslant 1 \tag{9.6}$$

for some a and $\nu(p)$ by Theorem 8.1 and the representation of γ_k as a linear combination of the $\beta_i, i \leqslant k$.

We set

$$\overline{U}_t(x, y) = \int_0^t G(X_t - X_r - x)\gamma_{k-1}(y, t),$$

and

$$\overline{M}_t(x, y) = \overline{U}_t(x, y) + \gamma_k(x, y, t),$$

where $x = x_1, y = (x_2, \cdots, x_{k-1})$. By (9.3), $\overline{M}_t(x, y)$ is a martingale.

Proposition 9.2. *There exist $\alpha > 0$ and $\nu(p)$ such that*

(*a*) $$E|\overline{U}_t(x, y) - \overline{U}_t(x', y)|^p \leqslant c(p)(G^\vee)^{\nu(p)} \mid x - x' \mid^{ap};$$

(*b*) $$E|\overline{U}_t(x, y) - \overline{U}_s(x, y)| \leqslant c(p)(G^\vee)^{\nu(p)} \mid t - s \mid^{ap};$$

(*c*) $$E|\overline{U}_t(x, y) - \overline{U}_t(x, y')| \leqslant c(p)(G^\vee)^{\nu(p)} \mid y - y' \mid^{ap};$$

Proof. The proof is again by induction. Note

$$E\Big|\int_0^t H_\zeta(X_t - X_r - x)\gamma^+_{k-1}(y, t)\Big|^p \leqslant c(p)(G^\vee)^{\nu(p)}\zeta^{bp}$$

and similarly with γ^+_{k-1} replaced by γ^-_{k-1} and with x replaced by x', using Proposition 5.2. If we connect x to x' by a curve Γ of length $\leqslant c \mid x - x' \mid$ so that Γ never gets closer to the point 0 than $\mid x \mid \wedge \mid x' \mid$,

$$\begin{aligned} &E\Big|\int_0^t [G_\zeta(X_t - X_r - x) - G_\zeta(X_t - X_r - x')]\gamma_k(y, dr)\Big|^p \\ &\leqslant c\zeta^{-p} \mid x - x' \mid^p E|\gamma^+_k(y, t) + \gamma^-_k(y, t)|^p \\ &\leqslant c(p)(G^\vee)^{\nu(p)} | x - x' \mid^p / \zeta^p. \end{aligned}$$

Adding our estimates and setting $\zeta =| x - x' |^{1/2}$, we get (a).

The proofs of parts (b) and (c) are similar, following the lines of the proofs of (8.2) and (8.3). □

Theorem 9.3. *$\gamma_k(x_1, \cdots, x_{k-1}, t)$ is jointly Hölder continuous in each variable on the set $(\mathbb{R}^2)^{k-1} \times [0,1]$.*

Proof. Let $z = (x_1, \cdots, x_{k-1}, t)$. From Propositions 9.2 and 6.2 and the triangle inequality, we get the existence of $a > 0$ and $\nu(p)$ such that

$$E|\gamma_k(z) - \gamma_k(z')|^p \leqslant c(p)(G^\vee)^{\nu(p)} \mid z - z' \mid^{ap} . \tag{9.7}$$

Fix p large enough so that $ap \geqslant 6k + 4$.

We now proceed to modify the standard chaining argument. Let $\mathcal{B}_n = \{x \in \mathbb{R}^2 \colon x \neq 0$ and both coordinates of x are integer multiples of $2^{-n}\}, n \geqslant 1$. Let $R \geqslant 1$ and let $\mathcal{A}_n = \{z = (x_1, \cdots, x_{k-1}, t) \colon | x_i | \leqslant R, x_i \in \mathcal{B}_n, i = 1, \ldots, k-1, t$ is an integer multiple of $2^{-n}\}, n \geqslant 1$. Let $\mathcal{A} = \bigcup_n \mathcal{A}_n$.

If $z \in \mathcal{A}$, let z_i be the point in $\mathcal{A}_i$ closest to z (with some convention for breaking ties). We write, for any i_0,

$$\gamma_k(z) = \sum_{i=i_0}^{\infty} [\gamma_k(z_{i+1} - \gamma_k(z_i)] + \gamma_k(z_{i_0}), \tag{9.8}$$

where the sum is actually finite, since $z \in \mathcal{A}$. We do the same for $\gamma_k(z')$. Note $\#\mathcal{A}_i \leqslant c2^{2jk}$.

Let $\lambda > 0$. If $| z - z' | < \delta$, and $| \gamma_k(z) - \gamma_k(z') | > \lambda$, then either (a) $| \gamma_k(z_{i_0}) - \gamma_k(z'_{i_0}) | > \lambda/2$ or (b) for some $j \geqslant i_0$ and some $w \in \mathcal{A}_j, w' \in \mathcal{A}_{j+1}$ with $| w - w' | \leqslant c2^{-j}$, we have $| \gamma_k(w) - \gamma_k(w') | \geqslant \lambda/40j^2$. So

$$\begin{aligned} &(9.9) P(| \gamma_k(z) - \gamma_k(z') | > \lambda \text{ for some } z, z' \in \mathcal{A} \text{ with } | z - z' | < \delta) \\ &\qquad \leqslant (\#\mathcal{A}_{i_0}) \sup_{\substack{z, z' \in \mathcal{A}_{i_0} \\ |z - z'| \leqslant c\delta}} P(| \gamma_k(z) - \gamma_k(z') | > \lambda/2) \\ &\qquad\quad + \sum_{j=i_0}^{\infty} (\#\mathcal{A}_j)(\#\mathcal{A}_{j+1}) \\ &\qquad\quad \times \sup\{P(| \gamma_k(w) - \gamma_k(w') | > \lambda/40j^2 \colon w \in \mathcal{A}_j, w' \in \mathcal{A}_{j+1}, | w - w' | \leqslant c2^{-j}\}. \end{aligned}$$

Using Chebyshev with (9.7), we bound (9.9) by

$$c(p)2^{2i_0k}(1\vee\sup_{\mathcal{B}_{i_0}}\mid G\mid)^{\nu(p)}\delta^{ap}/\lambda^p+c(p)\sum_{j=i_0}^{\infty}2^{4jk}(1\vee\sup_{\mathcal{B}_j}\mid G\mid)^{\nu(p)}2^{-jap}(40j^2)^p/\lambda^p$$
$$\leqslant c(p)2^{2i_0k}(i_0)^{\nu(p)}\delta^{ap}/\lambda^p$$
$$+c(p)\sum_{j=i_0}^{\infty}2^{-2j}j^{\nu(p)}j^{2p}/\lambda^p$$

by our choice of p and the fact that $\sup_{\mathcal{B}_j}\mid G\mid\leqslant c\log(2^{-j})=cj$. Choosing δ so that $2^{-i_0}\leqslant\delta\leqslant 2^{-i_0+1}$, we see the series on the right is summable with a sum $\leqslant c(p)\delta^{a'p}/\lambda^p$. A standard Borel–Cantelli argument shows that $\gamma_k(z)$ is uniformly Hölder continuous on $\mathcal{A}$, a.s. By Proposition 9.1 and Theorem 8.1, we know that $\gamma_k(z)$ is Hölder continuous on $(\mathbb{R}^2-\{0\})^{k-1}\times[0,1]$. So we can extend $\gamma_k(z)$ to be continuous on $B(0,R)^{k-1}\times[0,1]$. Since R is arbitrary, this completes the proof. □

Remark. In the above proof, we obtained the estimate

$$P\Big(\sup_{\substack{|z-z'|<\delta\\ z,z'\in\mathcal{A}}}|\gamma_k(z)-\gamma_k(z')|>\lambda\Big)\leqslant c(p)\delta^{a'p}/\lambda^p. \tag{9.10}$$

Given $p_0\geqslant 1$, if we take $p=p_0+1$, multiply by $p_0\lambda^{p_0-1}$ and integrate from 0 to ∞, and then use the fact that γ_k is continuous, we get

$$E(\sup_{|z-z'|<\delta}|\gamma_k(z)-\gamma_k(z')|^{p_0})\leqslant c(p)\delta^{a'p/2}. \tag{9.11}$$

Remark. Theorem 9.3 was conjectured but not proved in [Sh].

10. Other Results.

A. Diffusions. With minor modifications, most of our results also hold for elliptic diffusions. We consider two cases:

Case 1. X_t corresponds to the operator

$$Lf(x)=\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial f}{\partial x_j}(x)\right)$$

where the a_{ij} is bounded and uniformly elliptic (no smoothness required).

Case 2. X_t corresponds to the operator

$$Lf(x) = \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} b_i(x) \frac{\partial f}{\partial x_i}(x),$$

where the a_{ij} are uniformly elliptic and the a_{ij}, b_i are C^1.

In both cases, it is known (see [FS] and [LSW] for Case 1, [Fr] for Case 2) that the Green function $g(x,y)$ for X_t and the Green function $g_R(x,y)$ for X_t killed on exiting $B(0,R)$ are comparable (that is, the ratio is bounded above and below by constants) to the corresponding Green functions for Brownian motion, provided x and y are close together and in the interior of $B(0,R)$. In fact, the transition densities are also comparable to those of Brownian motion if x and y are close together, so the local properties (e.g., modulus of continuity, time to exit a small ball, etc.) of X_t are similar to those of Brownian motion.

In Case 2, the gradient of the Green function is comparable to that of Brownian motion, hence one can prove an analog of Brosamler's theorem for measures μ on $\mathbb{R}^d$ satisfying

$$\mu(B(y,s)) \leqslant c \mid (s \wedge 1) \mid^{d-2+\gamma},$$

c, γ independent of y, s by starting with Ito's formula and taking limits. For Case 1, we can only assert, in general, that the Green function is Hölder continuous, and so we cannot get an explicit form for the martingale term in Case 1; the Hölder continuity follows from Moser's Harnack inequality by a standard argument (see [M]).

In Section 5, the proof of Proposition 5.1 needs to be modified. Let $D_1(z, Y_r)$ be the number of crossings from $\partial B(z,\sigma)$ to $\partial B(z, \sigma^{1-\varepsilon})$ by $Y_r, r \leqslant 1$. We need a bound on $D_1(x, X_1 - X_r)$. However, $D_1(x, X_1 - X_r) \leqslant \sup_z D_1(z, X_r)$. For a single z, an upper bound for $D_1(z, X_r)$ can be proved similarly to the Brownian case (in the proof of Proposition 5.1, we need to set $R_t = G(Y_t, z)$); the bound has the same form and is exponential. One can get an exponential bound for $\sup_z D_1(z, X_r)$ by techniques very similar to the proof of Lemma 2.1.

In the same way, if $T(z, Y_r)$ denotes the time for $Y_r - x$ to exceed $\sigma^{1-\varepsilon}$, then

$$T(x, X_1 - X_r) \leqslant \sup_z T(z, X_r).$$

We can get bounds for a single z similar to the bound for Brownian motion. Again using the techniques of Lemma 2.1, we get a bound for the sup in z.

With these observations, all the proofs of Sections 2–9 go through with only minor modifications. We obtain, for example, for self-intersections of X_t in Case 2 the Tanaka formula for ILT for k-multiple points:

$$\int_0^t [G(X_t, X_r + x_1) - G(X_r, X_r + x_1)]\beta_{k-1}(x_2, \cdots, x_{k-1}, dr)$$
$$= \int_0^t [\int_0^s \nabla G(X_s, X_r + x_1)\beta_{k-1}(x_2, \cdots, x_{k-1}, dr)] \cdot dX_s^{(M)} - \beta_k(x_y, \cdots, x_{k-1}, t).$$

Here $G(x, y) = \int_0^\infty p_t(x, y)dt$, where $p_t(x, y)$ is the transition density of X_t and $X_s^{(M)}$ denotes the martingale part of X_s. In Case 1, we get the same expression on the left-hand side, but we cannot give an explicit formula for the martingale term. In either case, if one defines renormalized ILT for double points by

$$\gamma_2(x, t) = \beta_2(x, t) - \int_0^t G(X_r, X_r + x)dr,$$

then γ_2 will be Hölder continuous in both variables. Similar results hold for renormalization of k multiple points. See [R] for what was known previously.

B. Stable Processes. If X_t is a symmetric stable process in $\mathbb{R}^2$ of order α, then X_t will have k multiple points (and k independent such processes will intersect) if $2 - 2/k < \alpha$. In this case, we expect some of the above to go through with appropriate modifications. For example, whenever we applied Hölder's inequality, one must be much more careful with the exponents used to make sure g_R^p and G^p are appropriately integrable (where g_R, G denote the Green function in $B(0, R)$ and Newtonian potential kernel in $\mathbb{R}^d$, respectively, for X_t). In Brosamler's formula and the Tanaka formula, one does not have an explicit form for the martingale term. (Actually, one could express the martingale in terms of a stochastic integral with respect to a Poisson point process, but this does not seem very useful.) Numerous other modifications will also be necessary.

C. ILTs in Certain Random Sets. Suppose X_t and Y_t are independent Brownian motions in $\mathbb{R}^2$. Suppose one has some sort of local time L_t for a certain random set. For example, L_t might measure the amount of some appropriate subset of cone points up to time t. Other possibilities might be local times for cut points or where one of the coordinates of X_t has a slow point. If one defines $\mu(A) = \int_0^u 1_A(X_s)dL_s$ and can prove appropriate estimates, the associated additive functional would measure the amount of time Y_t intersects X_s at points where X_s is "slow" or is a cone point. Without an application in mind, we do not pursue this further.

References.

[B1] Bass, R. F.: Joint continuity and representations of additive functionals of d-dimensional Brownian motion. Stoch. Proc. Applic. **17**, 211–227 (1984)

[B2] Bass, R.F.: A probabilistic approach to the boundedness of singular integral operators. In: Séminaire de Probabilités XXIV, (eds. J. Azèma, M. Yor), 15–40. New York: Springer 1980.

[BK1] Bass, R.F., Khoshnevisan, D.: Local times on curves and uniform invariance principles. Probab. Th. & rel. Fields, to appear.

[BK2] Bass, R.F., Khoshnevisan, D.: Strong approximations to Brownian local time. Preprint.

[BG] Blumenthal, R.M., Getoor, R.K.: Markov Processes and Potential Theory. New York: Academic Press 1968.

[Br] Brosamler, G. A.: Quadratic variation of potentials and harmonic functions. Trans. Amer. Math. Soc. **149**, 243–257 (1970)

[Ch] Chung, K.L.: A Course in Probability Theory, 2nd ed. New York: Academic Press, 1974.

[DM] Dellacherie, C., Meyer, P.-A.: Probabilités et Potentiel: Théorie des Martingales. Paris: Hermann 1980

[D1] Dynkin, E. B.: Self–intersection gauge for random walks for Brownian motion. Ann. Probab. **16**, 1–57 (1988)

[D2] Dynkin, E.B.: Regularized self–intersection local times of planar Brownian motion. Ann. Probab. **16**, 58-74, (1988).

[FS] Fabes, E.B., Stroock, D.W.: A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash. Arch. Rat. Mech. Anal. **96**, 327–338, (1986).

[Fr] Friedman, A.: Partial Differential Equations of Parabolic Type. Englewood Cliffs, NJ: Prentice–Hall 1964.

[L] LeGall, J.–F.: Some properties of planar Brownian motion. Ecole d'Eté de St. Flour XX. To appear.

[LSW] Littman, W., Stampacchia, G., Weinberger, H.F.: Regular points for elliptic equations with discontinuous coefficients. Ann. Scuola Norm. Sup. Pisa **17**, 43–77, (1963).

[M] Moser, J.: On Harnack's theorem for elliptic differential operators. Comm. Pure Appl. Math. **14**, 577–591, (1961).

[ReY] Revuz, D., Yor, M.: Continuous Martingales and Brownian Motion. New York: Springer 1991.

[R1] Rosen, J.: A renormalized local time for multiple intersections of planar Brownian motion. In: Séminaire de Probabilités XX (eds. Azèma, J., Yor, M.), 515–531. New York: Springer 1986.

[R2] Rosen, J.: Joint continuity of the intersection local time of Markov processes. Ann. Probab. **15**, 659–675, (1987).

[R3] Rosen, J.: Random walks and intersection local time. Ann. Probab. **18**, 959–977 (1990)

[RY] Rosen, J., Yor, M.: Tanaka formulae and renormalization for triple intersections of Brownian motion in the plane. Ann. Probab. **19**, 142–159, (1991).

[Sh] Shieh, N.–R.: White noise analysis and Tanaka formula for intersections of planar Brownian motion. Nagoya Math. J., to appear.

[Y] Yor, M.: Compléments aux formules de Tanaka–Rosen. In: Séminaire de Probabilités XIX (eds. Azèma, J., Yor, M.), 332–349. New York: Springer 1985.