PROGRESS IN PROBABILITY, **45**, 201–210, Birkhäuser-Verlag, Basel (1999)

ITERATED BROWNIAN MOTION AND ITS INTRINSIC SKELETAL STRUCTURE

By Davar Khoshnevisan† and Thomas M. Lewis
The University of Utah and *Furman University*

Abstract. This is an overview of some recent results on the stochastic analysis of iterated Brownian motion. In particular, we make explicit an intrinsic skeletal structure for the iterated Brownian motion which can be thought of as the analogue of the strong Markov property. As a particular application, we derive a change of variables (i.e., Itô's) formula for iterated Brownian motion.

§1. INTRODUCTION

Let $X^+ \triangleq \{X^+(t) : t \geq 0\}$, $X^- \triangleq \{X^-(t) : t \geq 0\}$, and $Y \triangleq \{Y(t) : t \geq 0\}$ be independent Brownian motions, starting from the origin. Let

$$X(t) \triangleq \begin{cases} X^+(t) & \text{if } t \geq 0; \\ X^-(-t) & \text{it } t < 0. \end{cases}$$

Hence $X \triangleq \{X(t) : t \in \mathbb{R}\}$ is a two–sided Brownian motion. *Iterated Brownian motion* (I.B.M.) Z is defined by

$$Z(t) \triangleq X(Y(t)), \qquad t \geq 0.$$

It can be shown that Z is not a Markov process in its natural filtration. It is, however, self–similar with parameter 1/4, which means that for any $c > 0$ the finite–dimensional distributions of $\{Z(ct) : t \geq 0\}$ and $\{c^{1/4} Z(t) : t \geq 0\}$ agree. The reader can find a host of results on I.B.M. and its close relative, the Bahadur–Kiefer process, in [1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 16, 17, 18, 20, 21, 22, 29, 30]. In particular, we mention the recent work of [7], where it is shown that, suitably interpreted, I.B.M. is the canonical motion in an independent Brownian fissure.

There is a common thread which runs through references [20, 21, 22]. In each of these articles, we analyze Z by means of certain stopping times for Y. In

† Research Partially Supported by grants from NSF and NSA

principle, this idea is quite simple: by stopping Y as it crosses certain levels, and by sampling Z at these times, one can effectively separate X from Y. Our most extensive use of this method has been in [22], where we have developed a stochastic calculus for I.B.M. In this work, the stopping–time method is used to construct what we call the *intrinsic skeletal structure* (I.S.S.) of Z, and our development of the stochastic integral is fundamentally based on the I.S.S. The main motivation for this paper is to give a concise account of this structure and indicate how it is used to develop the stochastic integral.

According to [5], the sample paths of Z have finite quartic variation almost surely. An immediate consequence of this fact is that Z has infinite quadratic variation on a set of full measure. Thus Z is not a semi-martingale; nor is it a Dirichlet process. As such, our construction of a stochastic integral with respect to Z is necessarily nontrivial. Through the I.S.S., we derive the following fundamental theorem of calculus for I.B.M.: if $f : \mathbb{R} \to \mathbb{R}$ is sufficiently smooth, then

$$f\big(Z(t)\big) = f(0) + \int_0^t f'\big(Z(s)\big)\partial Z(s) \tag{1.1}$$

almost surely, where ∂Z is what we call the *Stratonovich differential of* Z. At present, we do not have a good understanding of the connection between formula (1.1) and the notion that I.B.M. is the canonical process on a Brownian crack. There is, however, a loose relationship between I.B.M. and the biLaplacian (see [4, 13]), and, as such, our work may have a fourth–order P.D.E. interpretation. A variety of different attempts at a probabilistic description of such P.D.E.'s can be found in [8, 9, 13, 14, 15, 24, 26, 27, 28].

Recently T. Lyons [25] has developed another approach to stochastic integration with respect to “rough signals”, which is based on the beautiful idea that if one can construct enough “stochastic areas”, then one can obtain stochastic integrals as a by–product. In fact, our methods can be used to construct such area integrals; however, our theory is self–contained and yields additional information, equation (1.1) being a case in point.

We conclude this section with a brief outline of this paper. In §2, we describe the I.S.S. §3 is an overview of some of the main results of [22]. In particular, we describe the ∂ operator and we present a collection of weak and strong limit theorems concerning the variations of I.B.M., including a surprising connection between these variations and H. Kesten and F. Spitzer's Brownian motion in random scenery (see [19] for further information on Brownian motion in random scenery). While there are no problems in constructing a Stratanovich integral with respect to I.B.M., defining an Itô integral is another matter, as the quadratic variation is unbounded. In Theorem 4.1 of §4, we describe the nature of this infinity in terms of a renormalization method. It is interesting that in a similar context, K. Nishioka [27] obtains solutions to related P.D.E.'s with “infinities” such as our Theorem 4.1. It can be shown that Theorem 4.1 is equivalent to (1.1). This fact is due to a cancellation of infinities. Is this related to the work of Nishioka? In §5, we conclude this note with some remarks and some problems for further investigation.

PROGRESS IN PROBABILITY, **45**, 201–210, Birkhäuser-Verlag, Basel (1999)

§2. THE INTRINSIC SKELETAL STRUCTURE (I.S.S.)

For $n \geq 1$ and $k \in \mathbb{Z}$, let $r_{k,n} \triangleq k2^{-n/2}$. Of course, $\mathcal{D}_n \triangleq \{r_{k,n} : k \in \mathbb{Z}\}$, $n \geq 1$, is an equi-partition of $\mathbb{R}$ with mesh size $2^{n/2}$. For each positive integer n, let $T_{0,n} \triangleq 0$ and iteratively define

$$T_{j,n} \triangleq \inf\big\{s > T_{j-1,n} \;\big|\; Y(s) \in \mathcal{D}_n \setminus \{Y(T_{j-1,n})\}\big\}, \qquad j \geq 1.$$

It can be shown that as n tends to infinity, the collection $\mathcal{T}_n \triangleq \{T_{j,n} : j \geq 0\}$ approximates the common dyadic partition $\{j2^{-n} : j \geq 0\}$ (see [22, Lemma 2.2] for a precise statement and proof of this result). For $n \geq 1$ and $j \geq 0$, let $S_{j,n} = Y(T_{j,n})$. Then, for each admissible n, $\mathcal{P}_n \triangleq \{S_{j,n} : j \geq 0\}$ is a simple symmetric random walk on $\mathcal{D}_n$. The *intrinsic skeletal structure* of Z is the collection

$$\text{I.S.S.} \triangleq \big\{(\mathcal{D}_n, \mathcal{P}_n, \mathcal{T}_n) \;: n \geq 1\big\}.$$

The principle use of the I.S.S. is through the following decomposition. For each integer $n \geq 1$, $k \in \mathbb{Z}$ and $t \geq 0$, let $U_{k,n}(t)$ (resp. $D_{k,n}(t)$) denote the number of upcrossings (resp. downcrossings) of the interval $[r_{k,n}, r_{k+1,n}]$ within the first $[2^n t]$ steps of the random walk $\mathcal{P}_n$. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$. We will say that φ is symmetric (resp. skew–symmetric) provided that $\varphi(x,y) = \varphi(y,x)$ for $x, y \in \mathbb{R}$ (resp. $\varphi(x,y) = -\varphi(y,x)$ for $x, y \in \mathbb{R}$).

Proposition 2.1. ([22, Lemma 2.4]) *If φ is symmetric, then*

$$\sum_{k=0}^{[2^n t]-1} \varphi\big(Z(T_{k,n}), Z(T_{k+1,n})\big) = \sum_{j \in \mathbb{Z}} \varphi\big(X(r_{j,n}), X(r_{j+1,n})\big)\big(U_{j,n}(t) + D_{j,n}(t)\big).$$

If φ is skew–symmetric, then

$$\sum_{k=0}^{[2^n t]-1} \varphi\big(Z(T_{k,n}), Z(T_{k+1,n})\big) = \sum_{j \in \mathbb{Z}} \varphi\big(X(r_{j,n}), X(r_{j+1,n})\big)\big(U_{j,n}(t) - D_{j,n}(t)\big).$$

As we will see in the next section, the form

$$\sum_{k=0}^{[2^n t]-1} \varphi\big(Z(T_{k,n}), Z(T_{k+1,n})\big)$$

is a recurring theme in [22]. The significant feature of this decomposition is that it separates X from Y, rendering this form amenable to analysis.

§3. STRATONOVICH INTEGRATION

In this section, we define our Stratonovich integral with respect to I.B.M. and study the variations of I.B.M. Since a detailed account of this material can be found in [22], we will state our results without formal proof.

For $k \geq 0$, let C_b^k denote the set of all functions $f : \mathbb{R} \to \mathbb{R}$ whose first k derivatives are bounded and continuous. Recall from §1, that X^+ and X^- are independent standard Brownian motions, and that X is the two–sided Brownian motion constructed from X^+ and X^-. For $f \in C_b^2$ and $t \in \mathbb{R}$, define (in the Itô sense)

$$\int_0^t f(X(s))dX(s) \triangleq \begin{cases} \int_0^t f(X^+(s))dX^+(s), & \text{if } t \geq 0; \\ \int_0^{-t} f(X^-(s))dX^-(s), & \text{if } t \leq 0. \end{cases}$$

The corresponding two–sided Stratanovich integral is given by

$$\int_0^t f(X(s))\partial X(s) \triangleq \int_0^t f(X(s))dX(s) + \frac{1}{2}\mathrm{sgn}(t) \int_0^t f'(X(s))ds.$$

For $f \in C_b^2$ and $t \geq 0$, define

$$\int_0^t f(Z(s))\partial Z(s) \triangleq \int_0^{Y(t)} f(X(s))\partial X(s). \tag{3.1}$$

Equation (3.1) may be taken as the definition of the Stratonovich integral with respect to I.B.M., and, as our first theorem shows, this definition is both sensible and consistent.

Theorem 3.1. ([22, Theorem 2.1]) *Let $f \in C_b^2$ and let $t \geq 0$. Then*

$$\lim_{n\to\infty} \sum_{k=0}^{[2^n t]-1} f\Big(\frac{Z(T_{k+1,n}) + Z(T_{k,n})}{2}\Big)\big(Z(T_{k+1,n}) - Z(T_{k,n})\big) = \int_0^t f(Z(s))\partial Z(s)$$

almost surely and in $L^2(P)$.

In other words, by taking our partition of integration from the I.S.S., we obtain Stratonovich ∂Z–integrals by the familiar midpoint rule. A few additional remarks are in order. First, it can be shown that the convergence in Theorem 3.1 holds uniformly on t–compacta. Furthermore, we observe that (3.1) and Theorem 3.1 imply that for all $t \geq 0$, $I_t : f \mapsto \int_0^t f(Z(s))\partial Z(s)$ is a random linear operator from C_b^2 into $\mathbb{R}$. (It is possible to extend the domain of this definition.) The nature of the stochastic evolution of $t \mapsto I_t$ is an important problem which has yet to be studied.

Next, we discuss the variations of I.B.M. To simplify the notation, let us fix a nonnegative real number t. For integers $n, k \geq 0$ and $f \in C_b^2$, let

$$V_n^{(k)}(f) = \sum_{j=0}^{[2^n t]-1} f\Big(\frac{Z(T_{k+1,n}) + Z(T_{k,n})}{2}\Big)\big(Z(T_{k+1,n}) - Z(T_{k,n})\big)^k. \tag{3.3}$$

When f is identically equal to 1, $V_n^{(k)}(f)$ is the usual k-th variation of Z, although we must emphasize that the underlying partition is drawn from the I.S.S.

Our first results describe the behavior of the quadratic variation.

Theorem 3.2. ([22, Theorem 3.1]) *For each* $f \in C_b^2$,

$$\lim_{n\to\infty} 2^{-n/2} V_n^{(2)}(f) = \int_0^t f(Z(s))ds$$

almost surely and in $L^2(P)$.

The analysis of the rate of convergence in Theorem 3.2 reveals a striking connection to Brownian motion and random scenery, a process which was first described by H. Kesten and F. Spitzer [19]. Let B_1 be a two–sided Brownian motion, let B_2 be an independent Brownian motion, and let $\{L_t^x(B_2): \ t \geq 0, x \in \mathbb{R}\}$ denote the process of local times of B_2. Let

$$G(t) \triangleq \int_{-\infty}^{\infty} L_t^x(B_2) dB_1(x), \qquad t \geq 0. \tag{3.4}$$

The process $G \triangleq \{G(t) : t \geq 0\}$ is called *Brownian motion in random scenery* (B.M.R.S.).

In the statements of our next results, we will use $\mathbf{1}$ to denote the function which is identically equal to 1.

Theorem 3.3. ([22, Theorem 3.2]) *As* $n \to \infty$,

$$\frac{2^{n/4}}{\sqrt{2}}\big(2^{-n/2}V_n^{(2)}(\mathbf{1}) - t\big) \overset{D[0,1]}{\longrightarrow} G(t)$$

Next, we consider the tertiary and quartic variations of I.B.M.

Theorem 3.4. ([22, Theorems 4.2, 4.3, 4.4, 4.5]) *For* $f \in C_b^2$,

$$\lim_{n\to\infty} V_n^{(3)}(f) = 0 \quad \textit{and} \quad \lim_{n\to\infty} V_n^{(4)}(f) = 3\int_0^t f(Z(s))ds$$

almost surely and in $L^2(P)$*. Moreover, as* $n \to \infty$,

$$\frac{2^{n/2}}{\sqrt{15}} V_n^{(3)}(\mathbf{1}) \overset{D[0,1]}{\longrightarrow} Z(t) \quad \textit{and} \quad \frac{2^{n/4}}{\sqrt{96}}\big(V_n^{(4)}(\mathbf{1}) - 3t\big) \overset{D[0,1]}{\longrightarrow} G(t).$$

In particular, along the I.S.S., Z has quartic variation. In [5, Theorem 1], K. Burdzy has shown this fact along certain nonrandom partitions, and our results are consistent with his.

PROGRESS IN PROBABILITY, **45**, 201–210, Birkhäuser-Verlag, Basel (1999)

It can be shown that, suitably normalized, each even–ordered variation of I.B.M. converges in distribution to B.M.R.S., while each odd–ordered variation of I.B.M. converges in distribution to I.B.M. itself, which suggests the existence of a form of measure–theoretic duality between I.B.M. and B.M.R.S.

To conclude this section, let us make a brief remark on the proofs of these results. For each $k \geq 1$, the mapping

$$\varphi(x,y) = f\left(\frac{x+y}{2}\right)(y-x)^k$$

is either symmetric (k even) or skew–symmetric (k odd); consequently, the proofs of these results begin with Proposition 2.1, the fundamental decomposition result.

§4. RENORMALIZED ITÔ INTEGRALS

In this section, we investigate the Itô calculus analogues of the results of the §3. Throughout, let t be a nonnegative real number. Given integers $n, k \geq 1$ and a Borel function $f : \mathbb{R} \to \mathbb{R}$, let

$$I_n^{(k)}(f) \triangleq \sum_{j=0}^{[2^n t]-1} f\big(Z(T_{j,n})\big)\big(Z(T_{j+1,n}) - Z(T_{j,n})\big)^k. \tag{4.1}$$

Note that $I_n^{(k)}(f)$ differs from $V_n^{(k)}(f)$ (cf. (3.3)) in the use of the left–hand rule instead of the midpoint rule, which is precisely what distinguishes the Itô and Stratonovich theories of stochastic calculus.

In this section, we study the asymptotic behavior of $I_n^{(k)}(f)$. As we shall see, $I_n^{(1)}(f)$ and $I_n^{(2)}(f)$ are generally divergent as $n \to \infty$, and this divergence can be related directly, through renormalization, to the divergence of the quadratic variation of I.B.M.

Theorem 4.1. *For $f \in C_b^5$,*

$$\lim_{n\to\infty} \left[I_n^{(1)}(f) + \frac{1}{2}V_n^{(2)}(f')\right] = \int_0^t f(Z(s))\partial Z(s) - \frac{1}{16}\int_0^t f'''(Z(s))ds$$

$$\lim_{n\to\infty} \left[I_n^{(2)}(f) - V_n^{(2)}(f)\right] = \frac{3}{8}\int_0^t f''(Z(s))ds$$

$$\lim_{n\to\infty} I_n^{(3)}(f) = -\frac{3}{2}\int_0^t f'(Z(s))ds$$

$$\lim_{n\to\infty} I_n^{(4)}(f) = 3\int_0^t f(Z(s))ds.$$

almost surely and in $L^2(P)$.

It is noteworthy that the Itô and Stratonovich quartic variations agree while their tertiary variations do not.

PROGRESS IN PROBABILITY, **45**, 201–210, Birkhäuser-Verlag, Basel (1999)

By Theorems 3.2 and 4.1, it is easy to give the precise rate of divergence of $I_n^{(1)}(f)$ and $I_n^{(2)}(f)$. We have

$$\lim_{n\to\infty} 2^{-n/2} I_n^{(1)}(f) = -\frac{1}{2}\int_0^t f'(Z(s))ds$$

and

$$\lim_{n\to\infty} 2^{-n/2} I_n^{(2)}(f) = \int_0^t f(Z(s))ds$$

almost surely and in $L^2(P)$.

Proof of Theorem 4.1. For simplicity, let

$$\begin{aligned} M_{j,n} &\triangleq \frac{Z(T_{j+1,n}) + Z(T_{j,n})}{2} \\ \Delta Z_{j,n} &\triangleq Z(T_{j+1,n}) - Z(T_{j,n}). \end{aligned}$$

Since $f \in C_b^5$, we can apply Taylor's expansion to see that

$$\begin{aligned} f(Z(T_{j,n})) = f(M_{j,n}) - \frac{1}{2} f'(M_{j,n}) \cdot (\Delta Z_{j,n}) + \frac{1}{8} f''(M_{j,n}) \cdot (\Delta Z_{j,n})^2 \\ - \frac{1}{48} f'''(M_{j,n}) \cdot (\Delta Z_{j,n})^3 + b_{j,n} \cdot (\Delta Z_{j,n})^4, \end{aligned}$$

where $b_{j,n}$ is a random variable bounded by $\sup_{x\in\mathbb{R}} |f^{(4)}(x)|$. By (3.3) and (4.1), it follows that, for $k \geq 1$,

$$I_n^{(k)}(f) = V_n^{(k)}(f) - \frac{1}{2} V_n^{(k+1)}(f') + \frac{1}{8} V_n^{(k+2)}(f'') - \frac{1}{48} V_n^{(k+3)}(f''') + \varepsilon_n^{(k)}(f),$$

where $\lim_{n\to\infty} \varepsilon_n^{(k)}(f) = 0$ almost surely and in $L^2(P)$. By appealing to Theorems 3.1, 3.3 and 3.4, we obtain the stated result. ////

§5. EPILOGUE

§5.1. Universality. An important open problem is to determine whether I.B.M. is the canonical process on a Brownian crack. The results of [7] clearly indicate that this may be so, and the work of [22] gives further evidence for this conclusion. To obtain a better sense of universality, it would be useful to show that the I.S.S. is not a special partitioning algorithm. Thus we pose the following problem:

Problem 5.1. *Let $f \in C_b^\infty$, let t be a nonnegative real number, and let $\Pi = \{t_j : 1 \leq j \leq J\}$ be a partition of $[0, t]$. Is it true that*

$$\sum_{j=1}^{J} f\Big(\frac{Z(t_{j+1}) + Z(t_j)}{2}\Big) \cdot \big(Z(t_{j+1}) - Z(t_j)\big) \to \int_0^t f(Z(s))\partial Z(s)$$

in probability as mesh$(\Pi) \to 0$*?*

§**5.2. Brownian motion in random scenery.** It is easy to show that G (see (3.4)) is self–similar with index 3/4, that is, for all $c > 0$, $\{G(ct) : t \geq 0\}$ and $\{c^{3/4}G(t) : t \geq 0\}$ have the same finite dimensional distributions. You will recall that Z is self–similar with index 1/4. Roughly speaking, this suggests that G and Z might have "dual" dimensional properties. As an example, recall from [6] that the level sets of Z have Hausdorff dimension 3/4 (cf. [16] for further results and refinements). This motivates the following conjecture:

Conjecture 5.2. *Let* dim *denote Hausdorff dimension. Then*

$$\dim \Big\{ s \geq 0 : G(s) = 0 \Big\} = \frac{1}{4} \qquad \text{a.s.}$$

We also suspect that some form of duality between G and Z will be evident in the functional forms of their respective laws of the iterated logarithm, perhaps a duality between their reproducing kernel Hilbert spaces. To this end, we have demonstrated the following provisional result:

Theorem 5.3. ([23, Theorem 1.1]) *There exists a constant* $0 < C_1 < \infty$ *such that*

$$\limsup_{t\to\infty} \frac{G(t)}{(t \log\log t)^{3/4}} = C_1 \qquad \text{a.s.}$$

This should be compared to a result of K. Burdzy concerning I.B.M.

Theorem 5.4. ([4, Theorem 3]) *There exists a constant* $0 < C_2 < \infty$ *such that*

$$\limsup_{t\to\infty} \frac{Z(t)}{t^{1/4}(\log\log t)^{3/4}} = C_2 \qquad \text{a.s.}$$

It should be noted that the value of C_2 is known; however, at present, we do not know the value of C_1.

While the functional analogue of Theorem 5.4 has been demonstrated [10], the functional analogue of Theorem 5.3 has not. Thus we offer the following problem:

Problem 5.5. *For each integer* $n \geq 1$, *let*

$$G_n(t) \triangleq \frac{G(nt)}{(n \log\log n)^{3/4}}, \qquad 0 \leq t \leq 1.$$

From Theorem 5.3 and tightness arguments, it can be shown that the limit points (in the uniform topology) of $\{G_n : n \geq 1\}$ *form a compact set. Is there an energy characterization of this set?*

PROGRESS IN PROBABILITY, **45**, 201–210, Birkhäuser-Verlag, Basel (1999)

REFERENCES

[1] Arcones, M.A. (1995). On the law of the iterated logarithm for Gaussian processes, *J. Th. Prob.*, **8**, 877–903

[2] Bertoin, J. (1996). Iterated Brownian motion and stable (1/4) subordinator, *Stat. Prob. Lett.*, **27**(2), 111–114

[3] Bertoin, J. and Shi, Z. (1996). Hirsch's integral test for iterated Brownian motion. *Sém. de Prob., XXX*, Lecture Notes in Math., **#1626**, 361–368 (ed's: J. Azéma, M. Emery, P.A. Meyer and M. Yor)

[4] Burdzy, K. (1993) Some path properties of iterated Brownian motion. *Sem. Stoch. Processes 1992*, 67–87, Birkhäuser, Boston (ed's: E. Çinlar, K. L. Chung and M. Sharpe)

[5] Burdzy, K. (1993). Variation of iterated Brownian motion, *Workshop and Conf. on Measure–Valued Processes, Stoch. Partial Diff. Eq.'s and Interacting Particle Systems.* CRM Proceedings and Lecture Notes

[6] Burdzy, K. and Khoshnevisan, D. (1995). The level sets of iterated Brownian motion, *Sém. de Prob., XXIX*, Lecture Notes in Math., **#1613**, 231–236 (ed's: J. Azéma, M. Emery, P.A. Meyer and M. Yor)

[7] Burdzy, K. and Khoshnevisan, D. (1997). Brownian motion in a Brownian crack. Preprint

[8] Burdzy, K. and Mądrecki, A. (1995). An asymptotically 4–stable process. *J. Fourier Analysis. Appl.* (Special Issue). Proceedings of the Conference in Honor of J.-P. Kahane, 97–117

[9] Burdzy, K. and Mądrecki, A. (1996). Ito formula for an asymptotically 4–stable process. *Ann. Appl. Prob.* **6**, 200–217

[10] Csáki, E., Csörgő, M., Földes, A. and Révész, P. (1995). Global Strassen type theorems for iterated Brownian motion. *Stoch. Proc. Appl.*, **59**, 321–341

[11] Csáki, E., Csörgő, M., Földes, A. and Révész, P. (1996). The local time of iterated Brownian motion, *J. Th. Prob.*, **9**(3), 717–743

[12] Deheuvels, P. and Mason, D.M. (1992). A functional LIL approach to pointwise Bahadur–Kiefer theorems, *Prob. in Banach Spaces*, **8**, 255–266 (ed's: R.M. Dudley, M.G. Hahn and J. Kuelbs)

[13] Funaki, T. (1979). A probabilistic construction of the solution of some higher order parabolic differential equations, *Proc. Japan Acad.*, **55**, 176–179

[14] Gaveau, B. and Sainty, P. (1988). A path integral formula for certain fourth–order elliptic operators, *Lett. Math. Phys.*, **15**, 345–350

[15] Hochberg, K. (1978). A signed measure on path space related to Wiener measure. *Ann. Prob.*, **6**, 433–458

[16] Hu, Y. (1997). Hausdorff and packing functions of the level sets of iterated Brownian motion. Preprint

[17] Hu, Y., Pierre-Lotti-Viaud, D. and Shi, Z. (1995). Laws of the iterated logarithm for iterated Wiener processes, *J. Th. Prob.*, **8**(2), 303–319

[18] Hu, Y. and Shi, Z. (1995). The Csörgő–Révész modulus of nondifferentiability of iterated Brownian motion, *Stoch. Proc. Appl.*, **58**, 267–279

[19] Kesten, H. and Spitzer, F. (1979). A limit theorem related to a new class of self–similar processes, *Z. Wahr. Verw. Geb.*, **50**, 327–340

[20] Khoshnevisan, D. and Lewis, T.M. (1996). The uniform modulus of continuity for iterated Brownian motion, *J. Th. Prob.*, **9**(2), 317–333

[21] Khoshnevisan, D. and Lewis, T.M. (1996). Chung's law of the iterated logarithm for iterated Brownian motion. *Ann. Inst. Henri Poincaré: Prob. et Stat.*, **32**(3), 349–359

[22] Khoshnevisan, D. and Lewis, T.M. (1997). Stochastic calculus for Brownian motion on a Brownian fracture. *Ann. Applied Prob.* (To Appear)

[23] Khoshnevisan, D. and Lewis, T.M. (1997). A law of the iterated logarithm for stable processes in random scenery. Preprint

[24] Krylov, V.Yu., (1960). Some properties of the distribution corresponding to the equation $\partial u/\partial t = (-1)^{q+1}\partial^{2q}u/\partial x^{2q}$, *Soviet Math. Dokl.*, **1**, 760–763

[25] Lyons, T. (1997). Differential Equations driven by rough signals, *Revista Math. Iberioamericana*, (To Appear)

[26] Nishioka, K. (1987). A stochastic solution of a high order parabolic equation. *J. Math. Soc. Japan*, **39**(2), 209–231

[27] Nishioka, K. (1987). A convergence of solutions of an inhomogeneous parabolic equation. *Proc. Japan Acad.*, **63**, Ser. A, No. 8, 292–294

[28] Nishioka, K. (1985). Stochastic Calculus for a class of evolution equations. *Japan J. Math.*, **11**(1), 59–102

[29] Shi, Z. (1995). Lower limit of iterated Wiener processes, *Stat. Prob. Lett.*, **23**, 259–270

[30] Xiao, Y. (1996). Local times and related properties of multi–dimensional iterated Brownian motion. *J. Th. Prob.* (To Appear)