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Escape rates for Lévy processes*

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Dedicated to Professor Endre Csáki, on the occasion of his sixtieth birthday

Abtract. We prove a space–time estimate for a Lévy process to hit a small set. As an application, we present escape rates for Lévy processes with strictly stable components.

§**1. Introduction.** Let X denote a d–dimensional Lévy process. It is a classical fact that a Borel set $A \subset \mathbb{R}^d$ is polar for X if and only if A has positive X–capacity; cf. Blumenthal and Getoor BG]. A sharper variant of the aforementioned fact is the consequence of more recent investigations such as those of Benjamini et al. [BPP], Fitzsimmons and Salisbury [FS], Peres [Pe] and Salisbury [Sa]. Roughly speaking, these results provide in a variety of different contexts, qualitiative estimates of the type: $\mathbb{P}^0\big(X_t \in A, \text{ for some } t > 0\big) \asymp e^{-1}(A)$, where $f \asymp g$ implies the existence of some universal $C > 1$, such that $C^{-1}g \le f \le Cf$ pointwise, and $e(A)$ is the X–energy integral associated with A. One of the many uses of such an estimate is that one can often approximate the chance that X ever hits a small set. Wishing to study escape rates, we present a different sort of a qualitative estimate below. Our notation is more or less that of Markov process theory.

(1.1) **Theorem.** *Suppose X is a d–dimensional Lévy process. For any $b > a > 0$ and $\varepsilon > 0$,*

$$\frac{1}{2}\frac{\int_a^b \mathbb{P}^0(|X_r| \le \varepsilon)dr}{\int_0^b \mathbb{P}^0(|X_r| \le \varepsilon)} \le \mathbb{P}^0\big(|X_r| \le \varepsilon, \text{ for some } a \le t \le b\big) \le \frac{\int_a^{2b-a} \mathbb{P}^0(|X_r| \le 2\varepsilon)dr}{\int_0^{b-a} \mathbb{P}^0(|X_r| \le \varepsilon)dr},$$

whenever the integrals exist and are nonzero.

The above extends the estimates of Perkins and Taylor [PT], Takeuchi [T1,T2] and Takeuchi and Watanabe [TW], to cite a few examples. To illustrate the use of such a general inequality, let us restrict attention to the class of processes described in Hendricks [H1,H2,H3]. Namely, we consider the case where X is a d–dimensional Lévy process with strictly stable components. In other words, there exists $\upsilon, \chi \in \mathbb{R}^d_+$ and $\alpha \in (0,2]^d$, such that for all $t > 0$ and all $\zeta \in \mathbb{R}^d$,

$$\mathbb{P}^0 \exp\big(i\zeta' X_t\big) = \exp\Big(-t\sum_{j=1}^d |\zeta_j \chi_j|^{1/\alpha_j} - i\sum_{j=1}^d \upsilon_j \operatorname{sgn}(\zeta_j)\Big). \tag{1.2}$$

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Throughout, we shall assume that the coordinate processes are not completely asymmetric, i.e,.

$$\left|\frac{\upsilon_j}{\chi_j}\right| < \tan\big(\pi\alpha_j/2\big),\ |\chi_j| > 0, \qquad \text{for all } j = 1, \cdots, d. \tag{1.3}$$

Viewed coordinate by coordinate, such processes scale, albeit differently in each. Define,

$$\beta = \sum_{j=1}^{d} \frac{1}{\alpha_j}. \tag{1.4}$$

Our intended application of Theorem (1.1) is the following:

(1.5) **Theorem.** *Suppose X is a Lévy process with stable components with parameters given by (1.2)–(1.4). When $\beta < 1$, X hits points. When $\beta = 1$, singletons are polar, but X is neighborhood recurrent. When $\beta > 1$, X is transient. For $\beta \geq 1$, let $\varphi : \mathbb{R}^1_+ \mapsto \mathbb{R}^1_+$ be an decreasing function and define*

$$\mathfrak{J}(\varphi) = \begin{cases} \int_1^\infty \varphi^{\beta-1}(t) t^{-1} dt, & \text{if } \beta > 1 \\ \int_1^\infty \big(t|\ln\varphi(t)|\big)^{-1} dt, & \text{if } \beta = 1 \end{cases}.$$

When $\beta \geq 1$, $\mathbb{P}^0$–almost surely,

$$\liminf_{\substack{t\to\infty \\ (t\to 0^+)}} \frac{\max_{1\leqslant j\leqslant d} |X_t^j|^{\alpha_j}}{t\varphi(t)} = \begin{cases} \infty, & \text{if } \mathfrak{J}(\varphi) < \infty \\ 0, & \text{if } \mathfrak{J}(\varphi) = \infty \end{cases}.$$

When α is a constant vector, the above appears to various degrees of generality in Dvoretsky and Erdős [DE], Spitzer [Sp], Takeuchi [T1,T2] and Takeuchi and Watanabe [TW]. When α is not a constant vector, a different but equivalent formulation can be found in Hendricks [H1] with a longer proof. Our formulation has two distinct advantages over the latter: (1) the large–time results and the small–time results are the same; (2) ours incorporates all the known results as one. Note that the critical case (i.e., $\beta = 1$) only applies to two cases: $d = 1$ and $\alpha = 1$ (Cauchy process on $\mathbb{R}^1$) or $d = 2$ and $\alpha_1 = \alpha_2 = 2$ (planar Brownian motion).

Above and throughout, we have used the notation: $\ln x \triangleq \log_e(x \vee 1)$, $x \geqslant 0$.

§2. The Proof of Theorem (1.1). Fix $0 < a < b$ and define $T \triangleq \inf\big(s > 0 : |X_s| \leq \varepsilon\big)$. Apply the strong Markov property at time T to see that

$$\begin{aligned} \mathbb{P}^0 \int_a^{2b-a} 1\big(|X_r| \leq 2\varepsilon\big) dr &\geq \mathbb{P}^0\Big(\int_a^{2b-a} 1\big(|X_r| \leq 2\varepsilon\big) dr \;\big|\; T \leq b\Big) \cdot \mathbb{P}^0(T \leq b) \\ &\geq \inf_{|x|\leq\varepsilon} \mathbb{P}^x \int_0^{b-a} 1\big(|X_r| \leq 2\varepsilon\big) dr \cdot \mathbb{P}^0(T \leq b) \\ &\geq \mathbb{P}^0 \int_0^{b-a} 1\big(|X_r| \leq \varepsilon\big) dr \cdot \mathbb{P}^0(T \leq b). \end{aligned}$$

Divide to obtain the upper bound. The lower bound follows along similar lines. Consider,

$$\begin{aligned} \mathbb{P}^0\Big(\int_a^b 1\big(|X_r| \leq \varepsilon\big) dr\Big)^2 &= 2\mathbb{P}^0 \int_a^b \int_a^r 1\big(|X_r| \leq \varepsilon, |X_s| \leq \varepsilon\big) ds dr \\ &\leq 2\mathbb{P}^0 \int_a^b \int_a^r 1\big(|X_s| \leq \varepsilon\big) 1\big(|X_r - X_s| \leq 2\varepsilon\big) ds dr \\ &= 2 \int_a^b \int_s^b \mathbb{P}^0(|X_s| \leq \varepsilon) \mathbb{P}^0(|X_{r-s}| \leq 2\varepsilon) dr ds \\ &\leq 2\mathbb{P}^0 \int_a^b 1\big(|X_r| \leq \varepsilon\big) dr \cdot \int_0^b \mathbb{P}^0(|X_r| \leq 2\varepsilon) dr. \end{aligned} \tag{2.1}$$

By the Cauchy–Schwartz inequality,

$$\mathbb{P}^0 \int_a^b 1\big(|X_r| \le \varepsilon\big) dr = \mathbb{P}^0\Big(\int_a^b 1\big(|X_r| \le \varepsilon\big) dr; T \le b\Big)$$
$$\le \sqrt{\mathbb{P}^0\Big(\int_a^b 1\big(|X_r| \le \varepsilon\big) dr\Big)^2} \cdot \sqrt{\mathbb{P}^0(T \le b)}.$$

Use (2.1) and solve to obtain the desired lower bound. ♢

(2.2) **Remark.** An inspection of the proof shows that for any $x \in \mathbb{R}^d$,

$$\mathbb{P}^x\big(|X_t| \le \varepsilon, \text{ for some } a \le t \le b\big) \le \frac{\int_a^{2b-a} \mathbb{P}^x\big(|X_r| \le 2\varepsilon\big) dr}{\int_0^{b-a} \mathbb{P}^0\big(|X_r| \le \varepsilon\big) dr}.$$

§**3. The Proof of Theorem (1.5).** Throughout this section, X denotes a Lévy process with strictly stable components given by (1.2)–(1.4). Let us start with a technical lemma.

(3.1) **Lemma.** *The random variable $|X_1|$ has a bounded $\mathbb{P}^a$–density, uniformly over all $a \in \mathbb{R}^d$. When $a = 0$, this density g is positive on some neighborhood of 0. Moreover, $\sup_x g(x)/g(0) < \infty$.*

Proof. By properties of convolutions, it suffices to show that each component of X has the given properties. The lemma follows from the inversion theorem for Fourier transforms. ♢

For the rest of this section, define,

$$S(x) \triangleq \max_{1 \leqslant j \leqslant d} |x_j|^{\alpha_j}, \qquad x \in \mathbb{R}^d.$$

(3.2) **Lemma.** *For all $r, a > 0$,*

$$\mathbb{P}^0\big(S(X_r) \le a\big) \asymp \big(r^{-1}a \wedge 1\big)^\beta.$$

Proof. Since the components X^j are independent α_j–stable processes, by Lemma (3.1),

$$\mathbb{P}^0\big(S(X_r) \leqslant \varepsilon\big) = \prod_{j=1}^d \mathbb{P}\big(|X_1^j| \leqslant \varepsilon^{1/\alpha_j} r^{-1/\alpha_j}\big)$$
$$\asymp \Big(\frac{\varepsilon}{r}\Big)^\beta \wedge 1.$$

This proves the lemma. ♢

Since $\alpha_j \leqslant 2$ for all j, it is not hard to show that for all $x, y \in \mathbb{R}^d$, $S(x+y) \leqslant 4\big(S(x) + S(y)\big)$. As such, $S(x)$ behaves much like $|x|$. Going through the proof of Theorem (1.1) and using Lemma (3.2), the following estimate emerges:

(3.3) **Corollary.** *For all $0 < a < b$ and all $\varepsilon > 0$ small,*

$$\mathbb{P}^0\big(S(X_t) \le \varepsilon, \textit{ for some } a \le t \le b\big) \asymp h(\varepsilon),$$

where

$$h(\varepsilon) = \begin{cases} 1, & \text{if } \beta < 1 \\ \big(\ln(1/\varepsilon)\big)^{-1}, & \text{if } \beta = 1 \\ \varepsilon^{\beta-1}, & \text{if } \beta > 1 \end{cases}.$$

We are now ready to prove Theorem (1.5). We shall do so for $t \to \infty$. The case $t \to 0^+$ is done similarly. The case $\beta < 1$ follows immediately from Corollary (3.3). Let us restrict attention to the case $\beta \geq 1$. We shall assume without loss of generality that $\varphi(x) \downarrow 0$ as $x \to \infty$. (When $\inf_x \varphi(x) > 0$, the result is simpler and also follows from the proof given below.)

Define $t_n \triangleq 2^n$, $\varphi_n \triangleq \varphi(t_n)$ and

$$E_n \triangleq \Big\{ \inf_{t_n \leq t \leq t_{n+1}} S(X_t) \leq 2^n \varphi_n \Big\}. \tag{3.4}$$

Note that from Corollary (3.3),

$$\mathbb{P}^0(E_n) \asymp h(\varphi_n). \tag{3.5}$$

From the definition of $\mathfrak{J}(\varphi)$ given in (1.5), it follows that $\sum_n \mathbb{P}^0(E_n) < \infty$ if and only if $\mathfrak{J}(\psi) < \infty$, when $\psi(x) \triangleq A\varphi(Bx)$, for any $A, B > 0$.

Suppose $\mathfrak{J}(\varphi) < \infty$. The previous paragraph shows that for any $c > 0$,

$$\sum_n \mathbb{P}^0\big(\inf_{t_n \leq t \leq t_{n+1}} |X_t| \leq c 2^n \varphi(2^{n-1}) \big) < \infty.$$

Since c is arbitrary and φ is decreasing, by the Borel–Cantelli lemma,

$$\liminf_{t \to \infty} \frac{|X_t|}{t\varphi(t)} = \infty,$$

$\mathbb{P}^0$–a.s. . Now suppose $\mathfrak{J}(\varphi) = \infty$. Note that Remark 2.2 holds with $|X_\cdot|$ replaced by $S(X_\cdot)$ everywhere. Using the Markov property and Lemma (3.1), for all n large enough,

$$\mathbb{P}^0(E_n \cap E_{n+k}) \leq \mathbb{P}^0(E_n) \sup_x \mathbb{P}^x(E_{n+k}) \asymp \mathbb{P}^0(E_n)\mathbb{P}^0(E_{n+k}).$$

Theorem (1.5) follows from (3.4), (3.5), Kolmogorov's 0–1 law and the Kochen–Stone lemma ([KS]). ♢

References.

[BPP] I. Benjamini, Y. Peres and R. Pemantle (1995). Martin capacity for Markov chains. *Ann. Prob.*, **23**, 1332–1346

[BG] R. Blumenthal and R.K. Getoor (1968). *Markov Processes and Potential Theory.* Academic Press, N.Y.

[DE] A. Dvoretsky and P. Erdős (1950). Some problems on random walk in space, *Proc. Second Berkeley Symp.*, 353–367

[FS] P.J. Fitzsimmons and T.S. Salisbury (1989). Capacity and energy for multiparameter Markov processes, *Ann. Inst. Henri Poicaré: prob. et stat.*, **25**, 325–350

STUDIA SCI. MATH. HUNGARICA **33**, 177–183.

[H1] W.J. HENDRICKS (1970). Lower envelopes near zero and infinity for processes with stable components, *Z. Wahr. verw. Geb.*, **16**, 261–278

[H2] W.J. HENDRICKS (1972). Hausdorff dimension in a process with stable components – an interesting counter–example, *Ann. Math. Stat.*, **43**, 690–694

[H3] W.J. HENDRICKS (1974). Multiple points for a process in $\mathbb{R}^2$ with stable components, *Z. Wahr. verw. Geb.*, **28**, 113–128

[KS] S.B. KOCHEN AND C.J. STONE (1964). A note on the Borel–Cantelli lemma, *Ill. J. Math.*, **8**, 248–251

[Pe] Y. PERES (1995). Intersection–equivalence of Brownian paths and certain branching processes. *Comm. Math. Phys.* (To appear)

[PT] E.A. PERKINS AND S.J. TAYLOR (1987). Uniform measure results for the image of subsets under Brownian motion, *Prob. Th. Rel. Fields*, **76**, 257–289

[Sa] T.S. SALISBURY (1995). Energy, and intersection of Markov chains. *Proceedings of the IMS Workshop on Random Discrete Structures* (To appear)

[Sp] F. SPTIZER (1958). Some theorems concerning 2–dimensional Brownian motion, *Trans. Amer. Math. Soc.*, **87**, 187–197

[T1] J. TAKEUCHI (1964). On the sample paths of the symmetric stable processes in spaces, *J. Math. Soc. Japan*, **16**, 109–127

[T2] J. TAKEUCHI (1964). A local asymptotic law for transient stable processes, *Proc. Japan Acad.*, **40**, 141–144

[TW] J. TAKEUCHI AND S. WATANABE (1964). Spitzer's test for the Cauchy process on the line, *Z. Wahr. Verw. Geb.*, **3**, 204–210