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DEVIATION INEQUALITIES FOR CONTINUOUS MARTINGALES *

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Abstract. We consider a broad class of continuous martingales whose local modulus of continuity is in some sense deterministic. We show that such martingales have Gaussian probability tails, provided we appropriately normalize them by their quadratic variation. As other applications of our methods, we provide energy inequalities and prove a new sufficient condition for the joint continuity of continuous additive functionals of Brownian motion indexed by their Revuz measures.

§**1. Introduction.** Suppose $(M_t; 0 \le t \le \infty)$ is a continuous martingale (including the terminal point at infinity to make the notation simpler) which has a finite moment generating function. That is, for all $\zeta \in \mathbb{R}^1$,

$$\mathbb{P} \exp(\zeta M_\infty) < \infty. \tag{1.0}$$

We assume that $M_0 = 0$ and that the underlying filtration, $\big(\mathcal{F}_t; 0 \le t \le \infty\big)$, satisfies the usual hypotheses. We refer the reader to Revuz and Yor [RY] for the theory of continuous martingales. The motivation behind this work is the following result, essentially due to McKean [McK] (see also Freedman [Fr]):

(1.1) **Theorem.** *If (1.0) holds, for any $\alpha, \beta, \lambda > 0$,*

$$\mathbb{P}\Big(M_\infty \ge (\alpha + \beta \langle M \rangle_\infty)\lambda\Big) \le \exp\big(-2\alpha\beta\lambda^2\big).$$

One can effectively drop the assumption (1.0) but we are not concerned with such refinements here.

The main result of this paper states that if (M_t) has a locally deterministic modulus of continuity in a sense which will be described shortly (cf. (1.2) below), the above Gaussian bound is the correct one up to a constant. Other related results appear in Barlow, Jacka and Yor [BJY] and Dembo [De]. There is a relationship between Theorem (1.4) below and the main result of [De]. Indeed, Dembo is interested in the large–time behavior of $t \mapsto M_t$: when the large–time behavior of $t \mapsto \langle M \rangle_t$ is in some sense deterministic, a moderate deviations principle holds. Here, we are interested in fixed–time results which are (at least from a technical stand point) related to the local behavior of $t \mapsto M_t$ in a somewhat similar way.

Our key assumption is one about the local modulus of continuity of M: there exists an adapted continuous local martingale $\big(D_t; 0 \le t \le \infty\big)$, such that with probability one, $D_0 = 0$ and for all $s, t > 0$,

$$\underline{\mu}\big(D_t; [t, t+s]\big) \le \mathbb{P}\big((M_{t+s} - M_t)^2 \mid \mathcal{F}_t\big) \le \overline{\mu}\big([t, t+s]\big), \tag{1.2}$$

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where for any $a \in \mathbb{R}^1$, $A \mapsto \underline{\mu}(a;A)$ and $A \mapsto \overline{\mu}(A)$ are finite and positive, nonrandom, atomless measures on $\mathbb{R}^1_+$. Moreover, for any Borel $A \subset \mathbb{R}^1$, $a \mapsto \underline{\mu}(a;A)$ is convex. It is possible to show that the fundamental martingales studied in [McK] satisfy (1.2). Let $\underline{\mu}$ and $\overline{\mu}$ denote the total mass of $\underline{\mu}(0;\cdot)$ and $\overline{\mu}$, respectively. More precisely,

$$\begin{aligned} \underline{\mu} &\triangleq \underline{\mu}\big(0;[0,\infty)\big) \\ \overline{\mu} &\triangleq \overline{\mu}\big([0,\infty)\big). \end{aligned} \tag{1.3}$$

The main result of this paper is the following:

(1.4) **Theorem.** *Suppose (1.0) and (1.2) hold. Fix some $p \in (0,1)$. Then for any choice of $C < (1-p)\underline{\mu}/(\overline{\mu} - p\underline{\mu})$, there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$,*

$$\exp\big(-A_1\lambda^2\big) \ge \mathbb{P}\Big(M_\infty \ge \big(1 + (p\underline{\mu})^{-1}\langle M\rangle_\infty\big)\lambda\Big) \ge C\exp\Big(-A_2\lambda^2\Big),$$

where $A_1 \triangleq 2(p\underline{\mu})^{-1}$ and $A_2 \triangleq 2(p\underline{\mu})^{-2}\overline{\mu}$.

The ideas employed in the proofs are reminiscent of the change of measure method of Cramér [Cr] and the energy inequalities of Meyer [Me].

This paper is organized as follows. In the next section, we describe some preliminary estimates. In particular, we prove in Proposition (2.5) that (1.2) implies that $M \in H^\infty(\mathbb{P})$ and we provide an explicit estimate for the H^∞ norm of M. In Section 3, we use the estimates of Section 2 in order to demonstrate Theorem (1.4). The next two sections are devoted to other consequences of (1.2). In the fourth section, we provide energy inequalities and results on the smoothness of the sample functions of $t \mapsto M_t$. The inequalities developed in Section 4 are in turn used in Section 5 to give estimates for the smoothness of continuous additive functionals of multidimensional Brownian motion viewed as functions of their Revuz measures. This extends and compliments some of the work of Bass and Khoshnevisan [BK]. In this connection, see also Marcus and Rosen [MR1,MR2].

Let us mention some examples.

(1.5) **Example.** Suppose $\langle M\rangle_t = \alpha(t)$ is a deterministic, bounded and increasing process. Then (1.2) holds with $\overline{\mu}(A) = \underline{\mu}(x;A) = \int_A \alpha(ds)$, for any x. In this case, Theorem (1.4) implies that for all $C, p \in (0,1)$, there exists $\lambda_0 > 0$ such that for all $0 < \lambda_0 < \lambda$,

$$\exp\Big(-\frac{2\lambda^2}{p\alpha(\infty)}\Big) \ge \mathbb{P}\big(M_\infty \ge (1+p^{-1})\lambda\big) \ge C\exp\Big(-\frac{2\lambda^2}{p^2\alpha(\infty)}\Big).$$

In particular,

$$\lim_{\lambda\to\infty} \lambda^{-2}\ln\mathbb{P}\big(M_\infty \ge \lambda\big) = -\frac{1}{2\alpha(\infty)}. \tag{1.6}$$

On the other hand, by Lévy's representation theorem (cf. Revuz and Yor [RY]), there exists a Brownian motion B such that $M_t = B_{\alpha(t)}$. Thus (1.6) agrees with well–known results about B.

(1.7) **Example.** Suppose $M_t = \int_0^t f(B_s)dB_s$, where B is a Brownian motion. We are interested in obtaining Gaussian estimates for $\mathbb{P}\big(M_1 \ge (\alpha + \beta\langle M\rangle_1)\lambda\big)$. Suppose that there exist $0 < \alpha_0 \le \alpha_1$ such that for all $x \in \mathbb{R}^1$, $\alpha_0 \le f^2(x) \le \alpha_1$. Then (1.2) holds with $\underline{\mu}(a;A) = \alpha_0 \cdot \mathrm{Leb}(A)$ and $\overline{\mu}(A) = \alpha_1 \cdot \mathrm{Leb}(A)$, where Leb denotes one–dimensional Lebesgue measure. Applying the proof of (1.4), we obtain the following: for all $p \in (0,1)$ and all $C < (1-p)\alpha_0/(\alpha_1 - p\alpha_0)$, there exists $\lambda_0 > 0$ such that for all $0 < \lambda_0 < \lambda$,

$$\exp\big(-2(p\alpha_0)^{-1}\lambda^2\big) \geq \mathbb{P}\big(M_1 \geq (1+(p\alpha_0)^{-1}\langle M\rangle_1)\lambda\big) \geq C\exp\big(-2(p\alpha_0)^{-2}\alpha_1\lambda^2\big).$$

(1.8) **Example.** Let B be one–dimensional Brownian motion and define $\tau \triangleq \inf\big(s>0 \,:\, |B_s|=1\big)$. Let $M_t \triangleq B_{t\wedge\tau}$; in particular, $\langle M\rangle_t = t\wedge\tau$. Note that (1.0) holds while it is not hard to see that (1.2) does not. On the other hand, (1.1) need not be sharp for M. Indeed,

$$\mathbb{P}\big(|M_\infty| \geq (\alpha+\beta\langle M\rangle_\infty)\lambda\big) = \mathbb{P}\big(\tau \leq \beta^{-1}(\lambda^{-1}-\alpha)\big),$$

which is zero unless $\alpha < \lambda^{-1}$. Suppose next, that $\alpha = 0$ and $\beta = 1$ (say). By the reflection principle, the probability in question becomes,

$$\mathbb{P}\big(|M_\infty| \geq \langle M\rangle_\infty\lambda\big) = \mathbb{P}(\tau \leq \lambda^{-1}) \asymp \exp\big(-\lambda/2\big).$$

Here, $f(\lambda) \asymp g(\lambda)$ means that $\lim_{\lambda\to\infty} \ln f(\lambda)/\ln g(\lambda) = 1$. Thus, the correct decay rate of the probability is different from the Gaussian bounds of (1.1). (Of course, (1.1) still holds but is non-informative when $\alpha = 0$.)

(1.9) **Example.** In this example, we will show how the decay rate of the deviation probabilities in question can be altered in some cases by changing the value of β in (1.1). Let B be a one–dimensional Brownian motion and define σ to be the first $s \in (0,1)$, such that $B_s = 1$. If such an s does not exist, let $\sigma = 2$. Define $M_t = B_{t\wedge\sigma}$ and observe that $\langle M\rangle_t = t\wedge\sigma$. (Note that (1.0) holds in this case.) We will look at two different cases where the behaviors of the deviation probabilities in question are very different from each other and from (1.1).

Case (i) In this case, we consider the parameters: $\beta = 0$, $\alpha = 1$ and consider $\lambda > 1$ large. Then

$$\begin{aligned}\mathbb{P}\big(|M_\infty| \geq \lambda\big) &= \mathbb{P}\big(|B_\sigma| \geq \lambda\big)\\ &= \mathbb{P}\big(\sigma = 2, B_2 \leq -\lambda\big)\\ &\leq \mathbb{P}\big(|B_2| \geq \lambda\big)\\ &\asymp \exp(-\lambda^2/4).\end{aligned}$$

Case (ii) In this case, we consider the choices: $\beta = 1$, $\alpha = 0$ and $\lambda > 1$ large. Then,

$$\begin{aligned}\mathbb{P}\big(|M_\infty| \geq \langle M\rangle_\infty\lambda\big) &= \mathbb{P}\big(\sigma < \lambda^{-1}\big) + \mathbb{P}\big(|B_2| \geq 2\lambda, \sigma = 2\big)\\ &\triangleq \mathrm{I} + \mathrm{II}.\end{aligned}$$

Note that $\mathrm{I} \asymp \exp\big(-\lambda/2\big)$ while $\mathrm{II} \leq \mathbb{P}\big(|B_2| \geq 2\lambda\big) \asymp \exp\big(-2\lambda^2\big)$. Thus,

$$\mathbb{P}\big(|M_\infty| \geq \langle M\rangle_\infty\lambda\big) \asymp \exp\big(-\lambda/2\big),$$

which is a very different rate than that provided by Case (i) or the Gaussian rates which one may expect. Note that (1.2) fails here too.

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§**2. Estimates.** For any $a \in \mathbb{R}^1$, define $\mathcal{E}_t^a \triangleq \exp\big(aM_t - a^2\langle M\rangle_t/2\big)$ be the exponential local martingale and define Cramér measures, $\mathbb{Q}^a A \triangleq \mathbb{P}\big(\mathcal{E}_t^a; A\big)$, for all $A \in \mathcal{F}_t$ and all $t > 0$. In other words, $\mathbb{Q}^a$ are probability measures whose Radon–Nykodym derivative (with respect to $\mathbb{P}$) is given by

$$\frac{d\mathbb{Q}^a}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} = \mathcal{E}_t^a, \qquad \text{a.s. .}$$

Let us begin with some preliminary observations which we shall take for granted thoughout the rest of the paper. By (1.0) and properties of submartingales,

$$\sup_t \mathbb{P}\exp(\zeta|M_t|) = \mathbb{P}\exp(\zeta|M_\infty|) \le 2\mathbb{P}\cosh(\zeta M_\infty) < \infty,$$

for all $\zeta \in \mathbb{R}^1$. This, in turn, implies that $\sup_t \mathbb{P}|M_t|^p < \infty$ for all $t, p > 0$. By Doob's inequality, $\mathbb{P}\sup_t |M_t|^p < \infty$ and by the Burkholder–Davis–Gundy inequality (cf. Revuz and Yor [RY]), $\langle M\rangle_\infty \in L^p(\mathbb{P})$ for all $p > 0$. Also note that $\mathcal{E}_t^a \le \exp(aM_t)$ and as argued above, $\sup_t \mathcal{E}_t^a$ and $\langle \mathcal{E}^a\rangle_\infty$ are both in $L^p(\mathbb{P})$ for all $a \in \mathbb{R}^1$ and $t, p > 0$. In particular, note that $\big(\mathcal{E}_t^a; 0 \le t \le \infty\big)$ is an $L^p(\mathbb{P})$–bounded martingale for all $p > 0$. In the language of Kazamaki [Ka], both M and $\mathcal{E}^a$ are in H^p for all $p > 0$. We shall see later that (1.2) implies that $M \in H^\infty$; see Proposition (2.5) below.

Throughout this section, $\{t_{j,n}; 1 \le j \le m_n\}$ denotes a finite partition of $[0,t]$ whose mesh size goes to 0 as $n \to \infty$. More precisely, $0 = t_{1,n} < t_{2,n} < \cdots < t_{m_{n-1},n} < t_{m_n,n} = t$ with

$$\lim_{n\to\infty} \max_{1\le j\le m_n-1} |t_{j+1,n} - t_{j,n}| = 0.$$

(2.1) **Lemma.** *If (1.0) and (1.2) are in force, for all* $0 \le s, t \le \infty$,

$$\underline{\mu}\big(M_t; [t, t+s]\big) \le \mathbb{P}\big(\langle M\rangle_{t+s} - \langle M\rangle_t \mid \mathcal{F}_t\big) \le \overline{\mu}\big([t, t+s]\big).$$

Proof. By polarization,

$$\mathbb{P}\big(\langle M\rangle_{t+s} - \langle M\rangle_t \mid \mathcal{F}_t\big) = \mathbb{P}\big((M_{t+s} - M_t)^2 \mid \mathcal{F}_t\big).$$

The lemma follows from (1.2). ◇

(2.2) **Lemma.** *If (1.0) and (1.2) are in force, for all* $0 \le t \le \infty$ *and* $a \in \mathbb{R}^1$, $\underline{\mu}\big(0; [0,t]\big) \le \mathbb{Q}^a\langle M\rangle_t \le \overline{\mu}\big([0,t]\big)$.

Proof. We begin with the proof of the lower bound. Integration by parts for stochastic integrals implies that

$$\begin{aligned}
\mathbb{Q}^a\langle M\rangle_t &= \mathbb{P}\big(\mathcal{E}_t^a\langle M\rangle_t\big) \\
&= \mathbb{P}\int_0^t \mathcal{E}_s^a d\langle M\rangle_s \\
&= \lim_{n\to\infty} \mathbb{P}\sum_{j=1}^{m_n-1} \mathcal{E}_{t_{j,n}}^a\big(\langle M\rangle_{t_{j+1,n}} - \langle M\rangle_{t_{j,n}}\big) \\
&= \lim_{n\to\infty} \mathbb{P}\sum_{j=1}^{m_n-1} \mathcal{E}_{t_{j,n}}^a\mathbb{P}\big(\langle M\rangle_{t_{j+1,n}} - \langle M\rangle_{t_{j,n}} \mid \mathcal{F}_{t_{j,n}}\big) \\
&\ge \lim_{n\to\infty} \mathbb{P}\sum_{j=1}^{m_n-1} \mathcal{E}_{t_{j,n}}^a\underline{\mu}\big(D_{t_{j,n}}; [t_{j,n}, t_{j+1,n}]\big), \qquad (2.3)
\end{aligned}$$

by (2.1). Now fix $1 \leq j \leq m_n - 1$ and define $N_s \triangleq \underline{\mu}\big(D_s; [t_{j,n}, t_{j+1,n}]\big)$, $s \geq 0$. Due to the convexity assumption on $a \mapsto \underline{\mu}(a; \cdot)$, $\big(N_s;\ s \geq 0\big)$ is a local semi–martingale. Suppose first that D, $\underline{\mu}'(\cdot; [t_{j,n}, t_{j+1,n}])$ and $\underline{\mu}''(\cdot; [t_{j,n}, t_{j+1,n}])$ are all bounded, where $\underline{\mu}'$ and $\underline{\mu}''$ are the derivatives in the first variable. We then obtain the following upon integration by parts:

$$\mathbb{P}\big(\mathcal{E}_t^a N_t\big) = \underline{\mu}\big(0; [t_{j,n}, t_{j+1,n}]\big) + \mathbb{P}\int_0^t \mathcal{E}_s^a dN_s. \tag{2.4}$$

By the Itô–Wang formula and the assumed convexity,

$$\mathbb{P}\int_0^t \mathcal{E}_s^a dN_s = \mathbf{12}\mathbb{P}\int_{\mathbf{0}}^{\mathbf{t}} \underline{\mu}''\big(\mathbf{D_s}; [\mathbf{t_{j,n}}, \mathbf{t_{j+1,n}}]\big)\mathbf{d}\langle \mathbf{D}\rangle_{\mathbf{s}} \geq \mathbf{0}.$$

By (2.4), $\mathbb{P}\big(\mathcal{E}_t^a N_t\big) \geq \underline{\mu}\big(0, [t_{j,n}, t_{j+1,n}]\big)$. Plugging this inequality in (2.3), we obtain

$$\begin{aligned}\mathbb{Q}^a \langle M\rangle_t &\geq \lim_{n\to\infty} \sum_{j=1}^{m_n-1} \underline{\mu}\big(0; [t_{j,n}, t_{j+1,n}]\big)\\ &= \underline{\mu}\big(0; [0,t]\big).\end{aligned}$$

In general, D or the derivatives of $\underline{\mu}$ may not be bounded. The above can then be proved by localization. This proves the lower bound. To prove the upper bound, we again integrate by parts to see that

$$\begin{aligned}\mathbb{Q}^a \langle M\rangle_t &= \mathbb{P}\int_0^t \mathcal{E}_s^a d\langle M\rangle_s\\ &= \lim_{n\to\infty} \mathbb{P} \sum_{j=1}^{m_n-1} \mathcal{E}_{t_{j,n}}^a \big(\langle M\rangle_{t_{j+1,n}} - \langle M\rangle_{t_{j,n}}\big)\\ &= \lim_{n\to\infty} \mathbb{P} \sum_{j=1}^{m_n-1} \mathcal{E}_{t_{j,n}}^a \mathbb{P}\big(\langle M\rangle_{t_{j+1,n}} - \langle M\rangle_{t_{j,n}} \mid \mathcal{F}_{t_{j,n}}\big)\\ &\leq \lim_{n\to\infty} \mathbb{P} \sum_{j=1}^{m_n-1} \mathcal{E}_{t_{j,n}}^a \overline{\mu}\big([t_{j,n}, t_{j+1,n}]\big)\\ &= \overline{\mu}\big([0,t]\big).\end{aligned}$$

This proves the upper bound and hence the lemma. ◊

Somewhat surprisingly, (1.2) implies the boundedness of the quadratic variation process $\big(\langle M\rangle_t\big)$ as the following result shows.

(2.5) **Proposition.** *If (1.0) and (1.2) hold, for all $a \in \mathbb{R}^1$, $M \in H^\infty(\mathbb{Q}^a)$. More precisely, for all $a \in \mathbb{R}^1$,*

$$\mathbb{Q}^a\Big(\langle M\rangle_t \leq \overline{\mu}([0,t]),\ \textit{for all } 0 \leq t \leq \infty\Big) = 1.$$

Proof. Recall that $t \mapsto \langle M\rangle_t$ is increasing and continuous. Moreover, $t \mapsto \overline{\mu}([0,t])$ is increasing and right continuous. Therefore, it suffices to show that for each $t > 0$ and $a \in \mathbb{R}^1$, $\mathbb{Q}^a\big(\langle M\rangle_t \leq$

$\overline{\mu}([0,t])\big) = 1$. Integrating by parts,

$$\begin{aligned}
\mathbb{Q}^a \langle M\rangle_t^k &= \mathbb{P}\big(\mathcal{E}_t^a \langle M\rangle_t^k\big) \\
&= k\mathbb{P}\int_0^t \mathcal{E}_s^a \langle M\rangle_s^{k-1} d\langle M\rangle_s \\
&= k \lim_{n\to\infty} \mathbb{P} \sum_{j=1}^{m_n-1} \mathcal{E}_{t_{j,n}}^a \langle M\rangle_{t_{j,n}}^{k-1} \big(\langle M\rangle_{t_{j+1,n}} - \langle M\rangle_{t_{j,n}}\big) \\
&= k \lim_{n\to\infty} \mathbb{P} \sum_{j=1}^{m_n-1} \mathcal{E}_{t_{j,n}}^a \langle M\rangle_{t_{j,n}}^{k-1} \mathbb{P}\big(\langle M\rangle_{t_{j+1,n}} - \langle M\rangle_{t_{j,n}} \mid \mathcal{F}_{t_{j,n}}\big) \\
&\le k \lim_{n\to\infty} \mathbb{P} \sum_{j=1}^{m_n-1} \mathcal{E}_{t_{j,n}}^a \langle M\rangle_{t_{j,n}}^{k-1} \overline{\mu}\big([t_{j,n}, t_{j+1,n}]\big) \\
&= k \lim_{n\to\infty} \sum_{j=1}^{m_n-1} \mathbb{Q}^a \langle M\rangle_{t_{j,n}}^{k-1} \overline{\mu}\big([t_{j,n}, t_{j+1,n}]\big) \\
&= k \int_0^t \mathbb{Q}^a \langle M\rangle_s^{k-1} \overline{\mu}(ds).
\end{aligned}$$

By Lemma (2.2) and induction, we see that for all $k \ge 1$,

$$\mathbb{Q}^a \langle M\rangle_t^k \le k \int_0^t \big(\overline{\mu}([0,s])\big)^{k-1} \overline{\mu}(ds) = \big(\overline{\mu}([0,t])\big)^k.$$

Therefore, for all $\varepsilon > 0$,

$$\mathbb{Q}^a \big(\langle M\rangle_t \ge (1+\varepsilon)\overline{\mu}([0,t])\big) \le (1+\varepsilon)^{-k}.$$

Letting $k \to \infty$, the result follows. ◊

Next, we prove an elementary probability bound for random variables.

(2.6) **Lemma.** *Let X be a positive random variable on a probability space $\big(\Upsilon, \mathcal{G}, \mathbb{Q}\big)$. Suppose there exist $0 < L \le R < \infty$ such that $\mathbb{Q}X \ge L$ while $\mathbb{Q}(X \le R) = 1$. Then for any $p \in (0,1)$,*

$$\mathbb{Q}\big(Lp \le X \le R\big) \ge \frac{L(1-p)}{R - Lp}.$$

Proof. Note that

$$\begin{aligned}
\mathbb{Q}X &= \mathbb{Q}\big(X; Lp \le X \le R\big) + \mathbb{Q}\big(X; X \le Lp\big) \\
&\le R\mathbb{Q}\big(Lp \le X \le R\big) + Lp\big(1 - \mathbb{Q}(Lp \le X \le R)\big) \\
&= (R - Lp)\mathbb{Q}(Lp \le X \le R) + Lp.
\end{aligned}$$

Since $\mathbb{Q}X \ge L$, solve for $\mathbb{Q}\big(Lp \le X \le R\big)$ to finish. ◊

Recalling (1.3), define $X \triangleq \langle M\rangle_\infty$, $L \triangleq \underline{\mu}$ and $R \triangleq \overline{\mu}$. The following is then a direct consequence of Lemmas (2.2) and (2.6) and Proposition (2.5):

(2.7) **Lemma.** *If (1.0) and (1.2) are in force, for all $a \in \mathbb{R}^1$ and all $p \in (0,1)$,*

$$\mathbb{Q}^a \Big(p\underline{\mu} \le \langle M \rangle_\infty \le \overline{\mu} \Big) \ge \frac{(1-p)\underline{\mu}}{\overline{\mu} - p\underline{\mu}}.$$

§**3. The Proof of Theorem (1.4).** The upper bound is a consequence of Theorem (1.1) upon letting $\alpha \triangleq 1$ and $\beta \triangleq (p\underline{\mu})^{-1}$. We proceed with the proof of the lower bound. Note that for any $\beta, \lambda > 0$ and $\rho > 1$,

$$\begin{aligned}\mathbb{P}\big(M_\infty \ge (1+\beta\langle M\rangle_\infty)\lambda\big) &\ge \mathbb{P}\big(M_\infty - \beta\lambda\langle M\rangle_\infty \in [\lambda, \rho\lambda]\big) \\ &\ge \exp\big(-2\beta\rho\lambda^2\big)\mathbb{P}\Big(\mathcal{E}_\infty^{2\beta\lambda}; \widetilde{M}_\infty - \mathbf{12}\langle\widetilde{\mathbf{M}}\rangle_\infty \in [\mathbf{2}\beta\lambda^{\mathbf{2}}, \mathbf{2}\beta\rho\lambda^{\mathbf{2}}]\Big),\end{aligned}$$

where $\widetilde{M}_t \triangleq 2\beta\lambda M_t$ is also a martingale. Let $N_t \triangleq \widetilde{M}_t - \langle\widetilde{M}\rangle_t$. By Girsonov, N is a $\mathbb{Q}^{2\beta\lambda}$–martingale with $\langle N\rangle_t = \langle\widetilde{M}\rangle_t = 4\beta^2\lambda^2\langle M\rangle_t$. Hence,

$$\mathbb{P}\Big(M_\infty \ge (1+\beta\langle M\rangle_\infty)\lambda\Big) \ge \exp\big(-2\beta\rho\lambda^2\big)\mathbb{Q}^{2\beta\lambda}\Big(N_\infty + \mathbf{12}\langle\mathbf{N}\rangle_\infty \in [\mathbf{2}\beta\lambda^{\mathbf{2}}, \mathbf{2}\beta\rho\lambda^{\mathbf{2}}]\Big).$$

Suppose we could prove the following: for all $\varepsilon > 0$,

$$\lim_{\lambda\to\infty} \mathbb{Q}^{2\beta\lambda}\big(|N_\infty| \ge \lambda^2\varepsilon\big) = 0. \tag{3.1}$$

Then by Slutsky's theorem,

$$\begin{aligned}\liminf_{\lambda\to\infty} \exp(2\beta\rho\lambda^2)\mathbb{P}\Big(M_\infty \ge (1+\beta\langle M\rangle_\infty)\lambda\Big) &\ge \liminf_{\lambda\to\infty} \mathbb{Q}^{2\beta\lambda}\big(\mathbf{12}\lambda^{-\mathbf{2}}\langle\mathbf{N}\rangle_\infty \in [\mathbf{2}\beta, \mathbf{2}\beta\rho]\big) \\ &= \liminf_{\lambda\to\infty} \mathbb{Q}^{2\beta\lambda}\Big(2\beta^2\langle M\rangle_\infty \in [2\beta, 2\beta\rho]\Big) \\ &\ge \inf_a \mathbb{Q}^a\Big(\langle M\rangle_\infty \in [\beta^{-1}, \rho\beta^{-1}]\Big).\end{aligned}$$

The above holds for any $\rho > 1$ and $\beta > 0$. Consider $\beta \triangleq (p\underline{\mu})^{-1}$ and $\rho \triangleq \overline{\mu}(p\underline{\mu})^{-1}$. Together with Lemma (2.7), this choice of β and ρ proves the theorem provided we establish (3.1). It is this which we shall do next. As remarked earlier, N_t is a centered $\mathbb{Q}^{2\beta\lambda}$–martingale. Hence,

$$\begin{aligned}\mathbb{Q}^{2\beta\lambda} N_\infty^2 &= \mathbb{Q}^{2\beta\lambda}\langle N\rangle_\infty \\ &= 4\beta^2\lambda^2\mathbb{Q}^{2\beta\lambda}\langle M\rangle_\infty \\ &\le 4\beta^2\lambda^2\overline{\mu},\end{aligned}$$

by Proposition (2.5). By Chebychev's inequality,

$$\mathbb{Q}^{2\beta\lambda}\big(|N_\infty| \ge \lambda^2\varepsilon\big) \le \frac{4\beta^2\overline{\mu}}{\varepsilon^2\lambda^2},$$

which goes to 0 as $\lambda \to \infty$. This proves (3.1) and hence the theorem. ♢

§**4. Energy inequalities and the modulus of continuity.** In this section we discuss some of the implications of condition (1.2). Let us begin with the following energy inequality which is more or less contained in Bass [Ba1], Meyer [Me] and Kazamaki [Ka].

(4.1) **Proposition.** *Suppose (1.0) and (1.2) hold (say with $\underline{\mu} \equiv 0$). Then for all even $k \geq 2$, and all $t, s > 0$,*

$$\mathbb{P}\big(M_{t+s} - M_t\big)^k \leq 2^{-k/2} k! \big(\overline{\mu}([t, t+s])\big)^{k/2}.$$

The other inequality in which we are interested is a considerably sharper version of the above energy type inequality:

(4.2) **Proposition.** *Suppose (1.0) and (1.2) hold. Suppose there exist an $\varepsilon > 0$ and an increasing function $\theta : \mathbb{R}^1_+ \mapsto \mathbb{R}^1_+$ such that $\theta(0+) = 0$ and for all intervals $I \subseteq \mathbb{R}^1_+$ with length $|I| < \varepsilon$, $\overline{\mu}(I) \leq \theta(|I|)$. Then for all $t, s > 0$ and all even integers $k \geq 2$,*

$$\mathbb{P}\big(M_{t+s} - M_t\big)^k \leq \frac{k!}{(k/2)!} 2^{-k/2} \big(\theta(s)\big)^{k/2}.$$

It is not hard to see that the constant in (4.2) is the best possible.

As consequences of the above results, we mention (without proofs) two results about the modulus of continuity of M_t. Proofs can be put together using Lévy's method for Brownian motion. See Revuz and Yor [RY].

(4.3) **Corollary.** *Fix some $T > 0$ and define for all $t > 0$,*

$$h_t(\delta) \triangleq \sup\big(s : \overline{\mu}([t, t+s]) \leq \delta\big),$$
$$H_T(\delta) \triangleq \sup\big(s : \sup_{r \leq T} \overline{\mu}([r, r+s]) \leq \delta\big).$$

Assume (1.0) and (1.2) hold and that $\lim_{\delta \to 0^+} H_T(\delta) = 0$. Then with probability one,

$$\limsup_{\delta \to 0^+} \sup_{0 \leq s \leq \delta} \frac{|M_{t+s} - M_t|}{\sqrt{h_t(\delta) \ln\ln\big(1/h_t(\delta)\big)}} \leq 1/\sqrt{2},$$

and

$$\limsup_{\delta \to 0^+} \sup_{\substack{0 \leq s \leq \delta \\ 0 \leq t \leq T}} \frac{|M_{t+s} - M_t|}{\sqrt{H_T(\delta) \ln\big(1/H_T(\delta)\big)}} \leq \sqrt{2}.$$

(4.4) **Corollary.** *Fix some $T > 0$ and assume the domination condition of Proposition (4.2). With probability one,*

$$\limsup_{\delta \to 0^+} \sup_{0 \leq s \leq \delta} \frac{|M_{t+s} - M_t|}{\sqrt{g(\delta) \ln\ln\big(1/g(\delta)\big)}} \leq \sqrt{2},$$

and

$$\limsup_{\delta \to 0^+} \sup_{\substack{0 \leq s \leq \delta \\ 0 \leq t \leq T}} \frac{|M_{t+s} - M_t|}{\sqrt{g(\delta) \ln\big(1/g(\delta)\big)}} \leq 2\sqrt{2},$$

where $g(\delta) \triangleq \sup\big(s : \theta(s) \leq \delta\big)$.

The Proof of Proposition (4.1). Apply Itô's formula to $N_s \triangleq M_{t+s} - M_t$ to see that for all even integers $k \geq 0$, $N_t^k - \frac{k(k-1)}{2} \int_0^s N_r^{k-2} d\langle N \rangle_r$ is a mean zero martingale. Taking expectations,

$$\mathbb{P}\big(M_{t+s} - M_t\big)^k = \frac{k(k-1)}{2} \lim_{n \to \infty} \mathbb{P} \sum_{j=1}^{S_n} \big(M_{t+s_{j,n}} - M_t\big)^{k-2} \Big(\langle M \rangle_{t+s_{j+1,n}} - \langle M \rangle_{t+s_{j,n}}\Big),$$

where $(s_{j,n}; 1 \le j \le S_n)$ is a partition of $[0, s]$ whose mesh size goes to 0 as $n \to \infty$. By (1.2) and conditioning,

$$\mathbb{P}\big(M_{t+s} - M_t\big)^k \le \frac{k(k-1)}{2} \int_0^s \mathbb{P}\big(M_{t+r} - M_t\big)^{k-2} \overline{\mu}_t(dr),$$

where $\overline{\mu}_t(A) \triangleq \overline{\mu}(t + A)$. Let $F_k(r) \triangleq \mathbb{P}\big(M_{t+r} - M_t\big)^k$. We have proven the following:

$$\begin{aligned} F_k(s) &\le \frac{k(k-1)}{2} \int_0^s F_{k-2}(r) \overline{\mu}_t(dr) \\ &\le \frac{k(k-1)}{2} \sup_{0 \le r \le s} F_{k-2}(r) \overline{\mu}([t, t+s]). \end{aligned}$$

Properties of submartingales dictate that $\sup_{r \le s} F_{k-2}(r) = F_{k-2}(s)$. Therefore,

$$F_k(s) \le \frac{k(k-1)}{2} F_{k-2}(s) \overline{\mu}([t, t+s]).$$

The result follows from induction. $\diamondsuit$

The Proof of Proposition (4.2). As in the proof of Proposition (4.1), let $F_k(t) \triangleq \mathbb{P}\big(M_{t+s} - M_t\big)^k$. From the latter proof, it follows that

$$\begin{aligned} F_k(s) &\le \frac{k(k-1)}{2} \int_0^s F_{k-2}(r) \theta(dr) \\ &\le \frac{k(k-1)}{2} \cdot \frac{(k-2)(k-3)}{2} \int_0^s \int_0^r F_{k-4}(u) \theta(du) \theta(dr) \\ &= \frac{k!}{(k-4)!} 2^{-2} \int_0^s \big(\theta(s) - \theta(u)\big) F_{k-4}(u) \theta(du) \\ &\le \frac{k!}{(k-6)!} 2^{-3} \int_0^s \int_0^u \big(\theta(s) - \theta(u)\big) F_{k-6}(r) \theta(dr) \theta(du) \\ &= \frac{k!}{(k-6)!} 2^{-3} \int_0^s \int_r^s \big(\theta(s) - \theta(u)\big) \theta(du) F_{k-6}(r) \theta(dr) \\ &= \frac{k!}{(k-6)!} 2^{-3} \frac{1}{2!} \int_0^s \big(\theta(s) - \theta(r)\big)^2 F_{k-6}(r) \theta(dr). \end{aligned}$$

(The second and the fourth lines follow from induction on k.) By induction, we see that for any integer $p \le k/2$,

$$F_k(s) \le \frac{k!}{(k-2p)!} 2^{-p} \frac{1}{(p-1)!} \int_0^s \big(\theta(s) - \theta(u)\big)^{p-1} F_{k-2p}(u) \theta(du), \tag{4.5}$$

since for all $s > r > 0$ and all positive integers q,

$$\int_r^s \big(\theta(s) - \theta(u)\big)^q \theta(du) = (1+q)^{-1} \big(\theta(s) - \theta(r)\big)^{1+q}. \tag{4.6}$$

Letting $p \triangleq k/2$ in (4.5) and applying (4.6) with $q \triangleq \frac{k}{2} - 1$, we obtain the result. $\diamondsuit$

§**5. Continuous Additive Functionals of Brownian Motion.** Let (Z_t) denote a d dimensional Brownian motion with $d > 2$. To expedite the presentation, we only consider $d \ge 3$. To consider planar Brownian motion, our methods should be applied to the process Z appropriately killed.

Let g be the Green's function for Z given by $g(x,y) \triangleq c(d)|x-y|^{2-d}$, for all $x,y \in \mathbb{R}^d$. The value of $c(d)$ is $(2\pi)^{-d/2}\Gamma(-1+d/2)$ but is of no consequence to us. Let μ be a positive Radon measure on $\mathbb{R}^d$ and suppose

$$\sup_{x\in\mathbb{R}^d} \int |x-y|^{2-d}\mu(dy) < \infty. \tag{5.1}$$

We define the μ–potential,

$$g_\mu(x) \triangleq \int_{\mathbb{R}^d} g(x,y)\mu(dy).$$

This is an excessive function and has a Riesz representation (cf. Bass [Ba1] or Sharpe [Sh]). We shall use the probababilistic form of it which is nowadays known as Brosamler's formula, first discovered in [Br]; see Bass [Ba2] for a different proof. Brosamler's formula states that almost surely for all $t>0$,

$$g_\mu(Z_t) = g_\mu(Z_0) + \int_0^t \nabla g_\mu(Z_s)\cdot dZ_s + L_t^\mu, \tag{5.2}$$

where (L_t^μ) is a continuous additive functional with Revuz measure μ. The latter means that (L_t^μ) is a continuous additive functional which is determined by its potential $\mathbb{P}^x L_\infty^\mu = g_\mu(x)$, for all $x\in\mathbb{R}^d$; cf. Sharpe [Sh]. In this section, we use another consequence of (1.2) (namely Proposition (2.5)), to give a condition which will insure that $\big(L_t^\mu; t\in[0,1], \mu\in\mathfrak{M}\big)$ is jointly continuous, where $\mathfrak{M}$ is an appropriate collection of Revuz measures. Our contribution complements those of Bass and Khoshnevisan [BK] and Marcus and Rosen [MR1,MR2].

In order to state and prove the main result of this section, we need some further notation. Let $\mu_i,\ i=1,2$ be positive Radon measures both satisfying the following with μ replaced by μ_i:

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} |x-y|^{1-d}\mu(dy) < \infty. \tag{5.3}$$

It is not difficult to see that the following is then well–defined:

$$\partial(\mu_1,\mu_2) \triangleq \|\nabla g_{\mu_1} - \nabla g_{\mu_2}\|_\infty. \tag{5.4}$$

We offer the following result:

(5.5) **Theorem.** *Suppose $\mathfrak{M}$ is a collection of positive Radon measures such that for all $\mu\in\mathfrak{M}$, $\|g_\mu\|_\infty < \infty$ and (5.3) holds. Let $H_{\mathfrak{M},\partial}(\varepsilon)$ denote the minimal number of ∂–balls of radius ε required to cover $\mathfrak{M}$. If*

$$\int_{0+} \sqrt{\ln H_{\mathfrak{M},\partial}(\varepsilon)}\,d\varepsilon < \infty,$$

there exists an almost surely jointly continuous modification of $\big(L_t^\mu; t>0, \mu\in\mathfrak{M}\big)$ with respect to the pseudo–distance given by $\partial_1(\mu,\nu) \triangleq \|g_\mu - g_\nu\|_\infty + \partial(\mu,\nu)$.

Our proof also implies the following estimate:

(5.6) **Corollary.** *In the set–up of Theorem (5.5), for any $t>0$, we have some $\delta_0=\delta_0(t)>0$ such that for all $\delta\in(0,\delta_0)$,*

$$\mathbb{P}\sup_{0\le s\le t}\ \sup_{\substack{\mu,\nu\in\mathfrak{M}\\ \partial(\mu,\nu)\le\delta}} |L_s^\mu - L_s^\nu| \le \|g_\mu - g_\nu\|_\infty + \delta_0^{-1}\int_0^\delta \sqrt{\ln H_{\mathfrak{M},\partial}(\varepsilon)}\,d\varepsilon.$$

Some remarks are in order.

(5.7) **Remark.** By the celebrated lemma of Frostman (cf. Kahane [K], for example), (5.3) implies that μ is very smooth. Indeed, the carrying dimension of μ can be no less than $d-1$. When the carrying dimension of μ is smaller than $d-1$, the situation seems to be different. See [BK] and [MR2] for some results.

(5.8) **Remark.** The estimates used in the proof of Theorem (5.5) involve metric entropy; see Dudley [Du]. In doing so, one assumes that the space is more or less homogeneous in the pseudo–metric $\partial(\cdot,\cdot)$. A refinement can be obtained by assuming the existence of a majorizing measure. Indeed, the metric entropy integral condition of Theorem (5.5) can be reduced to assuming the existence of a probability measure m on $\mathfrak{M}$, such that

$$\sup_{\mu\in\mathfrak{M}} \int_{0+} \sqrt{\ln \frac{1}{m\big(B_\partial(\mu,\varepsilon)\big)}}\; d\varepsilon < \infty,$$

where $B_\partial(\mu,\varepsilon)$ is the ∂–ball of radius ε about μ and $\mathfrak{M}$ is topologized by the weak-* topology. See Fernique [Fe] for details.

(5.9) **Remark.** Suppose there exists some $K>0$, such that for any $\mu\in\mathfrak{M}$, supp $\mu\subset[-K,K]^d$. Then (5.3) always implies $\|g_\mu\|_\infty<\infty$. Alternatively, one can consider Brownian motion killed when it leaves $[-K,K]^d$.

(5.10) **Remark.** If (Z_t) is a symmetric transient Markov process with Green's function g, we suspect that under a suitable re-interpretation of (5.3), the analogue of (5.4) still holds with $|\nabla g_{\mu_1}-\nabla g_{\mu_2}|^2$ replaced by $\Gamma(g_{\mu_1}-g_{\mu_2}, g_{\mu_1}-g_{\mu_2})$, where Γ is the trace of the opérateur carré du champ defined by $\Gamma(f,g)=A(fg)-fA(g)-gA(f)$, where A is the generator of Z and $f,g\in\mathfrak{D}(A)$. However, the correct statement and hence the proof eludes us.

(5.11) **Remark.** Suppose $\mathfrak{M}$ is a collection of measures all of whom are absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$ and satisfy (5.3). Abusing notation somewhat, we write for $\mu\in\mathfrak{M}$, $\mu(dx)=\mu(x)dx$. Then for $\mu,\nu\in\mathfrak{M}$,

$$\begin{aligned}\partial(\mu,\nu) &= \sup_{a\in\mathbb{R}^d} \big|\nabla\big(g_\mu-g_\nu\big)\big|(a)\\ &= c(d)\sup_{a\in\mathbb{R}^d}\Big|\nabla\int |a-y|^{2-d}\big(\mu(y)-\nu(y)\big)dy\Big|\\ &\le (d-2)c(d)\sup_{a\in\mathbb{R}^d}\int |a-y|^{1-d}\;|\mu(y)-\nu(y)|dy\\ &\triangleq \widetilde{\partial}(\mu,\nu).\end{aligned}$$

Hence, the statement of Theorem (5.5) remains true if we replace ∂_1 by $\partial_2(\mu,\nu)\triangleq\|g_\mu-g_\nu\|_\infty+\widetilde{\partial}(\mu,\nu)$, everywhere. The point is that while it is somewhat weaker, ∂_2 is a more manageable norm than ∂_1.

Proof of Theorem (5.5). For any $\mu\in\mathfrak{M}$ define the martingale,

$$N_t^\mu \triangleq \int_0^t \nabla g_\mu(Z_s)\cdot dZ_s.$$

Note that $\langle N^\mu\rangle_t\le t\|\nabla g_\mu\|_\infty$, which is bounded on compact t–sets. By Freedman [Fr], (1.0) holds for N_t^μ. Next, let $\mu,\nu\in\mathfrak{M}$, fix $T>0$ and define $M_t\triangleq N^\mu_{t\wedge T}-N^\nu_{t\wedge T}$. Then, (1.0) holds for M and

$$\mathbb{P}\Big((M_{t+s}-M_t)^2\;\Big|\;\mathfrak{F}_t\Big)=\mathbb{P}\Big(\int_{t\wedge T}^{(t+s)\wedge T}|\nabla g_\mu(Z_r)-\nabla g_\nu(Z_r)|^2dr\;\Big|\;\mathfrak{F}_t\Big)$$

$$\leq \mathbb{P}^{Z_t} \int_0^{s\wedge T} |\nabla g_\mu(Z_r) - \nabla g_\nu(Z_r)|^2 dr$$
$$\leq s \cdot \sup_{a\in\mathbb{R}^d} \big|\nabla g_\mu(a) - \nabla g_\nu(a)\big|^2$$
$$\triangleq \overline{\mu}\big([t,t+s]\big),$$

where $\overline{\mu}(A)$ is the Lebesgue measure of A times $\sup_a |\nabla g_\mu - \nabla g_\nu|^2(a)$. In other words, (1.2) holds with $\underline{\mu} \equiv 0$ and $\overline{\mu}$ as given above. By Proposition (2.5), almost surely, $\langle M\rangle_t \leq \overline{\mu}([0,t]) = t\partial^2(\mu,\nu)$ for all $t > 0$. Applying Proposition (4.2) with $\theta(s) = s\partial^2(\mu, \nu\partial^2(\mu,\nu)$, Theorem (1.1) can be used to show that for all $\alpha, \beta > 0$ and all $0 < t < T$,

$$\mathbb{P}\Big(|N_t^\mu - N_t^\nu| \geq \big(\alpha + t\beta\partial^2(\mu,\nu)\big)\lambda\Big) \leq 2\exp(-2\alpha\beta\lambda^2).$$

Picking $\alpha \triangleq t^{1/2}\partial(\mu,\nu)/2$ and $\beta \triangleq \big(2t^{1/2}\partial(\mu,\nu)\big)^{-1}$, it follows from the above and the usual maximal extension of Theorem (1.1) that,

$$\mathbb{P}\big(\sup_{0\leq s\leq T} |N_s^\mu - N_s^\nu| \geq t^{1/2}\partial(\mu,\nu)\lambda\big) \leq 2e^{-\lambda^2/2}.$$

To finish, use (5.2) together with the metric entropy method of Dudley [Du]. ♢

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