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On a Problem of Erdős and Taylor

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Abstract. Let $\{S_n, n \geqslant 0\}$ be a centered d–dimensional random walk ($d \geqslant 3$) and consider the so–called future infima process, $J_n \stackrel{\mathrm{df}}{=} \inf_{k \geqslant n} \|S_k\|$. This paper is concerned with obtaining precise integral criteria for a function to be in the Lévy upper class of J. This solves an old problem of Erdős and Taylor (1960), who posed the problem for the simple symmetric random walk on $\mathbb{Z}^d$, $d \geqslant 3$. These results are obtained by a careful analysis of the future infima of transient Bessel processes and using strong approximations. Our results belong to a class of Ciesielski–Taylor theorems which relate d and $(d-2)$ dimensional Bessel processes.

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1. Introduction

Throughout this paper, we will adopt the following notation: given $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $\|x\|$ will denote the (Euclidean) ℓ_2–norm of x, i.e.,

$$\|x\| \stackrel{\text{df}}{=} (x_1^2 + \cdots + x_d^2)^{1/2}.$$

Given $x \in \mathbb{R}^1$, we will write $\log_e$ for the natural logarithm, $\ln(x) = \ln_1(x) \stackrel{\text{df}}{=} \log_e(x \vee e)$ and, for $k \geqslant 1$, $\ln_{k+1}(x) \stackrel{\text{df}}{=} \ln(\ln_k(x))$.

Let $\{S_n,\, n \geqslant 0\}$ denote an $\mathbb{R}^d$–valued simple symmetric random walk. Erdős and Taylor (1960) present a variety of results on the fine structure of the sample paths of $\{S_n,\, n \geqslant 0\}$. Especially noteworthy is their complete characterization of the number of returns to origin by time n for the simple planar random walk. When $d \geqslant 3$, the simple random walk is transient, and, as such, it has only a finite number of returns to the origin. Instead they study the behavior of the associated *future infimum process*: for all $n \geqslant 0$, let

$$J_n \stackrel{\text{df}}{=} \inf_{k \geqslant n} \|S_k\|.$$

In their Theorems 8 and 9 (p. 154), Erdős and Taylor establish the following laws of the iterated logarithm for $\|S_n\|$ and J_n :

$$\limsup_{n \to \infty} \frac{\|S_n\|}{\sqrt{2n \ln_2(n)}} = d^{-1/2}, \qquad \text{a.s. }, \tag{1.1a}$$

$$\limsup_{n \to \infty} \frac{J_n}{\sqrt{2n \ln_2(n)}} = d^{-1/2}, \qquad \text{a.s. }. \tag{1.1b}$$

Let $0 < c < 1$. Then (1.1b) demonstrates that for almost all ω there is an increasing sequence $\{n_k,\, k \geqslant 1\}$ of integers such that

$$J_{n_k}(\omega) = \|S_{n_k}(\omega)\| \geqslant c\sqrt{2n_k \ln_2(n_k)}. \tag{1.2}$$

In words, J_n can be as large as $\|S_n\|$ when $\|S_n\|$ is near its upper envelope. A complete characterization (in the sense of the integral test of Erdős (1942)) of the upper functions of $\{J_n,\, n \geqslant 0\}$ is left unresolved (see the remarks on top of p. 155). The main purpose of this paper is to present this characterization for centered $\mathbb{R}^d$–valued random walks which satisfy certain integrability conditions (in particular, we make no assumptions about the support of the distribution of the increments). In this way we can find increasing functions $f,\, g : [1,\infty) \to \infty$ with $x \mapsto g(x) - f(x)$ increasing without bound such that for almost all ω there is an increasing sequence $\{n_k,\, k \geqslant 1\}$ and integer N such that

(a) $J_n(\omega) \leqslant f(n)$ for all $n \geqslant N$, while

(b) $\|S_{n_k}(\omega)\| > g(n_k)$.

In contrast with (1.2), this shows that whenever S_n leaves a centered ball of radius $g(n)$, at the very least it must return to a ball of radius $f(n)$ at some future time.

Our approach consists of the following: first we solve the analogous problem for transient Bessel processes. Then, by way of a strong approximation argument, we solve the problem for random walks. Although we will comment on this at greater length in Section 2, our results demonstrate that the upper class of the future infimum of a d–dimensional Bessel process ($d > 2$) is identical to the upper class of a $(d-2)$–dimensionl Bessel process: this relates our work to a class of results sometimes referred to as Ciesielski–Taylor theorems.

Future infima processes such as the ones in this paper occur naturally in the more general setting of Markov processes. As a sample see Aldous (1992), Burdzy (1994), Chen and Shao (1993), Khoshnevisan (1994), Khoshnevisan et al. (1994), Millar (1997) and Pitman (1975).

In conclusion, we would like to add that in the case of the simple walk in dimension $d \geq 3$ (i.e., the problem of Erdős and Taylor), one can adapt the method of Section 3 to obtain a completely classical proof of the Erdős–Taylor problem mentioned above. We will not discuss this approach, as it only seems to work for random walks which live on a sublattice of $\mathbb{Z}^d$.

2. Statements of results

Let us first state our results for Bessel processes. Throughout this paper, $X \stackrel{\text{df}}{=} \{X_t,\ t \geqslant 0\}$ will denote a d-dimensional Bessel process with $d > 2$. This is a diffusion on $[0, \infty)$ whose generator, $\mathcal{G}$, is given by

$$\mathcal{G}f(x) \overset{\text{df}}{=} \frac{1}{2}f''(x) + \frac{d-1}{2x}f'(x).$$

(Since $d > 2$, there is no need to specify a boundary condition at 0.) The definition and properties of X can be found in Revuz and Yor (1991).

It should be noted that when d is an integer, the radial part of a standard d–dimensional Brownian motion is a d–dimensional Bessel process. Of particular importance to us is that the condition $d > 2$ implies that X is transient. We will define two processes associated with X. First, we define the *future infimum process*: for all $t \geqslant 0$,

$$I_t \overset{\text{df}}{=} \inf_{s \geqslant t} X_s.$$

Next let $L_0 = 0$ and for $r > 0$, define the *escape process*:

$$L_r \overset{\text{df}}{=} \sup\{s : \; X_s \leqslant r\}.$$

I and L inherit scaling laws from X. For $c > 0$, $c^{-1/2} I_{ct}$ and I_t are equivalent processes, and $c^- 2L_{cr}$ and L_r are equivalent processes.

The processes I and L are inverses in the following natural sense:

$$\{\omega : \; L_r \leqslant t\} = \{\omega : \; I_t \geqslant r\}. \tag{2.1a}$$

We point out in passing that the transience of X trivially implies that $\lim_{t\to\infty} L_t = \lim_{t\to\infty} I_t = \infty$, almost surely. Properties of the processes I and L are explored in Getoor (1979), Yor (1992a,b) and Khoshnevisan, Lewis and Li (1994).

Given a stochastic process $\{Z_t, \, t \geqslant 0\}$ and a nondecreasing function $f : \; [1, \infty) \to (0, \infty)$, we will say that f is in the *upper class* (respectively *lower class*) of Z if for almost all ω there is an integer $N = N(\omega)$ such that $Z_t(\omega) \leqslant f(t)$ (respectively $Z_t(\omega) \geqslant f(t)$) for all $t \geqslant N$. We shall write this in the compact form, $f \in \mathcal{U}(Z)$ ($f \in \mathcal{L}(Z)$, respectively).

Let $\varphi : [1, \infty) \to (0, \infty)$ be nonincreasing and let $\psi : [1, \infty) \to (0, \infty)$ be nondecreasing. Define,

$$\mathfrak{J}_1(\varphi) \overset{\text{df}}{=} \int_1^\infty \varphi^{1-d/2}(t) \exp\Big(- \frac{1}{2\varphi(t)} \Big) \frac{dt}{t}$$

$$\mathfrak{J}_2(\psi) \overset{\text{df}}{=} \int_1^\infty \psi^{d-2}(t) \exp\Big(- \frac{\psi^2(t)}{2} \Big) \frac{dt}{t}.$$

For future reference, we observe the relationship:

$$\mathfrak{J}_1(\psi^{-2}) = \mathfrak{J}_2(\psi). \tag{2.1b}$$

Our first theorem characterizes the lower class of the exit process, L.

THEOREM 2.1. *Let $\varphi:\ [1,\infty) \to (0,\infty)$ be a nonincreasing function. Then*

$$t \mapsto t^2\varphi(t) \in \mathcal{L}(L) \quad \textit{if and only if} \quad \mathfrak{J}_1(\varphi) < \infty.$$

Essentially due to (2.1a) and (2.1b), we obtain our next theorem, which characterizes the upper class of I.

THEOREM 2.2. *Let $\psi:\ [1,\infty) \to (0,\infty)$ be a nondecreasing function. Then*

$$t \mapsto \sqrt{t}\psi(t) \in \mathcal{U}(I) \quad \textit{if and only if} \quad \mathfrak{J}_2(\psi) < \infty.$$

REMARK 2.2.1. Theorem 2.2 had been independently conjectured by CHEN AND SHAO (1993) and KHOSHNEVISAN, LEWIS AND LI (1994). (In fact, Theorem 4.3(3) of the latter reference constitutes the so–called easy half of Theorem 2.2. There it is shown that if $\psi : [1,\infty) \to \infty$ is nondecreasing and $\mathfrak{J}_2(\psi) < \infty$, then $t \mapsto \sqrt{t}\psi(t) \in \mathcal{U}(I)$.)

For purposes of comparison, let us recall the celebrated Kolmogorov–Dvoretsky–Erdős integral test (see, e.g., ITÔ AND MCKEAN (1965), p. 163):

THEOREM A. LET $\psi:\ [1,\infty) \to (0,\infty)$ BE A NONDCREASING FUNCTION. *Then*

$$t \mapsto \sqrt{t}\psi(t) \in \mathcal{U}(X) \quad \textit{if and only if} \quad \int_1^\infty \psi^d(t)\exp\left(-\psi^2(t)/2\right)\frac{dt}{t} < \infty.$$

Theorems 2.2 and A together show that the upper class of I in dimension d is the same as the upper class of X in dimension $(d-2)$. It should be noted that when $d = 3$, the comparison is stronger still; see PITMAN (1975) and REVUZ AND YOR (1991) for details. Let $\tau_0 \stackrel{\text{df}}{=} 0$ and, for all $r > 0$, let $\tau_r \stackrel{\text{df}}{=} \inf\{s > 0 : X_s = r\}$: this defines the *exit time* of X

from the interval $[0, r]$. Then the lower class of L in dimension d is the same as the lower class of τ in dimension $(d-2)$. Results relating d and $(d-2)$–dimensional Bessel processes are sometimes known as Ciesielski–Taylor theorems and can be found, for example, in CIESIELSKI AND TAYLOR (1962), JAIN AND TAYLOR (1973) and YOR (1992b).

Now let us state our theorems concerning random walks. Throughout let ξ, $\{\xi_i,\ i \geqslant 1\}$ be a sequence of independent and identically distributed $\mathbb{R}^d$–valued random variables, $d \geqslant 3$. Let $S_0 \stackrel{\mathrm{df}}{=} 0$ and, for all $n \geqslant 1$, let

$$S_n \stackrel{\mathrm{df}}{=} \xi_1 + \ldots + \xi_n.$$

We will extend the definition of our random walk to continuous time by simply setting $S_t \stackrel{\mathrm{df}}{=} S_{[t]}$ for all $t \geqslant 0$. Here $[t]$ is the greatest integer less than or equal to t. For all $t \geqslant 0$, let

$$J_t \stackrel{\mathrm{df}}{=} \inf_{u \geqslant t} \|S_u\|.$$

THEOREM 2.3. *Let ξ have zero mean vector and identity covariance matrix and suppose there is a $\delta > 0$ such that $\mathbb{E}\left(\|\xi\|^{2+\delta}\right) < \infty$. Let $\psi : [0, \infty) \to (0, \infty)$ be an unbounded function such that $t \mapsto \sqrt{t}\psi(t)$ is increasing. Then*

$$t \mapsto \sqrt{t}\psi(t) \in \mathcal{U}(J) \quad \textit{if and only if} \quad \mathfrak{J}_2(\psi) < \infty.$$

REMARK 2.3.1. It is easy to see how the above includes the promised solution to the Erdős–Taylor problem. Indeed, suppose ξ is as in the statement of Theorem 2.3, except that its covariance matrix is $\sigma^2 \mathbf{I}$, where $\sigma > 0$ and $\mathbf{I}$ is the $d \times d$ identity matrix. Obviously, Theorem 2.3 holds with S_n and J_n replaced with S_n/σ and J_n/σ, respectively.

It is worth noting that the nature of the problem changes entirely when the random walk does not have identity covariance matrix; nonetheless, we can prove various laws of the iterated logarithm. Let $d \geqslant 3$ be an integer and $\mathbf{Q}$ denote a $d \times d$ symmetric positive definite matrix with eigenvalues

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_d > 0.$$

Let $W \stackrel{\text{df}}{=} \{W_t,\, t \geqslant 0\}$ denote a d–dimensional centered Brownian motion with covariance matrix $\mathbf{Q}$ and define the associated future infimum and escape processes by

$$I_{\mathbf{Q}}(t) \stackrel{\text{df}}{=} \inf_{s \geqslant t} \|W_s\| \quad \text{and} \quad L_{\mathbf{Q}}(r) \stackrel{\text{df}}{=} \sup\{t \geqslant 0 : \|W_t\| \leqslant r\},$$

respectively. Likewise, let $\{S_t : t \geqslant 0\}$ denote a d–dimensional centered random walk with covariance matrix $\mathbf{Q}$, and define the associated future infimum and exit processes by

$$J_{\mathbf{Q}}(t) \stackrel{\text{df}}{=} \inf_{u \geqslant t} \|S_u\| \qquad \text{and} \qquad K_{\mathbf{Q}}(r) \stackrel{\text{df}}{=} \sup\{t \geq 0 : \|S_t\| \leq r\},$$

respectively. Our theorem follows:

THEOREM 2.4. *Suppose ξ, $\{\xi_i,\, i \geqslant 1\}$ form a sequence of i.i.d. random vectors taking values in $\mathbb{R}^d$ with $d \geq 3$. We assume that ξ has mean vector zero, covariance matrix $\mathbf{Q}$ and that there is a $\delta > 0$ such that $\mathbb{E}\,(\|\xi\|^{2+\delta}) < \infty$. Let $\{S_t : t \geqslant 0\}$ denote the associated random walk and let $\{W_t,\, t \geqslant 0\}$ be a d–dimensional centered Brownian motion with covariance matrix $\mathbf{Q}$. Then*

(a) $\displaystyle \liminf_{t\to\infty} \frac{2\ln_2(t)}{t^2} K_{\mathbf{Q}}(t) = \liminf_{t\to\infty} \frac{2\ln_2(t)}{t^2} L_{\mathbf{Q}}(t) = \frac{1}{\lambda_1} \qquad \text{a.s.;}$

(b) $\displaystyle \limsup_{t\to\infty} \frac{J_{\mathbf{Q}}(t)}{\sqrt{2t\ln_2(t)}} = \limsup_{t\to\infty} \frac{I_{\mathbf{Q}}(t)}{\sqrt{2t\ln_2(t)}} = \sqrt{\lambda_1} \qquad \text{a.s.}$

An alternative approach is to view our processes as taking values in a certain Riemannian sub-manifold of $\mathbb{R}^d$. To do this, we define a metric which is compatible with the underlying process. Because $\mathbf{Q}$ has orthogonal eigenvectors, there is an invertible matrix $\mathbf{V}$ such that $\mathbf{Q} = \mathbf{V}\mathbf{V}^T$. The standardized random vector $\mathbf{V}^{-1}\xi$ has identity covariance matrix. For $x \in \mathbb{R}^d$, let

$$\|x\|_{\mathbf{V}} \stackrel{\text{df}}{=} \|\mathbf{V}^{-1}x\|,$$

which defines a norm on $\mathbb{R}^d$. Clearly, $\|\cdot\|_{\mathbf{V}}$ and $\|\cdot\|$ are equivalent norms and Theorem 2.3 holds if we measure distance by the norm $\|\cdot\|_{\mathbf{V}}$. The inner product on the aforementioned space is the obvious one, by polarization. Moreover, the intrinsic volume element is given by $dm \stackrel{\text{df}}{=} (\det\mathbf{Q})^{1/2}dx$. This notion of geometry, however, is not natural to the study of the paths of random walk.

The remainder of this paper is organized as follows. In Section 3, we develop the relevant probability estimates. In Section 4, we prove Theorems 2.1 and 2.2. In Section 5, we prove a strong approximation theorem which is then used to prove Theorem 2.3. Finally, in Section 6, we prove Theorem 2.4.

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3. Preliminary estimates

The main purpose of this section is to present some lemmata which will be useful in the proof of the theorems. For all $\beta \geqslant 0$, let

$$H(\beta) \stackrel{\text{df}}{=} \mathbb{P}\,(L_1 \leqslant \beta).$$

The density of L_1 is well known (see, for example, GETOOR (1979), YOR (1992b) or KHOSHNEVISAN, LEWIS AND LI (1994)):

$$H'(x) = \gamma_d x^{-d/2} \exp\Big(-\frac{1}{2x}\Big) I(x>0), \quad \text{where} \quad \frac{1}{\gamma_d} \stackrel{\text{df}}{=} 2^{(d-2)/2}\Gamma\Big(\frac{d-2}{2}\Big).$$

Thus, by an application of l'Hôpital's rule, we obtain the following asymptotic estimate for the small–ball probability of L_1:

$$H(\beta) \sim 2\gamma_d \beta^{2-\frac{d}{2}} \exp\left(-1/(2\beta)\right) \quad \text{as } \beta \to 0. \tag{3.1}$$

¿From this, it is clear that for any $T > 0$, there exists a positive constant c, which depends only on T, such that

$$\frac{1}{c} x^{2-\frac{d}{2}} \exp\left(-1/(2x)\right) \leqslant H(x) \leqslant c x^{2-\frac{d}{2}} \exp\left(-1/(2x)\right) \quad \text{for all } x \in [0,T]. \tag{3.2}$$

In the course of proving Theorem 2.1, it will be necessary to estimate the distribution of the increment $L_1 - L_s$ for $0 < s < 1$. Our first lemma is a useful step in this direction.

LEMMA 3.1. *Let $\beta, a > 0$ and $0 < s < 1$. Then*

$$\mathbb{P}\left(L_1 - L_s \leqslant \beta\right) \leqslant \frac{H(\beta + a)}{H(as^{-2})}.$$

Proof. For any $s \in (0,1)$, and all $\beta, a > 0$,

$$\{\omega : \; L_1 - L_s \le \beta\} \cap \{\omega : \; L_s \le a\} \subseteq \{\omega : \; L_1 \le \beta + a\}.$$

By a theorem of GETOOR (1979), $\{L_t,\, t \geqslant 0\}$ has independent increments. The rest of the proof follows from the scaling law for L, since L_s has the same distribution as $s^2 L_1$; hence, $\mathbb{P}\left(L_s \leqslant a\right) = H(as^{-2})$. $\square$

Our next lemma is a direct application of Lemma 3.1; it provides a useful estimate for the distribution of $L_1 - L_s$ when s is relatively small.

LEMMA 3.2. *Fix $\lambda > 0$. Then there exists a positive constant c, depending only on λ and d, such that*

$$\mathbb{P}\left(L_1 - L_s \leqslant \beta\right) \leqslant cH(\beta),$$

for all $0 \leqslant \beta \leqslant \lambda^{-1}$ and $0 \leqslant s \leqslant \lambda\beta$.

Proof. Since $s \mapsto \mathbb{P}\left(L_1 - L_s \leqslant \beta\right)$ is increasing for all $0 \leqslant s \leqslant \lambda\beta$, it suffices to prove the lemma for $s = \lambda\beta$. To this end let μ to be the median of L_1, i.e., $H(\mu) = 1/2$. Applying Lemma 3.1 with $a \stackrel{\text{df}}{=} \mu\lambda^2\beta^2$, we obtain:

$$\begin{aligned}\mathbb{P}\left(L_1 - L_{\lambda\beta} \leqslant \beta\right) &\leqslant \frac{H(\beta + \mu\lambda^2\beta^2)}{H(\mu)} \\ &= 2H(\beta + \mu\lambda^2\beta^2).\end{aligned}$$

Now fix c so that (3.2) holds for all $0 \leqslant x \leqslant \lambda^{-1} + \mu$. Since $\beta + \mu\lambda^2\beta^2 \leqslant \lambda^{-1} + \mu$, it follows that

$$\mathbb{P}\left(L_1 \leqslant \beta + \mu\lambda^2\beta^2\right) \leqslant c(\beta + \mu\lambda^2\beta^2)^{2-\frac{d}{2}} \exp\left(-\frac{1}{2\beta + 2\mu\lambda^2\beta^2}\right).$$

However,

$$(\beta+\mu\lambda^2\beta^2)^{2-\frac{d}{2}} \leqslant \begin{cases} \beta^{2-\frac{d}{2}} & \text{if } d \geqslant 4 \\ \beta^{2-\frac{d}{2}}(1+\mu\lambda)^{2-\frac{d}{2}} & \text{if } d < 4 \end{cases}.$$

Moreover,

$$\exp\left(-\frac{1}{2\beta+2\mu\lambda^2\beta^2}\right) \leqslant e^{-1/(2\beta)} \exp\left(\mu\lambda^2/2\right).$$

Finally by (3.2),

$$\frac{1}{c}\beta^{2-\frac{d}{2}} e^{-1/(2\beta)} \leqslant H(\beta),$$

which gives us the desired result. □

Our next lemma is an estimate for the distribution of the increment $L_1 - L_s$ when s is not necessarily small. In order to state this lemma, we will need the following definition. If $d \leqslant 4$, let $\beta^* \stackrel{\text{df}}{=} 1$. If $d > 4$, let

$$\beta^* \stackrel{\text{df}}{=} \sup\{0 < x \leqslant e^{-1} : \; 4(d-4)x\ln(x^{-1}) \leqslant 1/2\}.$$

Since $x \mapsto x\ln(x^{-1})$ is increasing on $(0, e^{-1})$, it follows that $\beta^* > 0$, which is all we will need.

Lemma 3.3. *Let $\varepsilon > 0$. Then there exists a positive constant c, depending only on ε and d, such that*

$$\mathbb{P}\left(L_1 - L_s \leqslant \beta\right) \leqslant c\,\exp\left(-\frac{(1-s)^2}{4\beta}\right),$$

for all $\beta \leqslant (2\varepsilon)^{-1} \wedge \beta^*$ and $0 \leqslant s \leqslant 1-\beta\varepsilon$.

Proof. First fix $c_1 > 0$ such that (3.2) holds for all $0 \leqslant x \leqslant 2/\varepsilon$. We will consider two cases: $\beta\varepsilon \leqslant s \leqslant 1-\beta\varepsilon$ and $0 \leqslant s \leqslant \beta\varepsilon$.

In the first case, apply Lemma 3.1 with $a \stackrel{\text{df}}{=} \beta s/(1-s)$ and observe that

$$\begin{aligned} \beta + a &= \frac{\beta}{1-s} \leqslant \frac{1}{\varepsilon} \\ as^{-2} &= \frac{\beta}{s} + \frac{\beta}{1-s} \leqslant \frac{2}{\varepsilon} \\ -\frac{1}{2(\beta+a)} + \frac{1}{2as^{-2}} &= -\frac{(1-s)^2}{2\beta} \\ \frac{\beta+a}{as^{-2}} &= s \end{aligned}$$

Thus, by (3.2) and our choice of c_1, we obtain

$$\mathbb{P}\left(L_1 - L_s \leqslant \beta\right) \leqslant c_1^2 s^{2-\frac{d}{2}} \exp\Big(- \frac{(1-s)^2}{2\beta}\Big). \tag{3.3}$$

In the second case, $0 \leqslant s \leqslant \beta\varepsilon$. Then, by (3.3) and the monotonicity of $r \mapsto L_r$, we obtain

$$\mathbb{P}\left(L_1 - L_s \leqslant \beta\right) \leqslant \mathbb{P}\left(L_1 - L_{\beta\varepsilon} \leqslant \beta\right) \leqslant c_1^2 (\beta\varepsilon)^{2-\frac{d}{2}} \exp\Big(- \frac{(1-\beta\varepsilon)^2}{2\beta}\Big).$$

However, since $0 < s \leqslant \beta\varepsilon$, it is easy to see that

$$\exp\Big(- \frac{(1-\beta\varepsilon)^2}{2\beta}\Big) \leqslant e^{\varepsilon} \exp\Big(- \frac{(1-s)^2}{2\beta}\Big).$$

Setting $c_2 = c_1^2 \varepsilon^{2-\frac{d}{2}} e^{\varepsilon}$, we obtain:

$$\mathbb{P}\left(L_1 - L_s \leqslant \beta\right) \leqslant c_2 \beta^{2-\frac{d}{2}} \exp\Big(- \frac{(1-s)^2}{2\beta}\Big). \tag{3.4}$$

If $d \leqslant 4$, then we are done, since s and β are bounded by 1. If, however, $d > 4$, then we need to consider two additional subcases. If $1/2 \leqslant s \leqslant 1 - \beta\varepsilon$, then, by bounding $s^{2-\frac{d}{2}}$ by $2^{-\frac{d}{2}+2}$ in (3.3), we obtain the desired result. If, however, $0 \leqslant s \leqslant 1/2$, then by (3.3) and (3.4) we obtain

$$\mathbb{P}\left(L_1 - L_s \leqslant \beta\right) \leqslant c_2 \exp\Big(- \frac{(1-s)^2}{2\beta} + \frac{d-4}{2} \ln(\beta^{-1})\Big).$$

This last exponent may be expressed as:

$$-\frac{(1-s)^2}{2\beta}\left[1 - \frac{(d-4)\beta \ln(\beta^{-1})}{(1-s)^2}\right].$$

However, since $0 < s \leqslant 1/2$ and $\beta \leqslant \beta^*$, it follows that

$$1 - \frac{(d-4)\beta \ln(\beta^{-1})}{(1-s)^2} \geqslant \frac{1}{2},$$

which is what we wished to show. $\square$

REMARK 3.3.1. Equations (3.2) and (3.4) already yield good estimates for the distribution of the increment $L_1 - L_s$; however, for our applications, it is better to place all of the dependency on ε and d into a constant. Although it complicates the proof of the lemma and weakens the result, it will make the application cleaner.

LEMMA 3.4. *Suppose $\alpha \geqslant 0$, ε_1, $\varepsilon_2 \in (0,1)$, and $0 < \beta < (2\varepsilon_2)^{-1} \wedge \beta^* \wedge \alpha$. Then for all $\varepsilon_1 \leqslant s \leqslant 1-\beta\varepsilon_2$ there is a constant c, depending only on ε_1, ε_2 and d, such that*

$$\mathbb{P}\left(L_s \leqslant s^2\alpha,\ L_1 \leqslant \beta\right) \leqslant cH(\alpha)\exp\left(-\frac{(1-s)}{12\beta}\right).$$

Proof. Choose c_1 so that (3.2) holds for all $0 \leqslant x \leqslant 1$. By the independence of increments and scaling we obtain:

$$\begin{aligned}\mathbb{P}\left(L_s \leqslant s^2\alpha,\ L_1 \leqslant \beta\right) &\leqslant \mathbb{P}\left(L_s \leqslant s^3\beta\right) + \mathbb{P}\left(s^3\beta < L_s \leqslant s^2\alpha,\ L_1 - L_s \leqslant \beta(1-s^3)\right)\\ &\leqslant \mathbb{P}\left(L_1 \leqslant \beta s\right) + H(\alpha)\cdot\mathbb{P}\left(L_1 - L_s \leqslant \beta(1-s^3)\right)\end{aligned}$$

We will estimate each of the terms on the right hand side. First, by (3.2),

$$\mathbb{P}\left(L_1 \leqslant \beta s\right) \leqslant c_1(\beta s)^{2-\frac{d}{2}}e^{-1/(2\beta s)}.$$

Since $0 \leqslant s < 1$, it easily follows that $s^{2-\frac{d}{2}} \leqslant 1 \vee \varepsilon_1^{2-\frac{d}{2}}$ and

$$e^{-1/(2\beta s)} \leqslant e^{-1/(2\beta)}\exp\left(-\frac{1}{2\beta}(1-s)\right).$$

Thus, by another application of (3.2) we see

$$\mathbb{P}\left(L_1 \leqslant \beta s\right) \leqslant c_1^2(1 \vee \varepsilon_1^{2-\frac{d}{2}})H(\beta)\exp\left(-\frac{1}{2\beta}(1-s)\right). \tag{3.5}$$

Next we will apply Lemma 3.3 to estimate $\mathbb{P}\left(L_1 - L_s \leqslant \beta(1-s^3)\right)$. The conditions of this lemma are satisfied; hence,

$$\mathbb{P}\left(L_1 - L_s \leqslant \beta(1-s^3)\right) \leqslant c_2 \exp\left(-\frac{(1-s)^2}{4\beta(1-s^3)}\right),$$

where c_2 depends only on ε_2 and d. However,

$$\frac{(1-s)^2}{4\beta(1-s^3)} = \frac{1-s}{4\beta(1+s+s^2)} \geqslant \frac{1-s}{12\beta}.$$

Consequently,

$$\mathbb{P}\left(L_1 - L_s \leqslant \beta(1-s^3)\right) \leqslant c_2 \exp\left(- \frac{(1-s)}{12\beta}\right). \tag{3.6}$$

To achieve the final form of the lemma, combine (3.5) and (3.6), noting that

$$\exp\left(- \frac{(1-s)}{2\beta}\right) \leqslant \exp\left(- \frac{(1-s)}{12\beta}\right)$$

and $H(\beta) \leqslant H(\alpha)$. $\square$

4. Proofs of Bessel process results

Before proceeding to the proof of Theorem 2.1, let us make a few prefunctory remarks. Essential to the proofs will be the function $x \mapsto t_x$, which is defined for all real x by

$$t_x \overset{\text{df}}{=} \exp\left(\frac{x}{\ln(x)}\right). \tag{4.1}$$

We note that this function is increasing in x. Primarily we will be interested in the values of t_x at the positive integers, but occasionally we will write, e.g., $t_{n+\ln(n)}$, whose meaning is given by (4.1). By an application of the mean value theorem, it follows that

$$\lim_{n\to\infty} \frac{t_{n+1}-t_n}{t_{n+1}} \ln(n) = 1.$$

Since t_{n+1}/t_n approaches one as n tends to infinity, evidently the same limit is obtained upon replacing t_{n+1} by t_n in the denominator. These considerations lead us to the following: there exist universal positive constants c_1 and c_2 such that for all $n \geqslant 1$,

$$\begin{aligned} \frac{1}{c_1 \ln(n)} &\leqslant \frac{t_{n+1}-t_n}{t_{n+1}} \leqslant \frac{c_1}{\ln(n)}, \\ \frac{1}{c_2 \ln(n)} &\leqslant \frac{t_{n+1}-t_n}{t_n} \leqslant \frac{c_2}{\ln(n)}. \end{aligned} \tag{4.2}$$

Recall from Khoshnevisan, Lewis and Li (1994), the following law of the iterated logarithm for $\{I_t,\, t \geqslant 0\}$:

$$\limsup_{t\to\infty} \frac{I_t}{\sqrt{2t\ln_2(t)}} = 1 \qquad \text{a.s.}$$

The above together with (2.1a) easily yield,

$$\liminf_{t\to\infty} \frac{2\ln_2(t)}{t^2} L_t = 1 \qquad \text{a.s.} \tag{4.3}$$

Let $\varphi : [1,\infty) \to (0,\infty)$ be nonincreasing and, for each positive integer n, let $\varphi_n \stackrel{\text{df}}{=} \varphi(t_n)$. In light of (4.3) (with respect to the proof of Theorem 2.1) we may assume without any loss of generality that there exists a universal positive constant c such that

$$\frac{1}{c\ln_2(t)} \leqslant \varphi(t) \leqslant \frac{c}{2\ln_2(t)}$$

Thus, without loss of generality, we may assume that there exists a universal positive constant c such that

$$\frac{1}{c\ln(n)} \leqslant \varphi_n \leqslant \frac{c}{\ln(n)} \tag{4.4}$$

First we will prove

$$\mathfrak{J}_1(\varphi) < \infty \Longrightarrow \mathbb{P}\left(L_t \geqslant t^2\varphi(t), \text{ eventually}\right) = 1. \tag{4.5}$$

For each positive integer n, define the event

$$F_n \stackrel{\text{df}}{=} \{\omega : \ L_{t_n} \leqslant t_{n+1}^2 \varphi_n\}.$$

First we will demonstrate that $\sum_{n=1}^{\infty} \mathbb{P}\left(F_n\right) < \infty$. To this end, observe that

$$\int_{t_n}^{t_{n+1}} \varphi^{1-d/2}(t) e^{-1/(2\varphi(t))} \frac{dt}{t} \geqslant \varphi_n^{1-d/2} e^{-1/(2\varphi_{n+1})} \frac{(t_{n+1}-t_n)}{t_{n+1}}.$$

Thus, by (4.2) and (4.4), there exists a universal positive constant c_1 such that for all $n \geqslant 1$, we obtain

$$\int_{t_n}^{t_{n+1}} \varphi^{1-d/2}(t) e^{-1/(2\varphi(t))} \frac{dt}{t} \geqslant c_1 \varphi_{n+1}^{2-d/2} e^{-1/(2\varphi_{n+1})}. \tag{4.6}$$

By scaling, (3.1) and some algebra, it follows that there exists a universal positive constant c such that

$$\mathbb{P}\left(F_n\right) \leqslant c((t_{n+1}/t_n)^2 \varphi_{n+1})^{2-\frac{d}{2}} e^{-1/(2\varphi_{n+1})} \exp\left(\frac{1}{2\varphi_{n+1}}(1-(t_n/t_{n+1})^2)\right).$$

This last exponent is easily estimated: observe that

$$\frac{1}{2\varphi_{n+1}}\left(1-\frac{t_n^2}{t_{n+1}^2}\right)\leqslant\frac{1}{\varphi_{n+1}}\left(\frac{t_{n+1}-t_n}{t_{n+1}}\right).$$

Now (4.2) and (4.4) show that this exponent is uniformly bounded in n. Thus, there exists a universal positive constant c_2 such that

$$\mathbb{P}\left(F_n\right)\leqslant c_2\varphi_{n+1}^{2-d/2}e^{-1/(2\varphi_{n+1})}. \tag{4.7}$$

Combining (4.6) and (4.7) with $\mathfrak{J}_1(\varphi)<\infty$ we see that $\sum_{n=1}\mathbb{P}\left(F_n\right)<\infty$. Thus, by the first Borel-Cantelli lemma, it follows that $\mathbb{P}\left(F_n,\ \text{i.o.}\right)=0$. Consequently, for almost all ω, there exists a (random) index N beyond which

$$L_{t_n}>t_{n+1}^2\varphi(t_n).$$

Finally, given $n\geqslant N$ and $s\in[t_n,\,t_{n+1}]$ it follows that, with probability one,

$$L_s\geqslant L_{t_n}>t_{n+1}^2\varphi(t_n)\geqslant s^2\varphi(s),$$

where we have used the fact that $s\mapsto\varphi(s)$ is nonincreasing. This demonstrates (4.5).

Next we will show that

$$\mathfrak{J}_1(\varphi)=\infty\Longrightarrow\mathbb{P}\left(L_t\geqslant t^2\varphi(t),\ \text{eventually}\right)=0. \tag{4.8}$$

To this end, for each positive integer n, define the event

$$E_n\stackrel{\text{df}}{=}\{\omega:\ L_{t_n}\leqslant t_n^2\varphi_n\}.$$

To demonstrate (4.8), it suffices to show that $\mathbb{P}\left(E_n,\ \text{i.o.}\right)=1$. Since $\{E_n,\text{i.o.}\}$ is a 0-1 event, by Kochen-Stone (1964), it is enough to show that there is a universal positive constant c such that

$$\sum_{n=1}^{\infty}\mathbb{P}\left(E_n\right)=\infty \tag{4.9}$$

$$\sum_{1\leqslant k<n\leqslant N}\mathbb{P}\left(E_k\cap E_n\right)\leqslant c\left(\sum_{k=1}^{N}\mathbb{P}\left(E_k\right)\right)^2. \tag{4.10}$$

The verification of (4.9) is straightforward. For all n sufficiently large, there exists a constant c_1 such that

$$\int_{t_n}^{t_{n+1}} \varphi^{1-d/2}(t) e^{-1/(2\varphi(t))} \frac{dt}{t} \leqslant \varphi_{n+1}^{1-d/2} e^{-1/(2\varphi_n)} \frac{t_{n+1}-t_n}{t_n} \leqslant c_1 \varphi_n^{2-d/2} e^{-1/(2\varphi_n)}, \tag{4.11}$$

where we have used (4.2). By scaling and (3.2), it follows that there exists a universal positive constant c_2 such that

$$\mathbb{P}(E_n) \geqslant c_2 \varphi_n^{2-d/2} e^{-1/(2\varphi_n)}. \tag{4.12}$$

Combining (4.11) and (4.12) with $\mathfrak{J}_1(\varphi)=\infty$ demonstrates (4.9).

We are left to demonstrate (4.10). To this end, we introduce the following sets of indices:

$$\begin{aligned}
\mathcal{G}_n^N &= \{1 \le k \le N : \ k \ge \ln(n)\cdot \ln_2(n)\} \\
\mathcal{B}_n^N &= \{1 \le k \le N : \ \ln(n) \le k < \ln(n)\cdot \ln_2(n)\} \\
\mathcal{U}_n^N &= \{1 \le k \le N : \ 1 \le k < \ln(n)\} \\
\mathcal{A}_n^N(j) &= \left\{1 \le k \le N : \ \frac{j}{\sqrt{\ln(n)}} \le 1 - \frac{t_n}{t_{n+k}} \le \frac{j+1}{\sqrt{\ln(n)}}\right\} \\
\widetilde{\mathcal{A}}_n^N(j) &= \left\{1 \le k \le N : \ \frac{j}{\ln(n)} \le 1 - \frac{t_n}{t_{n+k}} \le \frac{j+1}{\ln(n)}\right\}.
\end{aligned}$$

In fact, we will prove that there exist universal positive constants c_1, c_2 and c_3 such that for all $N \geqslant 1$ we have

$$\sum_{n=1}^{N} \sum_{k\in\mathcal{G}_n^N} \mathbb{P}(E_n \cap E_{n+k}) \le c_1 \left(\sum_{n=1}^{N} \mathbb{P}(E_n)\right)^2 \tag{4.13}$$

$$\sum_{n=1}^{N} \sum_{k\in\mathcal{B}_n^N} \mathbb{P}(E_n \cap E_{n+k}) \le c_2 \sum_{n=1}^{N} \mathbb{P}(E_n) \tag{4.14}$$

$$\sum_{n=1}^{N} \sum_{k\in\mathcal{U}_n^N} \mathbb{P}(E_n \cap E_{n+k}) \le c_3 \sum_{n=1}^{N} \mathbb{P}(E_n), \tag{4.15}$$

which would suffice to verify (4.10). The verifications of (4.13) through (4.15) involve various counting arguments, which are contained in the following lemmas.

LEMMA 4.1. *There exists universal constant c such that*

$$\frac{1}{c} t_{n+\ln(n)\ln_2(n)} \leqslant t_n \ln(n) \leqslant c t_{n+\ln(n)\ln_2(n)}.$$

Proof. Let

$$q_n \stackrel{\mathrm{df}}{=} \frac{t_n \ln(n)}{t_{n+\ln(n)\ln_2(n)}}.$$

It suffices to show that $\lim_{n\to\infty} q_n = 1$. However, by some algebra,

$$\ln(q_n) = \frac{\ln\left(1 + \frac{\ln(n)\ln_2(n)}{n}\right)}{\frac{\ln(n)\ln_2(n)}{n}} \times \frac{n\ln_2(n) + \ln(n)(\ln_2(n))^2}{n\ln(n + \ln(n)\ln_2(n))}$$

Since $x^{-1}\ln(1+x) \to 1$ as $x \to 0$, it follows that $\ln(q_n) \to 0$ as $n \to \infty$. □

LEMMA 4.2. *There exists a universal positive constant c such that for all k, $n \geqslant 1$*

$$k \le c \ \ln(n) \left(\frac{t_{n+k} - t_n}{t_n}\right).$$

Proof. Observe that $j \mapsto t_j \cdot (\ln(j))^{-1}$ is increasing. From this and (4.2), it follows that

$$\begin{aligned} t_{n+k} - t_n &= \sum_{j=0}^{k-1} (t_{n+j+1} - t_{n+j}) \ge \sum_{j=0}^{k-1} \frac{t_{n+j}}{c\ln(n+j)} \\ &\ge \frac{k}{c} \cdot \frac{t_n}{\ln(n)}. \end{aligned}$$

Solving for k, we obtain the lemma. □

LEMMA 4.3. *There exists some $c > 0$ such that for all $N \ge n \ge 1$ and all $j \ge 0$,*

$$\#(\mathcal{B}_n^N \cap \mathcal{A}_n^N(j)) \le c(j+1)^4.$$

Proof. Let $k \in \mathcal{B}_n^N \cap \mathcal{A}_n^N(j)$. We will show that

$$k \le c(j+1)^4. \tag{4.16}$$

By Lemma 4.2,

$$\begin{aligned} k &\le c\ \ln(n)\left(\frac{t_{n+k}-t_n}{t_n}\right) \\ &= c\ \frac{t_{n+k}}{t_n}\ \ln(n)\left(1-\frac{t_n}{t_{n+k}}\right) \\ &\le c\ \frac{t_{n+k}}{t_n}\ (j+1)\sqrt{\ln(n)}. \end{aligned}$$

But for k in question,

$$t_{n+k} \le t_{n+\ln(n)\cdot\ln_2(n)} \le ct_n\cdot\ln(n),$$

by Lemma 4.1. Therefore,

$$k \le c(j+1)(\ln(n))^{3/2}. \tag{4.17}$$

On the other hand, by the definition of the annulus, $\mathcal{A}_n^N(j)$,

$$\begin{aligned} \frac{j+1}{\sqrt{\ln(n)}} &\ge 1-\frac{t_n}{t_{n+k}} \\ &\ge 1-\frac{t_n}{t_{n+\ln(n)}} \\ &\sim 1-\frac{1}{e}. \end{aligned}$$

Hence, for some $c>0$, $(j+1)\ge c\sqrt{\ln(n)}$. Equivalently, $\ln(n)\le c(j+1)^2$. By (4.17), we obtain (4.16) and hence the result. □

LEMMA 4.4. *There exists some $c>0$ such that for all $N\ge n\ge 1$ and all $j\ge 0$,*

$$\#(\mathcal{U}_n^N\cap\widetilde{\mathcal{A}}_n^N(j))\le c(j+1).$$

Proof. Let $k\in\mathcal{U}_n^N\cap\widetilde{\mathcal{A}}_n^N(j)$. By Lemma 4.2,

$$\begin{aligned} k &\le c\ \ln(n)\left(\frac{t_{n+k}-t_n}{t_n}\right) \\ &= c\ \ln(n)\ \frac{t_{n+k}}{t_n}\left(1-\frac{t_n}{t_{n+k}}\right) \\ &\le c\ \frac{t_{n+\ln(n)}}{t_n}\ (j+1) \\ &\le c(j+1). \end{aligned}$$

Therefore $\#(\mathcal{U}_n^N \cap \widetilde{\mathcal{A}}_n^N(j)) \le c(j+1)$, thus proving the result. $\square$

In the verification of (4.13) and (4.14), we will use the following inequality: for $0 \leqslant x < y$ and $a,\, b > 0$ we have

$$\mathbb{P}\,(L_x \leqslant a,\; L_y \leqslant b) \leqslant \mathbb{P}\,(L_x \leqslant a)\mathbb{P}\,(L_y - L_x \leqslant b), \tag{4.19}$$

where we have used the fact that $t \mapsto L_t$ is nondecreasing and that the process $\{L_t,\, t \geqslant 0\}$ has independent increments; see Getoor (1979), for example.

Verification of (4.13). By (4.19) and scaling we have

$$\begin{aligned}\mathbb{P}\,(E_n \cap E_{n+k}) &\leqslant \mathbb{P}\,(L_{t_n} \leqslant t_n^2 \varphi_n)\mathbb{P}\,(L_{t_{n+1}} - L_{t_n} \leqslant t_{n+k}^2 \varphi_{n+k})\\ &= \mathbb{P}\,(E_n)\mathbb{P}\,(L_1 - L_{t_n/t_{n+k}} \leqslant \varphi_{n+k}).\end{aligned}$$

We wish to demonstrate the conditions of Lemma 3.2 prevail with $s = t_n/t_{n+k}$ and $\beta = \varphi_{n+k}$. To this end we need to show that there is a universal λ for which

$$\frac{t_n}{t_{n+k}} \leqslant \lambda \varphi_{n+k}$$

for all $N \geqslant 1$, $n \geqslant 1$ and $k \in \mathcal{G}_n^N$. Since $j \mapsto t_j \varphi_j$ is increasing, it is enough to show that

$$\limsup_{n\to\infty} \frac{t_n}{t_{n+\ln(n)\ln_2(n)}\varphi_{n+\ln(n)\ln_2(n)}} < \infty. \tag{4.19}$$

This, however, follows immediately from (4.4) and Lemma 4.1. By Lemma 3.2 and the definition of $x \mapsto H(x)$, we obtain,

$$\begin{aligned}\mathbb{P}\,(L_1 - L_{t_n/t_{n+k}} \leqslant \varphi_{n+k}) &\leqslant cH(\varphi_{n+k})\\ &= c\mathbb{P}\,(E_{n+k}).\end{aligned}$$

In this way we have shown that there is a universal positive constant c such that

$$\mathbb{P}\,(E_n \cap E_{n+k}) \leqslant c\mathbb{P}\,(E_n)\mathbb{P}\,(E_{n+k}),$$

for all $n \geqslant 1$ and $k \in \mathcal{G}_n^N$. This verifies (4.13).

Verification of (4.14). It suffices to prove the result for N sufficiently large. By (4.19) and Lemma 3.3, there exists $n_0 \ge 1$ such that for all $N \ge n \ge n_0$ and all $k \in \mathcal{B}_n^N$,

$$\begin{aligned}\mathbb{P}\,(E_n \cap E_{n+k}) &\le \mathbb{P}\,(E_n) \cdot \mathbb{P}\,(L_1 - L_{t_n/t_{n+k}} \le \varphi_{n+k})\\ &\le c_1\; \mathbb{P}\,(E_n) \cdot \exp\big(-\tfrac{1}{4}\ln(n) \cdot (1 - (t_n/t_{n+k}))^2\big),\end{aligned} \tag{4.20}$$

since by (4.4), if n_0 is large enough, for all $N \geq n \geq n_0$ and all $k \in \mathcal{B}_n^N$,

$$\frac{t_n}{t_{n+k}} \leq \frac{t_n}{t_{n+\ln(n)}} \leq \frac{1}{2e} \leq 1 - \frac{c_1}{\ln(2n)} \leq 1 - c_2 \varphi_{n+k}.$$

By (4.20) and Lemma 4.3, for all $N \geq n \geq n_0$,

$$\begin{aligned}
\sum_{n=1}^{N} \sum_{k \in \mathcal{B}_n^N} \mathbb{P}\,(E_n \cap E_{n+k}) &\leq \sum_{n=n_0}^{N} \sum_{j=0}^{\infty} \sum_{k \in \mathcal{B}_n^N \cap \mathcal{A}_n^N(j)} \mathbb{P}\,(E_n) \exp\left(-c \;\; \ln(n) \;\; (1 - (t_n/t_{n+k}))^2\right) \\
&\qquad + \ln(n_0) \cdot \ln_2(n_0) \cdot \sum_{n=1}^{n_0} \mathbb{P}\,(E_n) \\
&\leq \sum_{n=1}^{N} \sum_{j=0}^{\infty} \#(\mathcal{B}_n^N \cap \mathcal{A}_n^N(j)) \cdot \mathbb{P}\,(E_n) \cdot \exp\left(-c_1 j^2\right) + c_2 \sum_{n=1}^{N} \mathbb{P}\,(E_n) \\
&\leq c \sum_{n=1}^{N} \mathbb{P}\,(E_n),
\end{aligned}$$

verifying (4.14).

Verification of (4.15). It suffices to prove the result when N is sufficiently large. Take $k \in \mathcal{U}_n^N$. We want to use Lemma 3.4 with $s = t_n/t_{n+k}$, $\alpha = \varphi_n$ and $\beta = \varphi_{n+k}$. By (4.2) and (4.4), as $n \to \infty$,

$$\begin{aligned}
s &\geq \frac{t_n}{t_{n+\ln(n)}} \sim \frac{1}{e} \\
s &\leq \frac{t_n}{t_{n+1}} \sim \left(1 - \frac{1}{\ln(n)}\right) \leq 1 - c\varphi_n.
\end{aligned}$$

Thus there exist $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $n_0 \geq 1$, such that for all $N \geq n \geq n_0$ and all $k \in \mathcal{G}_n^N$, $\varepsilon_1 < s < 1 - \varepsilon_2 \beta$. By scaling $H(\varphi_n) = \mathbb{P}\,(E_n)$. Hence, by Lemma 3.4, for all $N \geq n \geq n_0$ and all $k \in \mathcal{U}_n^N$,

$$\mathbb{P}\,(E_n \cap E_{n+k}) \leq c\mathbb{P}\,(E_n) \cdot \exp\left(- \frac{1}{12\varphi_n} \left(1 - \frac{t_n}{t_{n+k}}\right)\right).$$

Therefore by (4.2) and Lemma 4.4, for all $N \geq n \geq n_0$,

$$\sum_{n=1}^{N} \sum_{k \in \mathcal{U}_n^N} \mathbb{P}\,(E_n \cap E_{n+k}) \leq \sum_{n=n_0}^{N} \sum_{j=0}^{\infty} \sum_{k \in \mathcal{U}_n^N \cap \widetilde{\mathcal{A}}_n^N(j)} \mathbb{P}\,(E_n)\exp\left(-(c/\varphi_n)(1-(t_n/t_{n+k}))\right) + \\ + \ln(n_0) \cdot \sum_{n=1}^{n_0} \mathbb{P}\,(E_n)$$

$$\leq \sum_{n=1}^{N} \sum_{j=0}^{\infty} \sum_{k \in \mathcal{U}_n^N \cap \widetilde{\mathcal{A}}_n^N(j)} \mathbb{P}\,(E_n)\exp\left(-c_1 \ \ln(n)(1-(t_n/t_{n+k}))\right) + c_2 \sum_{n=1}^{N} \mathbb{P}\,(E_n)$$

$$\leq \sum_{n=1}^{N} \sum_{j=0}^{\infty} \#(\mathcal{U}_n^N \cap \widetilde{\mathcal{A}}_n^N(j)) e^{-c_1 j} \mathbb{P}\,(E_n) + c_2 \sum_{n=1}^{N} \mathbb{P}\,(E_n)$$

$$\leq c \sum_{n=1}^{N} \mathbb{P}\,(E_n).$$

This verifies (4.15) and finishes the proof of Theorem 2.1. □

Finally, we offer the following:

Proof of Theorem 2.2. Recall (2.1b); since the case $\mathfrak{J}_2(\psi) < \infty$ is contained in the result of Khoshnevisan, Lewis and Li (1994), we need only consider the case $\mathfrak{J}_2(\psi) = \infty$. Let $\varphi(t) = \psi^{-2}(t)$. Then φ satisfies the conditions of Theorem 2.1. Moreover, by exactly the same proof as in Theorem 2.1:

$$L_{\psi_{n+1} t_{n+1}/\psi_n} \leq \frac{t_n^2}{\psi_{n+1}^2}, \qquad \text{i.o.} \tag{4.21}$$

where $t_n = \exp(n/\ln(n))$ and $\psi_n = \psi(t_n)$. Since $t \mapsto \psi(t)$ is nondecreasing, the interval $[t_n^2/\psi_{n+1}^2, t_{n+1}^2/\psi_n^2]$ is not empty. On the other hand, (4.21) is equivalent to the existence of a random and infinite set, $\mathcal{N}$, such that

$$I_{t_n^2/\psi_{n+1}^2} \geq t_{n+1} \frac{\psi_{n+1}}{\psi_n}, \qquad \text{for all } n \in \mathcal{N}.$$

Notice that for all $n \in \mathcal{N}$ and all $s \in [t_n^2/\psi_{n+1}^2, t_{n+1}^2/\psi_n^2]$,

$$I_s \geq I_{t_n^2/\psi_{n+1}^2} \geq t_{n+1} \frac{\psi_{n+1}}{\psi_n} \geq s^{1/2} \psi_{n+1} \geq s^{1/2} \psi(s),$$

which is what we wished to show. □

5. Proofs of random walk results

Throughout this section, let $d \geqslant 3$ be an integer and let $W \stackrel{\text{df}}{=} \{W_t,\, t \geqslant 0\}$ denote a standard d–dimensional Brownian motion. Given a collection $\{\xi_i,\, \geqslant 1\}$ of i.i.d. $\mathbb{R}^d$–valued random vectors, let $S_0 \stackrel{\text{df}}{=} 0$ and, for all $n \geqslant 1$, let

$$S_n \stackrel{\text{df}}{=} \xi_1 + \cdots + \xi_n.$$

We will extend the random walk to continuous time by setting $S_t \stackrel{\text{df}}{=} S_{[t]}$ for all $t \geqslant 0$, where $[t]$ denotes the greatest integer which is less than or equal to t. For real numbers $0 \leqslant t \leqslant T$, let

$$J_t \stackrel{\text{df}}{=} \inf_{u \geq t} \|S_u\|, \qquad J_{t,T} \stackrel{\text{df}}{=} \inf_{t \leqslant u \leq T} \|S_u\|;$$
$$I_t \stackrel{\text{df}}{=} \inf_{u \geq t} \|W_u\|, \qquad I_{t,T} \stackrel{\text{df}}{=} \inf_{t \leq u \leq T} \|W_u\|.$$

The main result of this section is the following strong approximation theorem:

THEOREM 5.1. *Let ξ be an $\mathbb{R}^d$–valued random variable with zero mean vector, identity covariance matrix and $\mathbb{E}\,(\|\xi\|^{2+\delta}) < \infty$, for some $\delta > 0$. Then on a suitable probability space we can reconstruct a sequence of i.i.d. random vectors $\{\xi_i,\, i \geqslant 1\}$ with $\mathcal{L}(\xi_1) = \mathcal{L}(\xi)$ and a Brownian motion W such that for some $q = q(\delta) \in (0, 1/2)$, almost surely,*

$$|I_t - J_t| = o(t^{1/2-q}).$$

REMARK 5.1.1. With more work, the condition $\mathbb{E}\,(\|\xi\|^{2+\delta}) < \infty$ can be relaxed to the following: there exists some $p > 1$ such that $\mathbb{E}\,(K(\xi)) < \infty$, where

$$K(x) \stackrel{\text{df}}{=} x^2 (\ln(x))^{\frac{2}{d-2}} (\ln_2(x))^{2p(\frac{d-1}{d-2})}.$$

The proof of Theorem 5.1 relies on a strong approximation result of EINMAHL (1987), which is implicit in the following lemma. Given $\delta > 0$, let

$$\eta \stackrel{\text{df}}{=} \frac{1}{2} - \frac{1}{2+\delta}. \tag{5.1}$$

LEMMA 5.2. *Under the hypothesis of Theorem 5.1 almost surely*

$$\sup_{0\leqslant u\leqslant t}\|S_u-W_u\|=o(t^{1/2-\eta}),$$

where η is given by (5.1).

Proof. Since $t\mapsto\sup_{0\leqslant u\leqslant t}\|S_u-W_u\|$ and $t\mapsto t^{1/2-\eta}$ are nondecreasing, it suffices to prove the result for integral values of t. Let n be a positive integer. By the triangle inequality,

$$\begin{aligned}\sup_{0\leqslant u\leqslant n}\|S_u-W_u\| &\leqslant \sup_{0\leqslant u\leqslant n}\|S_u-W_{[u]}\|+\sup_{0\leqslant u\leqslant n}\|W_u-W_{[u]}\|\\ &=\max_{0\leqslant k\leqslant n}\|S_k-W_k\|+\max_{0\leqslant k\leqslant n-1}\max_{k\leqslant s\leqslant k+1}\|W_s-W_k\|.\\ &=I_n+II_n,\end{aligned}$$

with obvious notation. We will estimate each of these terms in order.

A direct consequence of Theorem 2 of EINMAHL (1987) is the following: on a suitable probability space, one can reconstruct the sequence $\{\xi_i,\ i\geqslant 1\}$ and a sequence of independent standard normal random variables, $\{g_i,\ i\geqslant 1\}$, such that

$$\max_{k\leq n}\|S_k-G_k\|=o(n^{\frac{1}{2}-\eta}),\qquad \text{a.s.,}\tag{5.2}$$

where $G_k\stackrel{\text{df}}{=}\sum_{j=1}^k g_j$. To construct the Brownian motion W, enlarge the probability space (by introducing product spaces) so that it contains independent processes, $\{B_i(t);0\leq t\leq 1\}_{i\geq 1}$, where B_i is a standard Brownian motion starting at G_i conditioned to be G_{i+1} at time 1 (an interesting construction of such processes appears in PITMAN (1974)). On this extended probability space, define

$$W_t\stackrel{\text{df}}{=}\sum_{k=0}^{\infty}1_{[k,k+1)}(t)B_k(t-k).$$

It is easy to see that W is a standard Brownian motion and that $W_k=G_k$; consequently, (5.2) can be written as follows: as $n\to\infty$,

$$I_n=\max_{0\leqslant k\leq n}\|S_k-W_k\|=o\,(n^{\frac{1}{2}-\eta}),\qquad \text{a.s.}\tag{5.3}$$

Finally, by Theorem 1.7.1 of CSÖRGŐ AND RÉVÉSZ (1981), as $n \to \infty$,

$$II_n = \max_{0 \leqslant k \leqslant n-1} \sup_{k \leqslant s \leqslant k+1} \|W_s - W_k\| = O\left(\sqrt{\ln(n)}\right), \qquad \text{a.s.}$$

which, in light of (5.3), proves the lemma in question. □

A small but important step in proving Theorem 5.1 is the following:

LEMMA 5.3. *Let $\theta \in (0, \eta)$, where η is given by (5.1). Then, with probability one, $I_t \geq t^{\frac{1}{2}-\theta}$ and $J_t \geq t^{\frac{1}{2}-\theta}$ for all t sufficiently large.*

Proof. Let $f : \mathbb{R}^1_+ \to \mathbb{R}^1_+$ be nondecreasing. Then a real-variable argument shows that the following are equivalent:

(i) $J_t \geq f(t)$, eventually ($I_t \geq f(t)$, eventually)

(ii) $\|S_t\| \geq f(t)$, eventually ($\|W_t\| \geq f(t)$, eventually).

Let $\theta_0 > 0$. Then, either by direct calculation (using the Borel-Cantelli lemma and the explicit density of I_1, given in KHOSHNEVISAN, ET AL. (1994)) or by the theorem of Dvoretsky and Erdős (see MOTOO (1959)) on the rate of escape of $\|W_t\|$, it can be shown that eventually,

$$\|W_t\| \geq t^{\frac{1}{2}-\theta_0}, \qquad \text{a.s.} \tag{5.4}$$

Now the lemma follows by Lemma 5.2, (i), (ii), (5.4) and the choice of θ. □

Proof of Theorem 5.1. Construct a probability space in accordance with Lemma 5.2. Given $\delta > 0$, η is given by (5.1). Now, choose θ, $\varepsilon \in \mathbb{R}^+$ such that $0 < \theta < \eta$ and

$$0 < \varepsilon < \frac{\eta - \theta}{1 - 2\eta},$$

and set

$$\rho \stackrel{\text{df}}{=} \frac{1 + 2\varepsilon}{1 - 2\theta}.$$

Significantly,

$$\rho\left(\frac{1}{2} - \theta\right) = \frac{1}{2} + \varepsilon, \tag{5.5}$$

$$\rho\left(\frac{1}{2} - \eta\right) < \frac{1}{2}. \tag{5.6}$$

Our first task is to show that the infimum of $\|W_s\|$ for $s \geqslant t$ is actually attained for $s \in [t, t^\rho]$ (for all t sufficiently large). Indeed, by Lemma 5.3 and (5.5), it follows that

$$I_{t^\rho} \geqslant (t^\rho)^{1/2-\theta} \geqslant t^{1/2+\varepsilon}, \qquad \text{a.s.},$$

for all t sufficiently large. However, by the ordinary law of the iterated logarithm,

$$I_t \leqslant \|W_t\| < t^{1/2+\varepsilon}, \qquad \text{a.s.},$$

for all t sufficiently large. Since $I_t = I_{t,t^\rho} \wedge I_{t^\rho}$, these considerations yield: with probablity one,

$$I_t = I_{t,t^\rho}, \tag{5.7}$$

for all t sufficiently large. Likewise, with probability one,

$$J_t = J_{t,t^\rho}, \tag{5.8}$$

for all t sufficiently large.

Finally, it is easy to see that

$$|J_{t,t^\rho} - I_{t,t^\rho}| \leqslant \sup_{0 \leqslant u \leqslant t^\rho} \|S_u - W_u\|.$$

Thus, by (5.7), (5.8) and Lemma 5.2 we obtain: as $t \to \infty$,

$$|J_t - I_t| \leqslant \sup_{0 \leqslant u \leqslant t^\rho} \|S_u - W_u\| = o\,(t^{\rho(1/2-\eta)}), \qquad \text{a.s.}$$

By (5.6), this proves the theorem. □

LEMMA 5.4. *Let $\psi : [1,\infty) \to (0,\infty)$ be nondecreasing and unbounded. For all $t \geqslant 1$, let*

$$\psi_U(t) \stackrel{\text{df}}{=} \begin{cases} \psi(t) + (\psi(t))^{-1} & \text{if } \psi(t) \geqslant 1 \\ 2 & \text{if } 0 < \psi(t) < 1. \end{cases}$$

$$\psi_L(t) \stackrel{\text{df}}{=} \begin{cases} \psi(t) - (\psi(t))^{-1} & \text{if } \psi(t) \geqslant 2 \\ 1 & \text{if } 0 < \psi(t) < 2. \end{cases}$$

Then ψ_U and ψ_L are nonegative and nondecreasing and $\mathfrak{J}_2(\psi)$, $\mathfrak{J}_2(\psi_U)$ and $\mathfrak{J}_2(\psi_L)$ converge or diverge together.

Proof. That ψ_U and ψ_L are nonegative and nondecreasing follows trivially from the fact that the mappings $x \mapsto x + x^{-1}$ and $x \mapsto x - x^{-1}$ are nonnegative and increasing on $x \geqslant 1$ and $x \geqslant 2$, respectively.

Since ψ is unbounded, it is easy to see that, as $t \to \infty$,

$$\psi^{d-2}(t) \sim \psi_U^{d-2}(t) \sim \psi_L^{d-2}(t)$$

Moreover, there exist positive constants c_1, c_2, c_3 and c_4 such that

$$c_1 \exp\left(-\psi^2(t)/2\right) \leqslant \exp\left(-\psi_U^2(t)/2\right) \leqslant c_2 \exp\left(-\psi^2(t)/2\right)$$
$$c_3 \exp\left(-\psi^2(t)/2\right) \leqslant \exp\left(-\psi_L^2(t)/2\right) \leqslant c_4 \exp\left(-\psi^2(t)/2\right),$$

which proves the lemma. $\square$

Proof of Theorem 2.3. Let $\psi : [1, \infty) \to (0, \infty)$ be nondecreasing. As is customary, we will assume, without loss of generality, there there exists a positive constant c such that

$$\frac{1}{c}\sqrt{\ln_2(t)} \leqslant \psi(t) \leqslant c\sqrt{\ln_2(t)}.$$

If $\mathfrak{J}_2(\psi) < \infty$, by Lemma 5.4, $\mathfrak{J}_2(\psi_L) < \infty$. Consequently, for t sufficiently large,

$$I_t < \sqrt{t}\psi_L(t) \leqslant \sqrt{t}\psi(t) - c\sqrt{\frac{t}{\ln_2(t)}}, \qquad \text{a.s.}$$

By Theorem 5.1, $J_t \leqslant I_t + t^{1/2-q}$, where $0 < q < 1/2$. From this it follows that, for t sufficiently large,

$$J_t < \sqrt{t}\psi(t), \qquad \text{a.s.}$$

Likewise, if $\mathfrak{J}_2(\psi) = \infty$, by Lemma 5.4, $\mathfrak{J}_2(\psi_U) = \infty$, as well. Consequently, there exists a (random) increasing and unbounded sequence $\{t_n,\, n \geqslant 1\}$ for which

$$I_{t_n} \geqslant \sqrt{t_n}\psi_U(t_n) \geqslant \sqrt{t_n}\psi(t_n) + c\sqrt{\frac{t_n}{\ln_2(t_n)}}, \qquad \text{a.s.}$$

However, by Theorem 5.1, $J_t \geqslant I_t - t^{1/2-q}$, where $0 < q < 1/2$, from which it follows that

$$J_{t_n} \geqslant \sqrt{t_n}\psi(t_n), \qquad \text{a.s.}$$

This proves the theorem in question. $\square$

6. The general covariance case

Throughout this section, $d \geqslant 3$ will be an integer and $\mathbf{Q}$ will denote a $d \times d$ symmetric positive definite matrix with eigenvalues

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \lambda_d > 0$$

and corresponding orthonormal eigenvectors $\zeta_1, \ldots, \zeta_d$. Let $B_1, \ldots, B_d$ denote d independent standard 1–dimensional Brownian motions. Then for all $t \geqslant 0$,

$$W_{\mathbf{Q}}(t) \stackrel{\text{df}}{=} \sum_{i=1}^{d} \sqrt{\lambda_i} B_i(t) \zeta_i$$

defines a centered Brownian motion with covariance matrix $\mathbf{Q}$. ¿From this, it is clear that

$$\begin{aligned} \|W_{\mathbf{Q}}(t)\|^2 &= \sum_{i=1}^{d} \lambda_i B_i^2(t) \\ &= \|\mathbf{\Lambda} W(t)\|^2, \end{aligned}$$

where

$$W(t) \stackrel{\text{df}}{=} (B_1(t), \ldots, B_d(t))$$

is a standard d–dimensional Brownian motion, and $\mathbf{\Lambda}$ is the $d \times d$ diagonal matrix

$$\mathbf{\Lambda} \stackrel{\text{df}}{=} \begin{pmatrix} \sqrt{\lambda_1} & 0 & \ldots & 0 \\ 0 & \sqrt{\lambda_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \sqrt{\lambda_d} \end{pmatrix}$$

This demonstrates that it is sufficient to verify Theorem 2.4 for

$$L_{\mathbf{Q}}(r) \stackrel{\text{df}}{=} \sup\{t \geqslant 0 : \|\mathbf{\Lambda} W(t)\| \leqslant r\} \quad \text{and} \quad I_{\mathbf{Q}}(t) \stackrel{\text{df}}{=} \inf_{s \geqslant t} \|\mathbf{\Lambda} W(s)\|.$$

Our first task is to derive a large deviation estimate for $I_{\mathbf{Q}}(1)$, which, in turn, will yield the small–ball probability of $L_{\mathbf{Q}}(1)$. The following geometric argument will be useful in this regard. For every $t \geqslant 0$, let us define the ellipsoid

$$\mathcal{E}_t \stackrel{\text{df}}{=} \{x \in \mathbb{R}^d : \|\mathbf{\Lambda} x\| \leqslant t\}.$$

LEMMA 6.1. *Let*

$$r \stackrel{\text{df}}{=} \sqrt{\lambda_1}\Big(\frac{1}{\lambda_d} - \frac{1}{\lambda_1}\Big).$$

Then

$$\mathcal{E}_1 \subset \{x \in \mathbb{R}^d : \|x + re_1\| \leqslant r + 1/\sqrt{\lambda_1}\,\},$$

where e_1 is the unit vector $(1, 0, \ldots, 0)$.

Proof. In the case $d = 2$, parameterize the boundary of $\mathcal{E}_1$ by

$$x_1 \stackrel{\text{df}}{=} \frac{\cos(\theta)}{\sqrt{\lambda_1}} \quad \text{and} \quad x_2 \stackrel{\text{df}}{=} \frac{\sin(\theta)}{\sqrt{\lambda_2}}$$

where $\theta \in [0,\, 2\pi]$. Now the proof follows by some elementary algebra and trigonometry.

If $d \geqslant 3$ and $x \in \mathcal{E}_1$, then

$$\lambda_1 x_1^2 + \lambda_d y^2 \leqslant 1,$$

where $y \geqslant 0$ and

$$y^2 = \frac{\lambda_2}{\lambda_d} x_2^2 + \cdots + \frac{\lambda_{d-1}}{\lambda_d} x_{d-1}^2 + x_d^2.$$

Then, by the two–dimensional case and the ordering of the eigenvalues,

$$\begin{aligned} \|x + re_1\|^2 &= (x_1 + r)^2 + x_2^2 + \ldots + x_d^2 \\ &\leqslant (x_1 + r)^2 + y^2 \\ &\leqslant (r + 1/\sqrt{\lambda_1})^2, \end{aligned}$$

as was to be shown. □

Our next result is the aforementioned large deviation estimate for $I_{\mathbf{Q}}(1)$.

LEMMA 6.2.

(a) $\displaystyle\lim_{t\to\infty} t^{-2} \ln \mathbb{P}\left(I_{\mathbf{Q}}(1) \geqslant t\right) = -\frac{1}{2\lambda_1}.$

(b) $\displaystyle\lim_{\varepsilon\to 0^+} \varepsilon \ln \mathbb{P}\left(L_{\mathbf{Q}}(1) \leqslant \varepsilon\right) = -\frac{1}{2\lambda_1}.$

Proof. Since $\mathbb{P}\left(L_{\mathbf{Q}}(1) \leqslant \varepsilon\right) = \mathbb{P}\left(I_{\mathbf{Q}}(1) \geqslant 1/\sqrt{\varepsilon}\right)$, it is enough to prove (a). Given $t > 0$, let

$$D \stackrel{\text{df}}{=} \{x : \|x + tre_1\| \leqslant t(r + \lambda_1)\}.$$

Observe that if $x \in \mathcal{E}_t$, then $x/t \in \mathcal{E}_1$. Thus, by Lemma 6.1, $\mathcal{E}_t \subset D$. Consequently,

$$\begin{aligned}\mathbb{P}\left(I_{\mathbf{Q}}(1) \geqslant t\right) &= \mathbb{P}\left(W_s \notin \mathcal{E}_t \text{ for all } s \geqslant 1\right) \\ &\geqslant \mathbb{P}\left(W_s \notin D \text{ for all } s \geqslant 1\right).\end{aligned}$$

Now pick $\alpha > 1/\sqrt{\lambda_1}$ and let H be the following half-space

$$H \overset{\text{df}}{=} \{x \in \mathbb{R}^d : x_1 \geqslant \alpha t\}.$$

Then we have

$$\mathbb{P}\left(W_s \notin D \text{ for all } s \geqslant 1\right) \geqslant \int_H \mathbb{P}\left(W_s \notin D \text{ for all } s \geqslant 1 \,|\, W_1 = x\right) \mathbb{P}\left(W_1 \in dx\right). \qquad (6.1)$$

Notice that each of the conditional probabilities is the probability that W, started from a point in the exterior of the ball D, never hits D. Consequently, by a gambler's ruin calculation,

$$\mathbb{P}\left(W_s \notin D \text{ for all } s \geqslant 1 \,|\, W_1 = x\right) = 1 - \left(\frac{t(r + 1/\sqrt{\lambda_1})}{\|x + rte_1\|}\right)^{d-2}$$

Since $x \in H$, we know that $\|x + rte_1\| \geqslant t(r+\alpha)$. Consequently, we have the uniform lower bound:

$$\mathbb{P}\left(W_s \notin D \text{ for all } s \geqslant 1 \,|\, W_1 = x\right) \geqslant 1 - \left(\frac{r + 1/\sqrt{\lambda_1}}{r+\alpha}\right)^{d-2} \overset{\text{df}}{=} \kappa. \qquad (6.2)$$

Combining (6.1) and (6.2), we obtain:

$$\begin{aligned}\mathbb{P}\left(W_s \notin D \text{ for all } s \geqslant 1\right) &\geqslant \kappa \mathbb{P}\left(W_1 \in H\right) \\ &= \kappa \mathbb{P}\left(B_1(1) \geqslant \alpha t\right),\end{aligned}$$

where, in our notation, B_1 is the first coordinate of W; hence, a standard one–dimensional Brownian motion. Simple considerations yield:

$$\liminf_{t\to\infty} \frac{1}{t^2} \ln \mathbb{P}\left(I_{\mathbf{Q}}(1) \geqslant t\right) \geqslant -\frac{\alpha^2}{2}.$$

The lower bound is achieved by sending α to $1/\sqrt{\lambda_1}$.

The upper bound is easy: $I_{\mathbf{Q}}(1) \leqslant \|\mathbf{\Lambda} W(1)\|$ and, by a result of Zolotarev (1961),

$$\lim_{t\to\infty} \frac{1}{t^2} \ln \mathbb{P}\left(\|\mathbf{\Lambda} W(1)\| \geqslant t\right) = -\frac{1}{2\lambda_1}.$$

This proves the lemma in question. □

Proof of Theorem 2.4. It is sufficient to prove (a), since the results of part (b) follow by inversion and strong approximation respectively. The proof of (a) is standard.

Let $B > 1$ and let $t_n \stackrel{\mathrm{df}}{=} B^n$ for all $n \geqslant 1$. Let $0 < \varepsilon < 1$ and define the events: for all $n \geqslant 1$,

$$A_n \stackrel{\mathrm{df}}{=} \{\omega : L_{\mathbf{Q}}(t_n) \leqslant (1-\varepsilon) t_n^2 / (2\lambda_1 \ln_2(t_n))\}.$$

By scaling and Lemma 6.2, it follows that as $n \to \infty$,

$$\ln \mathbb{P}\,(A_n) \sim -\frac{1}{1-\varepsilon} \ln(n)$$

Thus $\sum_n \mathbb{P}\,(A_n) < \infty$ and, by the easy half of the Borel–Cantelli lemma, $\mathbb{P}\,(A_n \text{ i.o.}) = 0$. It follows that for almost all ω there is an integer N such that for all $n \geqslant N$ we have

$$\frac{2\lambda_1 \ln_2(t_n)}{t_n^2} L_{\mathbf{Q}}(t_n) \geqslant (1-\varepsilon). \tag{6.3}$$

Hence for $n \geqslant N$ and $t_n \leqslant s < t_{n+1}$ we obtain:

$$\frac{2\lambda_1 \ln_2(s)}{s^2} L_{\mathbf{Q}}(s) \geqslant \frac{(1-\varepsilon)}{B^2},$$

where we have used (6.3) and the fact that $t \mapsto L_{\mathbf{Q}}(t)$ is nondecreasing. This demonstrates that

$$\liminf_{s\to\infty} \frac{2\lambda_1 \ln_2(s)}{s^2} L_{\mathbf{Q}}(s) \geqslant 1, \qquad \text{a.s. .}$$

Let $\varepsilon > \alpha > 0$ be chosen. Let $t_1 \stackrel{\mathrm{df}}{=} e$ and, for all $n \geqslant 1$, let

$$t_{n+1} \stackrel{\mathrm{df}}{=} t_n \exp\left(n^\alpha\right).$$

It follows that

$$\begin{aligned} \ln(t_n) &\sim \frac{n^{1+\alpha}}{1+\alpha}, \\ \ln_2(t_n) &\sim (1+\alpha)\ln(n). \end{aligned} \tag{6.4a}$$

Before proceeding with Borel–Cantelli argument, let us observe that $\|W_{\mathbf{Q}}(t)\| \geqslant \lambda_d \|W(t)\|$ for all $t \geqslant 0$. Moreover, by a theorem of Dvoretsky and Erdős (see MOTOO (1959)), almost surely

$$\|W(t)\| \geqslant \frac{\sqrt{t}}{(\ln(t))^3},$$

for all t sufficiently large. Thus by a real–variable argument and inversion, it follows that with probability one there is a positive constant c such that

$$L_{\mathbf{Q}}(t) \leqslant c\,t^2(\ln(t))^6, \tag{6.4b}$$

for all t sufficiently large.

For every $n \geqslant 1$, let

$$\begin{aligned} A_n &\stackrel{\text{df}}{=} \{\omega : L_{\mathbf{Q}}(t_{n+1}) - L_{\mathbf{Q}}(t_n) \leqslant ((1+\varepsilon)t_{n+1}^2/(2\lambda_1 \ln_2(t_{n+1}))\} \\ &\supset \{\omega : L_{\mathbf{Q}}(t_{n+1}) \leqslant ((1+\varepsilon)t_{n+1}^2/(2\lambda_1 \ln_2(t_{n+1}))\}. \end{aligned}$$

Hence by Lemma 6.2 and (6.4a) we obtain: as $n \to \infty$,

$$\liminf_{n\to\infty} \frac{\ln \mathbb{P}\,(A_n)}{\ln(n)} \geqslant -\frac{1+\alpha}{1+\varepsilon} > -1,$$

from which it follows that $\sum_n \mathbb{P}\,(A_n) = \infty$. Since increments of the escape process are independent, the events $\{A_n,\, n \geqslant 1\}$ are independent. Thus, by the independence half of the Borel–Cantelli lemma, $P(A_n \text{ i.o.}) = 1$. Consequently, for almost all ω there is an infinite $\mathcal{N} \subset \mathbb{Z}^+$ such that $n \in \mathcal{N}$ implies

$$\frac{2\lambda_1 \ln_2(t_{n+1})}{t_{n+1}^2} L_{\mathbf{Q}}(t_{n+1}) \leqslant (1+\varepsilon) + \frac{2\lambda_1 \ln_2(t_{n+1})}{t_{n+1}^2} L_{\mathbf{Q}}(t_n). \tag{6.5}$$

¿From (6.4a) and (6.4b), it follows that

$$\lim_{n\to\infty} \frac{\ln_2(t_{n+1})}{t_{n+1}^2} L_{\mathbf{Q}}(t_n) = 0, \qquad \text{a.s. .}$$

¿From this, we obtain

$$\liminf_{t\to\infty} \frac{2\lambda_1 \ln_2(t)}{t^2} L_{\mathbf{Q}}(t) \leqslant (1+\varepsilon), \qquad \text{a.s. .}$$

The proof is completed upon sending ε to 0. □

THE ANNALS OF PROBABILITY **23**(3), 761–787 (1996).

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THE ANNALS OF PROBABILITY **23**(3), 761–787 (1996).

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