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Brownian Sheet and Capacity

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Summary. The main goal of this paper is to present an explicit capacity estimate for hitting probabilities of the Brownian sheet. As applications, we determine the escape rates of the Brownian sheet, and also obtain a local intersection equivalence between the Brownian sheet and the additive Brownian motion. Other applications concern quasi–sure properties in Wiener space.

Keywords. Capacity, hitting probability, escape rate, local intersection equivalence, Brownian sheet, additive Brownian motion.

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1. Introduction

Let $\big(B(\mathbf{u}); \mathbf{u} \in \mathbb{R}^N_+\big)$ denote an (N, d) Brownian sheet. That is, a centered continuous Gaussian process which is indexed by N real, positive parameters and takes its values in $\mathbb{R}^d$. Moreover, its covariance structure is given by the following: for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^N_+$ and all $1 \leqslant i, j \leqslant d$,

$$\mathbb{E}\big\{ B_i(\mathbf{u})B_j(\mathbf{v})\big\} = \begin{cases} \prod_{k=1}^N (u_k \wedge v_k), & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}.$$

Note that along lines which are parallel to the axes, B is a d–dimensional Brownian motion with a constant speed. To illustrate, let us fix $a_1, \cdots, a_{N-1} \in \mathbb{R}^1_+$ and for all $v \in \mathbb{R}^1_+$, define $\langle \mathbf{v} \rangle = (a_1, \cdots, a_{N-1}, v)$. Then, $\big([\prod_{j=1}^{N-1} a_j]^{-1/2} B(\langle \mathbf{v}\rangle); v \in \mathbb{R}^1_+\big)$ is a standard d–dimensional Brownian motion. This is best seen by checking the covariance structure. As such, $\big(B(\langle \mathbf{v}\rangle); v \in \mathbb{R}^1_+\big)$ is a Markov process; cf. [1] and [30] for the theory of one–parameter Markov processes. It turns out that Brownian sheet is a **temporally inhomogeneous** Markov process; cf. Lemma 3.1 below for a precise statement. Therefore, the methods of [6] or [10] do not readily apply. One of the goals of this paper is to provide an elementary proof of the following result:

Theorem 1.1. *Suppose $M > 0$ and $0 < a_k < b_k < \infty$ $(k = 1, \cdots, N)$ are fixed. Then there exists a finite positive constant K_0 which only depends on the parameters M, N, d, $\min_{1 \leqslant j \leqslant N} a_j$ and $\max_{1 \leqslant j \leqslant N} b_j$, such that for all compact sets $E \subset \big\{x \in \mathbb{R}^d : |x| \leqslant M\big\}$,*

$$K_0^{-1} \mathrm{Cap}_{d-2N}(E) \leqslant \mathbb{P}\Big(B([\mathbf{a}, \mathbf{b}]) \cap E \neq \varnothing \Big) \leqslant K_0 \mathrm{Cap}_{d-2N}(E),$$

where $[\mathbf{a}, \mathbf{b}] \triangleq \prod_{j=1}^N [a_j, b_j]$.

Remark 1.1.1. Due to compactness and sample function continuity, measurability problems do not arise in the above context. In order to obtain a full capacity theory (i.e., one that estimates hitting probabilities for Borel or even analytic sets), we need to either replace $\mathbb{P}$ by its Carathéodory outer measure extension $\mathbb{P}^\star$, or to appropriately enrich many of the filtrations in the proof of Theorem 2.1 below.

Remark 1.1.2. A more or less immediate consequence of Theorem 1.1 is that A is **polar** for the (N, d) Brownian sheet if and only if $\mathrm{Cap}_{d-2N}(A) = 0$. This completes the results of [15] and [17, Theorem 6.1].

It is time to explain the notation. Fix an integer $k \geqslant 1$ and consider a Borel set $A \subset \mathbb{R}^k$. For any $\mu \in \mathcal{P}(A)$ — the collection of all probability measures on A — and for all $\beta > 0$, define the β–**energy** of μ by,

$$\mathcal{E}_\beta(\mu) \triangleq \int\!\!\int |x - y|^{-\beta} \mu(dx)\mu(dy).$$

When $\beta = 0$, define for all $\mu \in \mathcal{P}(A)$,

$$\mathcal{E}_0(\mu) \triangleq \int\!\!\int \ln\Big(\frac{1}{|x-y|} \vee e\Big) \mu(dx)\mu(dy).$$

For all $\beta \geqslant 0$, the β–**capacity** of A can then be defined by,

$$\mathrm{Cap}_\beta(A) \triangleq \frac{1}{\inf_{\mu \in \mathcal{P}(A)} \mathcal{E}_\beta(\mu)}.$$

To keep from having to single out the $\beta < 0$ case, we define $\mathrm{Cap}_{-\beta}(A) = 1$, whenever $\beta > 0$. The notation of Theorem 1.1 should now be clear.

Theorem 1.1 belongs to a class of results in the potential theory of multi–parameter processes. The latter is a subject of vigorous current research; cf. [6, 10, 11, 29, 32] for some of the recent activity. An important multi–parameter process which is covered by most if not all of the above references is the **Ornstein–Uhlenbeck sheet** (written as the O–U sheet). One way to think of a d–dimensional, N–parameter O–U sheet $\{U(\mathbf{t});\ \mathbf{t} \in \mathbb{R}_+^N\}$ is as follows: given an (N, d) Brownian sheet, define

$$U(\mathbf{t}) \triangleq \exp\Big(-\frac{\sum_{j=1}^N t_j}{2}\Big) B(e^{\mathbf{t}}), \qquad \mathbf{t} \in \mathbb{R}_+^N,$$

where $e^{\mathbf{t}}$ denotes the N–dimensional vector whose i–th coordinate is e^{t_i} $(1 \leqslant i \leqslant N)$. Then, according to [32], for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}_+^N$ such that $a_k < b_k$ $(1 \leqslant k \leqslant N)$, for every $M > 1$ and for all compact sets $E \subset [-M, M]^N$, there exists a constant $K_0' > 1$ such that

$$\frac{1}{K_0'} \mathrm{Cap}_{d-2N}(E) \leqslant \mathbb{P}\big(U([\mathbf{a}, \mathbf{b}]) \cap E \neq \varnothing\big) \leqslant K_0' \mathrm{Cap}_{d-2N}(E). \tag{1.1}$$

(The upper bound on the above hitting probability is essentially contained in a variational form in [28, Theorem 3.2] and [32, Lemma 4.4], while the asserted lower bound can be shown to follow from [32, Theorem 5.2].) References [6, 10, 11, 29] contain extensions of such a result to a larger class of what are now aptly called "multi–parameter Markov processes".

As mentioned above, the proof of Eq. (1.1) (stated in a different form) is given in [32] (some of the ideas for the case $N = 2$ appear also in [33]); see also [28, Theorem 3.2] for a related result. The arguments of [32] are based on two novel ideas: the first is an appropriate use of Cairoli's maximal inequality ([35]); the second idea is to use facts about the potential theory of U to compute the "energy" of certain "continuous additive functionals". These facts rely on the stationarity of the increments of U, and in particular on the observation that the distribution of $\big(U(\mathbf{t}), U(\mathbf{t}+\mathbf{s})\big)$ and $\big(U(\mathbf{0}), U(\mathbf{s})\big)$ are the same for any $\mathbf{s}, \mathbf{t} \in \mathbb{R}_+^N$. (In the argument used to prove [32, Lemma 4.2], this is how Φ is approximated by suitably chosen "potentials" Φ_k.) While the processes U and B are closely related, their analyses markedly differ in this second step. Just as Ref. [32]'s proof of (1.1), our proof of Theorem 1.1 uses Cairoli's maximal inequality as a first step. The main portion of this paper is concerned with overcoming the nonstationarity of the increments of Brownian sheet. Our methods are elementary and quite robust; for example, they can be used to study the polar sets of more general, non–stationary multi–parameter processes. At the heart of our method lies the multi–parameter analogue of a "Markov property" of Brownian sheet which we will now sketch for the case $N = 2$; see [4, 14] for earlier appearances of such ideas in a different context.

Fix any $\mathbf{s} \in \mathbb{R}^2_+$ and consider the process $B \circ \theta_{\mathbf{s}} \triangleq \big(B(\mathbf{s}+\mathbf{t});\ \mathbf{t} \in \mathbb{R}^N_+\big)$, with the understanding that $B \circ \theta_{\mathbf{s}}(\mathbf{t}) = B(\mathbf{s}+\mathbf{t})$. (To borrow from the language of 1–parameter Markov processes, this is one of the two possible "post-$\mathbf{s}$" processes. Recall that $N = 2$ for this discussion). Then, it can be shown that the process $B \circ \theta_{\mathbf{s}}$ has the following decomposition:

$$B \circ \theta_{\mathbf{s}}(\mathbf{t}) = B(\mathbf{s}) + \sqrt{s_1}\beta_1(t_2) + \sqrt{s_2}\beta_2(t_1) + \widetilde{B}(\mathbf{t}), \qquad \mathbf{t} \in \mathbb{R}^2_+,$$

where, β_1 and β_2 are d–dimensional (1–parameter) Brownian motions, $\widetilde{B}$ is a 2–parameter Brownian sheet, and β_1, β_2, $\widetilde{B}$ and $\big\{B(\mathbf{r});\ \mathbf{0} \leqslant \mathbf{r} \leqslant \mathbf{s}\big\}$ are mutually independent. When $\mathbf{t}$ is coordinatewise small, $\beta_1(t_2)$ is of rough order $t_2^{1/2}$, $\beta_2(t_1)$ is of rough order $t_1^{1/2}$ and $\widetilde{B}(\mathbf{t})$ is of rough order $(t_1 t_2)^{1/2}$. By the asserted independence, for any fixed $\mathbf{s} \in \mathbb{R}^N_+$,

$$B \circ \theta_{\mathbf{s}}(\mathbf{t}) \simeq B(\mathbf{s}) + \sqrt{s_1}\beta_1(t_2) + \sqrt{s_2}\beta_2(t_1), \tag{1.2}$$

when $\mathbf{t} \simeq \mathbf{0}$. It is part of the folklore of Markov processes that potential theory is typically based on local properties. With this in mind, it should not be surprising that what is relevant is the behavior of the process $\mathbf{t} \mapsto B \circ \theta_{\mathbf{s}}(\mathbf{t})$ when $\mathbf{t}$ is close to $\mathbf{0}$. Recalling once more that $\mathbf{s}$ is fixed, we can "conclude" from (1.2) that despite the fact that B is non–stationary, it is "approximately locally stationary" in the following sense:

$$B \circ \theta_{(1,1)}(\mathbf{t}) \simeq B(1,1) + \beta_1(t_2) + \beta_2(t_1).$$

That is, in the notation of Section 6 below, 2–parameter, d–dimensional Brownian sheet locally resembles a 2–parameter, d–dimensional additive Brownian motion. As the latter is much easier to analyze, this relationship is a distinct simplification. While the preceeding discussion is a mere heuristic, it is the guiding light behind the estimates of Section 3. In fact, in the above notation, the process $Z_{\mathbf{s},\mathbf{t}}$ of Section 3 is none other than $B \circ \theta_{\mathbf{s}}(t) - B(\mathbf{s})$. Lemmas 3.6 and 3.7 below implicitly show that $Z_{\mathbf{s},\mathbf{t}}$ behaves like the (N, d) additive Brownian motion of Section 6. More will be said about this connection in Section 6.

We now explain some of the notation which is to be used in this paper. Throughout, $\log_+(x) \triangleq \ln(x \vee e)$ and for all $x \in \mathbb{R}^d$,

$$\kappa(x) \triangleq \begin{cases} |x|^{-d+2N}, & \text{if } d > 2N \\ \log_+(1/|x|), & \text{if } d = 2N \\ 1, & \text{if } d < 2N \end{cases}.$$

Thus, for any compact set $E \subset \mathbb{R}^d$ and any $\mu \in \mathcal{P}(E)$,

$$\mathcal{E}_{d-2N}(\mu) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \kappa(x-y)\mu(dx)\mu(dy).$$

We shall always impose the following partial order on $\mathbb{R}^N_+$: $\mathbf{s} \leqslant \mathbf{t}$ if and only if for all $1 \leqslant i \leqslant N$, $s_i \leqslant t_i$. In agreement with the above, all temporal variables will be in bold face while spatial ones are not. Somewhat ambiguously, expressions like $|x|$ refer to the Euclidean (ℓ^2) norm of the (spatial, or temporal when in bold face) vector x in any dimensions. Finally, we need some σ–fields. For all $\mathbf{t} \in \mathbb{R}_+^N$, we let $\mathcal{F}(\mathbf{t})$ denote the σ–field generated by the collection $\{B(\mathbf{r}); \mathbf{0} \leqslant \mathbf{r} \leqslant \mathbf{t}\}$ and $\mathcal{F} \triangleq \vee_{\mathbf{t} \geqslant \mathbf{0}} \mathcal{F}(\mathbf{t})$.

We close this section with concluding remarks on the organization of this paper. Section 2 contains some elementary facts about multi–parameter martingales of interest to us. In Section 3, we prove a few preliminary inequalities for some (conditioned and unconditioned) Gaussian laws. Section 4 contains the proof of Theorem 1.1. In Section 5, we address the question of escape rates, thus completing the earlier work of [24] and parts of the work of the authors in [17]. Section 6 is concerned with the closely related additive Brownian motion. It turns out that additive Brownian motion and Brownian sheet are equi-polar. For a precise quantitative statement see Corollary 6.2. We present some further assorted facts about the Markovian nature of additive Brownian motion. Section 7 contains some applications to analysis on Wiener space. In particular, we extend the results of Ref.'s [7, 18] on the quasi–sure transience of continuous paths in $\mathbb{R}^d$ $(d > 4)$ and those of [20] on the quasi–sure non–polarity of singletons in $\mathbb{R}^d$ $(d \leqslant 3)$.

2. Multi–Parameter Martingales

Throughout this section, for any $p > 0$, $\mathcal{L}_p$ denotes the collection of all random variables $Y : \Omega \mapsto \mathbb{R}^1$ such that Y is $\mathcal{F}$–measurable and $\mathbb{E}\big[|Y|^p\big] < \infty$. The multi–parameter martingales of this section are of the form:

$$\Pi_{\mathbf{t}} Y \triangleq \mathbb{E}\big[Y \bigm| \mathcal{F}(\mathbf{t})\big], \tag{2.1}$$

for $Y \in \mathcal{L}_1$. It is not too difficult to see that $\Pi_{\mathbf{t}}$ is a projection when $Y \in \mathcal{L}_2$. It is also easy to see that $\Pi_{\mathbf{t}} = \Pi^1_{t_1} \cdots \Pi^N_{t_N}$, where

$$\Pi^i_{t_i} Y \triangleq \mathbb{E}\Big[Y \Bigm| \bigvee_{t_j \geqslant 0: j \neq i} \mathcal{F}(\mathbf{t})\Big]. \tag{2.2}$$

Indeed, we have the following:

Lemma 2.1. *For all $Y \in \mathcal{L}_1$ and all $\mathbf{t} \in \mathbb{R}_+^N$,*

$$\Pi_{\mathbf{t}} Y = \Pi^1_{t_1} \cdots \Pi^N_{t_N} Y.$$

Proof. It suffices to show the above for $Y \in \mathcal{L}_2$ of the form:

$$Y = f\Big(\int h_1(\mathbf{s}) \cdot B(d\mathbf{s}), \cdots, \int h_k(\mathbf{s}) \cdot B(d\mathbf{s})\Big),$$

where $f : \mathbb{R}^k \mapsto \mathbb{R}^1_+$ and $h_i : \mathbb{R}^N_+ \mapsto \mathbb{R}^d$ $(1 \leqslant i \leqslant k)$ are Borel measurable and for all $1 \leqslant i \leqslant k$,

$$\int_{\mathbb{R}^N_+} \big|h_i(\mathbf{s})\big|^2 d\mathbf{s} < \infty.$$

As the integrand h is nonrandom, there are no problems with the definition (and existence, for that matter) of the stochastic integals in the definition of Y; they are all Bochner integrals. In analogy with Itô theory, much more can be done; see [35], for instance. By the Stone–Weierstrauss theorem, it suffices to prove the above for Y of the form:

$$Y = \exp\Big(\int h(\mathbf{s}) \cdot B(d\mathbf{s}) \Big), \tag{2.3}$$

where $h : \mathbb{R}_+^N \mapsto \mathbb{R}^d$ is Borel measurable with

$$\int_{\mathbb{R}_+^N} \big|h(\mathbf{s})\big|^2 d\mathbf{s} < \infty.$$

In this case, a direct evaluation yields,

$$\Pi_{\mathbf{t}} Y = \exp\Big(\int_{[\mathbf{0},\mathbf{t}]} h(\mathbf{s}) \cdot B(d\mathbf{s}) + \frac{1}{2} \int_{\mathbb{R}_+^N \setminus [\mathbf{0},\mathbf{t}]} |h(\mathbf{s})|^2 \, d\mathbf{s} \Big). \tag{2.4}$$

On the other hand, for our Y (i.e., given by (2.3)),

$$\Pi_{t_1}^1 Y = \exp\Big(\int_{[0,t_1]\times\mathbb{R}_+^{N-1}} h(\mathbf{s}) \cdot B(d\mathbf{s}) + \frac{1}{2} \int_{(t_1,\infty)\times\mathbb{R}_+^{N-1}} \big|h(\mathbf{s})\big|^2 d\mathbf{s} \Big).$$

Similarly,

$$\begin{aligned} \Pi_{t_2}^2 \Pi_{t_1}^1 Y = & \exp\Big(\int_{[0,t_1]\times[0,t_2]\times\mathbb{R}_+^{N-2}} h(\mathbf{s}) \cdot B(d\mathbf{s}) \\ & + \frac{1}{2} \int_{(0,t_1)\times(t_2,\infty)\times\mathbb{R}_+^{N-2}} \big|h(\mathbf{s})\big|^2 d\mathbf{s} + \frac{1}{2} \int_{(t_1,\infty)\times\mathbb{R}_+^{N-1}} \big|h(\mathbf{s})\big|^2 d\mathbf{s} \Big). \end{aligned}$$

In $\mathbb{R}_+^N$,

$$\Big((0,t_1) \times (t_2,\infty) \times \mathbb{R}_+^{N-2} \Big) \bigcup \Big((t_1,\infty) \times \mathbb{R}_+^{N-1} \Big) = \Big([0,t_1] \times [0,t_2] \times \mathbb{R}_+^{N-2} \Big)^c.$$

We obtain the result from induction. $\diamondsuit$

Cairoli's maximal inequality (Lemma 2.2 below) is an immediate consequence of the above. It can be found in various forms in Ref.'s [10, 19, 35]. We provide a proof for the sake of completeness.

Lemma 2.2. *Suppose $p > 1$ and $Y \in \mathcal{L}_p$. Then,*

$$\mathbb{E}\Big[\sup_{\mathbf{t}\in\mathbb{Q}_+^N} \big|\Pi_{\mathbf{t}} Y\big|^p \Big] \leqslant \Big(\frac{p}{p-1} \Big)^{Np} \mathbb{E}\big[|Y|^p\big].$$

Proof. By Lemma 2.1, simultaneuosly over all $\mathbf{t} \in \mathbb{Q}_+^N$,

$$\Pi_{\mathbf{t}} Y = \Pi^1_{t_1}\Big[\Pi^2_{t_2}\cdots\Pi^N_{t_N} Y\Big].$$

By Jensen's inequality, $\sup \Pi^2_{t_2}\cdots\Pi^N_{t_N} Y \in \mathfrak{L}_p$, where the supremum is taken over all positive rationals $t_2, \cdots, t_N$. Therefore, applying Doob's maximal inequality,

$$\begin{aligned}
\mathbb{E}\Big[\sup_{\mathbf{t}\in\mathbb{Q}_+^N} \big|\Pi_{\mathbf{t}} Y\big|^p\Big] &\leqslant \Big(\frac{p}{p-1}\Big)\mathbb{E}\Big[\sup_{t_2,\cdots,t_N\in\mathbb{Q}_+^1} \big|\Pi^2_{t_2}\cdots\Pi^N_{t_n} Y\big|^p\Big] \\
&\leqslant \Big(\frac{p}{p-1}\Big)\mathbb{E}\Big[\sup_{t_2\in\mathbb{Q}_+^1} \Pi^2_{t_2}\Big|\sup_{t_3,\cdots,t_N\in\mathbb{Q}_+^1} \Pi^3_{t_3}\cdots\Pi^N_{t_N} Y\Big|^p\Big] \\
&\leqslant \Big(\frac{p}{p-1}\Big)^2\mathbb{E}\Big[\sup_{t_3,\cdots,t_N\in\mathbb{Q}_+^1} \big|\Pi^3_{t_3}\cdots\Pi^N_{t_N} Y\big|^p\Big].
\end{aligned}$$

Iterating this procedure yields the Lemma. $\diamondsuit$

In fact, one can replace the quantifier "$\sup_{t\in\mathbb{Q}^N}$" by "$\sup_{t\in\mathbb{R}_+^N}$" in the statement of Lemma 2.2. The following is clearly more than sufficient for this purpose.

Proposition 2.3. *Suppose $p > 1$ and $Y \in \mathfrak{L}_p$. Then $\Pi_{\mathbf{t}} Y$ has an almost surely continuous modification.*

Proof. Suppose $Y \in \mathfrak{L}_2$ and is of the form given by (2.3) where $h : \mathbb{R}_+^N \mapsto \mathbb{R}^d$ is a C^∞ function with compact support. In this case,

$$\Pi_{\mathbf{t}} Y = \exp\Big(\int_{[\mathbf{0},\mathbf{t}]} h(\mathbf{s})\cdot B(d\mathbf{s}) + \frac{1}{2}\int_{\mathbb{R}_+^N\setminus[\mathbf{0},\mathbf{t}]} \big|h(\mathbf{s})\big|^2 d\mathbf{s}\Big),$$

which clearly is continuous, since the B with which we work has continuous sample paths. If $Y \in \mathfrak{L}_2$, take $h_n : \mathbb{R}_+^N \mapsto \mathbb{R}^d$ with $\int |h_n(\mathbf{s})|^2 d\mathbf{s} < \infty$ and $Y_n \triangleq \int h_n \cdot dB$ such that $\lim_n Y_n = Y$ in $\mathfrak{L}_2$. That is, $\lim_{n\to\infty}\mathbb{E}\big[(Y_n - Y)^2\big] = 0$. By Lemma 2.2 and Jensen's inequality,

$$\mathbb{E}\Big[\sup_{\mathbf{t}\in\mathbb{Q}_+^N} \big(\Pi_{\mathbf{t}} Y_n - \Pi_{\mathbf{t}} Y\big)^2\Big] \leqslant 4^N \mathbb{E}\big[(Y - Y_n)^2\big],$$

which goes to 0 as $n \to \infty$. We have proven the result in $\mathfrak{L}_2$ and thus in $\mathfrak{L}_p$ when $p \geqslant 2$. When $p \in (1, 2)$, we can take $Y'_n \triangleq Y \wedge n \vee (-n)$ and use Lemma 2.2 again to see that

$$\mathbb{E}\Big[\sup_{\mathbf{t}\in\mathbb{Q}_+^N} \big|\Pi_{\mathbf{t}} Y - \Pi_{\mathbf{t}} Y'_n\big|^p\Big] \leqslant \Big(\frac{p}{p-1}\Big)^{Np}\mathbb{E}\big[|Y|^p; |Y| > n\big],$$

which goes to 0 as $n \to \infty$. This proves the result. $\diamondsuit$

3. Preliminary Estimates

Our first result is a simple fact which can be gleaned from covariance considerations.

Lemma 3.1. *For all* $\mathbf{s}, \mathbf{t} \in \mathbb{R}_+^N$, $Z_{\mathbf{s},\mathbf{t}} \triangleq B(\mathbf{t}+\mathbf{s}) - B(\mathbf{s})$ *is independent of* $\mathcal{F}(\mathbf{s})$ *and is a* d*–dimensional multivariate normal with mean vector zero and covariance matrix* $\sigma^2(\mathbf{s}, \mathbf{t})$ *times the identity, where*

$$\sigma^2(\mathbf{s}, \mathbf{t}) \triangleq \prod_{j=1}^N (s_j + t_j) - \prod_{j=1}^N s_j.$$

We need to estimate $\sigma^2(\mathbf{s}, \mathbf{t})$ in terms of nicer (i.e., more manageable) quantities. First, we need a lemma from calculus.

Lemma 3.2. *For all* $\mathbf{x} \in \mathbb{R}_+^N$,

$$\exp\Big\{ \sum_{j=1}^N \Big(x_j - \frac{x_j^2}{2} \Big) \Big\} - 1 \leqslant \prod_{j=1}^N (1 + x_j) - 1 \leqslant \exp\Big\{ \sum_{j=1}^N x_j \Big\} - 1.$$

Proof. For all $x > 0$,

$$x - \frac{x^2}{2} \leqslant \ln(1+x) \leqslant x.$$

The lemma follows immediately. $\diamondsuit$

Next, we wish to prove that Brownian sheet locally looks like a stationary process. A useful way to make this statement precise is the following:

Lemma 3.3. *Suppose* $\mathbf{s} \in [1,2]^N$ *and* $\mathbf{t} \in [0,2]^N$*. Then*

$$\frac{1}{4} |\mathbf{t}| \leqslant \sigma^2(\mathbf{s}, \mathbf{t}) \leqslant N^{1/2} 2^N e^{2N} |\mathbf{t}|.$$

Proof. Of course,

$$\sigma^2(\mathbf{s}, \mathbf{t}) = \Big(\prod_{j=1}^N s_j \Big) \Big(\prod_{k=1}^N \Big[1 + \frac{t_k}{s_k} \Big] - 1 \Big).$$

Using $s_j \in [1,2]$ for all j,

$$\prod_{k=1}^N \Big(1 + \frac{t_k}{2} \Big) - 1 \leqslant \sigma^2(\mathbf{s}, \mathbf{t}) \leqslant 2^N \Big[\prod_{k=1}^N (1 + t_k) - 1 \Big].$$

By Lemma 3.2,

$$\exp\Big\{ \frac{1}{2} \sum_{j=1}^N t_j \Big(1 - \frac{t_j}{4} \Big) \Big\} - 1 \leqslant \sigma^2(\mathbf{s}, \mathbf{t}) \leqslant 2^N \Big\{ \exp\Big\{ \sum_{j=1}^N t_j \Big\} - 1 \Big\}.$$

Since $t_i \in [0,2]$ for all i,

$$\frac{1}{2}t_i\Big(1-\frac{t_i}{4}\Big) \geqslant \frac{1}{4}t_i.$$

Therefore, over the range in question,

$$\exp\Big\{\frac{1}{4}\sum_{j=1}^N t_j\Big\} - 1 \leqslant \sigma^2(\mathbf{s},\mathbf{t}) \leqslant 2^N\Big\{\exp\big\{\sum_{j=1}^N t_j\big\} - 1\Big\}. \tag{3.1}$$

Observe that $1+x \leqslant e^x \leqslant 1+xe^x$ for all $x>0$. Applying this in (3.1) and using the fact that $t_i \leqslant 2$ for all i, we obtain the following over the range in question:

$$\frac{1}{4}\sum_{j=1}^N t_j \leqslant \sigma^2(\mathbf{s},\mathbf{t}) \leqslant 2^N \sum_{i=1}^N t_i \exp\big(\sum_{j=1}^N t_j\big) \leqslant 2^N e^{2N}\sum_{i=1}^N t_i.$$

To finish, note that by the Cauchy–Schwarz inequality, $|\mathbf{t}| \leqslant \sum_{j=1}^N t_j \leqslant N^{1/2}|\mathbf{t}|$. This completes the proof. $\diamondsuit$

For all $r>0$, define

$$\varphi(r) \triangleq \int_{[0,r]^N} |\mathbf{s}|^{-d/2}\exp\Big(-\frac{1}{|\mathbf{s}|}\Big)d\mathbf{s}. \tag{3.2}$$

Recall the definition of the random field $Z_{\mathbf{s},\mathbf{t}}$ defined in Lemma 3.1. The significance of (3.2) is an estimate for the $L^2(\mathbb{P})$–norm of additive functionals of B to which we will come shortly. First, a few more technical lemmas are in order.

Lemma 3.4. *For any $\varepsilon>0$ there exists a constant $K_1(\varepsilon;N,d)\in(0,1)$ such that for all $r>2\varepsilon$,*

$$K_1(\varepsilon;N,d)\kappa(r^{-1/2}) \leqslant r^{d/2-N}\varphi(r) \leqslant K_1^{-1}(\varepsilon;N,d)\kappa(r^{-1/2}),$$

where $r^{-1/2} \triangleq (r^{-1/2},0,\cdots,0)\in\mathbb{R}^d$.

Proof. Let ω_N denote the area of the N–dimensional unit sphere $\{\mathbf{x}\in\mathbb{R}^N : |\mathbf{x}|=1\}$. By symmetry and a calculation in polar coordinates,

$$\begin{aligned}\varphi(r) &\geqslant \int_{|\mathbf{s}|\leqslant r} \mathbf{s}^{-d/2}\exp\Big(-\frac{1}{|\mathbf{s}|}\Big)d\mathbf{s}\\ &= \omega_N\int_0^r x^{-\frac{d}{2}+N-1}e^{-1/x}dx\\ &\geqslant \omega_N e^{-1/\varepsilon}\int_\varepsilon^r x^{-\frac{d}{2}+N-1}dx.\end{aligned}$$

The lemma's lower bound follows from a few elementary computations. The upper bound is proven along the same lines. $\diamondsuit$

An immediate but important corollary of the above is the following.

Lemma 3.5. *For any $c, M > 0$ there exists a constant $K_2(c, M; N, d) \in (0, 1)$ such that for all $b \in \mathbb{R}^d$ with $|b| \leqslant M$,*

$$K_2(c, M; N, d)\kappa(b) \leqslant |b|^{-d+2N} \varphi\Big(\frac{1}{c|b|^2}\Big) \leqslant K_2^{-1}(c, M; N, d)\kappa(b).$$

In complete analogy to 1–parameter potential theory, we need a lower bound for the occupation measure of B. For technical reasons, it turns out to be simpler to first consider a lower bound for the occupation measure of $Z_{\mathbf{s},\mathbf{t}}$ viewed as an N–parameter process in $\mathbf{t}$.

Lemma 3.6. *Suppose $g : \mathbb{R}^d \mapsto \mathbb{R}^1_+$ is a probability density on $\mathbb{R}^d$ whose support is in $\{x \in \mathbb{R}^d : |x| \leqslant M\}$ for some fixed $M > 0$. Then there exists $K_3(M; N, d) \in (0, \infty)$ such that*

$$\inf_{\mathbf{s} \in [1,3/2]^N \cap \mathbb{Q}^N} \mathbb{E}\Big[\int_{[0,1/2]^N} g(Z_{\mathbf{s},\mathbf{t}}) d\mathbf{t}\Big] \geqslant K_3(M; N, d) \int_{\mathbb{R}^d} \kappa(b) g(b) db.$$

Proof. For each $\mathbf{s} \in \mathbb{R}^N_+$, define the (expected) occupation measure, $\nu_{\mathbf{s}}$, by

$$\langle g, \nu_{\mathbf{s}} \rangle \triangleq \mathbb{E}\Big[\int_{[0,1/2]^N} g(Z_{\mathbf{s},\mathbf{t}}) d\mathbf{t}\Big].$$

The above uniquely defines $\nu_{\mathbf{s}}$ by its action on probability densities g on $\mathbb{R}^d$. By Lemma 3.1 and the exact form of the Gaussian density,

$$\langle g, \nu_{\mathbf{s}} \rangle = (2\pi)^{-d/2} \int_{[0,1/2]^N} \int_{\mathbb{R}^d} g(b) \frac{e^{-|b|^2/2\sigma^2(\mathbf{s},\mathbf{t})}}{\sigma^d(\mathbf{s},\mathbf{t})} db \, d\mathbf{t}.$$

By Lemma 3.3, for all $\mathbf{s} \in [1, 3/2]^N$,

$$\langle g, \nu_{\mathbf{s}} \rangle \geqslant \big(2^{N+1}\pi N^{1/2} e^{2N}\big)^{-d/2} \int_{[0,1/2]^N} \int_{\mathbb{R}^d} g(b) |\mathbf{t}|^{-d/2} e^{-2|b|^2/|\mathbf{t}|} db \, d\mathbf{t}.$$

Taking the infimum over $\mathbf{s} \in [1, 3/2]^N$ and using Fubini's theorem, we obtain:

$$\begin{aligned}\inf_{\mathbf{s} \in [1,3/2]^N \cap \mathbb{Q}^N} \langle g, \nu_{\mathbf{s}} \rangle &\geqslant \big(2^{N+1}\pi N^{1/2} e^{2N}\big)^{-d/2} \int_{\mathbb{R}^d} g(b) db \int_{[0,1/2]^N} |\mathbf{t}|^{-d/2} e^{-2|b|^2/|\mathbf{t}|} d\mathbf{t} \\ &= \big(2^{N+1}\pi N^{1/2} e^{2N}\big)^{-d/2} 2^{N-(d/2)} \int_{\mathbb{R}^d} g(b) |b|^{-d+2N} \varphi\Big(\frac{1}{4|b|^2}\Big) db.\end{aligned}$$

By Lemma 3.5, $|b|^{-d+2N} \varphi\big(1/(4|b|^2)\big) \geqslant K_2(4, M; N, d)\kappa(b)$. Define,

$$K_3(M; N, d) \triangleq \big(2^{N+1}\pi N^{1/2} e^{2N}\big)^{-d/2} 2^{N-(d/2)} K_2(4, M; N, d).$$

This proves the result. $\diamondsuit$

We can now state and prove the main result of this section:

Proposition 3.7. *Suppose $f : \mathbb{R}^d \mapsto \mathbb{R}^1_+$ is a probability density on $\mathbb{R}^d$ whose support is in $\{x \in \mathbb{R}^d : |x| \leqslant M/2\}$ for some $M \in (0, \infty)$. With probability one, for all $\mathbf{s} \in [1, 3/2]^N \cap \mathbb{Q}^N$,*

$$\begin{aligned}\mathbb{E}\Big[\int_{[1,2]^N} f\big(B(\mathbf{u})\big) d\mathbf{u} \;\Big|\; \mathcal{F}(\mathbf{s}) \Big] \\ \geqslant K_3(M; N, d) \int_{\mathbb{R}^d} \kappa(b) f\big(b + B(\mathbf{s})\big) db \; \mathbb{1}\big\{|B(\mathbf{s})| \leqslant M/2\big\}.\end{aligned} \tag{3.3}$$

Proof. If $|B(\mathbf{s})| > M/2$, there is nothing to prove. Therefore, we can and will assume that $|B(\mathbf{s})| \leqslant M/2$. (More precisely, we only work with the realizations ω, such that $|B(\mathbf{s})|(\omega) \leqslant M/2$.)

Let us fix some $\mathbf{s} \in [1, 3/2]^N$. By Lemma 3.1, $Z_{\mathbf{s},\mathbf{t}}$ is independent of $\mathcal{F}(\mathbf{s})$. Together with Lemma 3.6 this implies that for all probability densities g on $\{x \in \mathbb{R}^d : |x| \leqslant M\}$,

$$\mathbb{E}\Big[\int_{[0,1/2]^N} g(Z_{\mathbf{s},\mathbf{t}}) d\mathbf{t} \;\Big|\; \mathcal{F}(\mathbf{s}) \Big] \geqslant K_3(M; N, d) \int_{\mathbb{R}^d} \kappa(b) g(b) db, \qquad \text{a.s.} \tag{3.4}$$

Note that the above holds even when g is random, as long as it is $\mathcal{F}(\mathbf{s})$–measurable. Since f is non–negative, a few lines of algebra show that for any $\mathbf{s} \in [1, 3/2]^N$,

$$\begin{aligned}\int_{[1,2]^N} f\big(B(\mathbf{u})\big) d\mathbf{u} &\geqslant \int_{\substack{[1,2]^N \\ \mathbf{t} \geqslant \mathbf{s}}} f\big(B(\mathbf{t})\big) d\mathbf{t} \\ &\geqslant \int_{[0,1/2]^N} f\big(Z_{\mathbf{s},\mathbf{t}} + B(\mathbf{s})\big) d\mathbf{t}.\end{aligned}$$

Define,

$$g(x) \triangleq f\big(x + B(\mathbf{s})\big), \qquad x \in \mathbb{R}^d.$$

We have,

$$\int_{[1,2]^N} f\big(B(\mathbf{u})\big) d\mathbf{u} \geqslant \int_{[0,1/2]^N} g\big(Z_{\mathbf{s},\mathbf{t}}\big) d\mathbf{t}.$$

Note that g is measurable with respect to $\mathcal{F}(\mathbf{s})$. Moreover, on the set $\big\{|B(\mathbf{s})| \leqslant M/2\big\}$, g is a probability density on $\{x \in \mathbb{R}^d : |x| \leqslant M\}$. Therefore, (3.4) implies that (3.3) holds a.s., for each fixed $\mathbf{s} \in [1, 3/2]^N$. The result follows from this. $\diamondsuit$

Proposition 3.7 is fairly sharp. Indeed, one has the following $L^2(\mathbb{P})$–inequality:

Lemma 3.8. *For all $M > 0$, there exists a $K_4(M; N, d) \in (0, \infty)$ such that for all probability densities f on $\{x \in \mathbb{R}^d : |x| \leqslant M\}$,*

$$\mathbb{E}\Big[\Big(\int_{[1,2]^N} f\big(B(\mathbf{u})\big) d\mathbf{u} \Big)^2 \Big] \leqslant K_4(M; N, d) \mathcal{E}_{d-2N}(f),$$

where $\mathcal{E}_{d-2N}(f)$ denotes the $(d-2N)$–energy of the measure $f(x)dx$.

Proof. This lemma is basically a direct calculation. Since the details are similar to those in the proof of Proposition 3.7, we will merely sketch the essential ideas.

For any $\mathbf{s},\mathbf{t}\in[1,2]^N$, define

$$m(\mathbf{s},\mathbf{t}) \triangleq \prod_{j=1}^{N} \frac{s_j\wedge t_j}{s_j},$$
$$\tau^2(\mathbf{s},\mathbf{t}) \triangleq \Big(\prod_{j=1}^{N} t_j\Big)\Big(1-\prod_{k=1}^{N}\Big(\frac{s_k\wedge t_k}{s_k\vee t_k}\Big)\Big).$$

Using Gaussian regressions shows that for any $x\in\mathbb{R}^d$, the distribution of $B(\mathbf{t})$ conditional on $\big\{B(\mathbf{s})=x\big\}$ is Gaussian with mean vector $m(\mathbf{s},\mathbf{t})x$ and covariance matrix $\tau^2(\mathbf{s},\mathbf{t})$ times the identity. For all $\mathbf{s},\mathbf{t}\in[1,2]^N$,

$$\prod_{j=1}^{N}\frac{s_j\wedge t_j}{s_j} = \exp\Big\{\sum_{j=1}^{N}\ln\Big(1-\frac{(s_j-t_j)^+}{s_j}\Big)\Big\},$$
$$\prod_{k=1}^{N}\frac{s_k\wedge t_k}{s_k\vee t_k} = \exp\Big\{\sum_{k=1}^{N}\ln\Big(1-\frac{|s_k-t_k|}{s_k\vee t_k}\Big)\Big\}. \tag{3.5}$$

Note that

$$\begin{aligned}
&\ln(1-x)\geqslant -x(1+x), \qquad \text{for } 0\leqslant x\leqslant 1/2,\\
&\ln(1-y)\leqslant -y,\\
&1-e^{-y}\leqslant -y, \qquad \text{for } y\geqslant 0,\\
&1-e^{-z}\geqslant c_N\, z, \qquad \text{for } 0\leqslant z\leqslant N\ln 2,
\end{aligned}$$

where $c_N>0$ denotes a positive finite constant whose value depends only on the temporal dimension N. Since $|s_k-t_k|/(s_k\vee t_k)\leqslant 1/2$, applying this in (3.5) much like in the proof of Lemma 3.3, we arrive at the following:

$$1-m(\mathbf{s},\mathbf{t})\leqslant 2N^{1/2}\,|\mathbf{s}-\mathbf{t}|,$$
$$\frac{c_N}{2}|\mathbf{s}-\mathbf{t}|\leqslant \tau^2(\mathbf{s},\mathbf{t})\leqslant 2^N N^{1/2}\,|\mathbf{s}-\mathbf{t}|.$$

Therefore, for all $\mathbf{s},\mathbf{t}\in[1,2]^N$,

$$\begin{aligned}
\frac{\mathbb{P}\big(B(\mathbf{t})\in dy \;\big|\; B(\mathbf{s})=x\big)}{dy} &= \frac{1}{(2\pi)^{d/2}\tau^d(\mathbf{s},\mathbf{t})}\exp\Big(-\frac{|y-m(\mathbf{s},\mathbf{t})x|^2}{2\tau^2(\mathbf{s},\mathbf{t})}\Big)\\
&\leqslant (\pi c_N)^{-d/2}|\mathbf{s}-\mathbf{t}|^{-d/2}e^{4N^{3/2}|x|^2/c_N}\exp\Big(-\frac{|y-x|^2}{2^{N+1}N^{1/2}|\mathbf{s}-\mathbf{t}|}\Big).
\end{aligned}$$

Accordingly,

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_{[1,2]^N} f\big(B(\mathbf{u})\big)d\mathbf{u}\Big)^2\Big] \\
&\quad\leqslant \big(2\pi^2 c_N \prod_{j=1}^N s_j\big)^{-d/2}\int_{[1,2]^N} d\mathbf{s}\int_{[1,2]^N} d\mathbf{t}\int_{\mathbb{R}^d} dx\int_{\mathbb{R}^d} dy\ f(x)f(y)e^{4N^{3/2}|x|^2/c_N} \\
&\qquad\times |\mathbf{s}-\mathbf{t}|^{-d/2}\exp\Big(-\frac{|x|^2}{2\prod_{j=1}^N s_j}\Big)\exp\Big(-\frac{|y-x|^2}{2^{N+1}N^{1/2}|\mathbf{s}-\mathbf{t}|}\Big).
\end{aligned}$$

Write $K_5(M;N,d) \triangleq (2\pi^2 c_N)^{-d/2}e^{4N^{3/2}M^2/c_N}$ for brevity. Since f is supported in $\{x \in \mathbb{R}^d : |x| \leqslant M\}$,

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_{[1,2]^N} f\big(B(\mathbf{u})\big)d\mathbf{u}\Big)^2\Big] \\
&\quad\leqslant K_5(M;N,d)\int_{[1,2]^N} d\mathbf{s}\int_{[1,2]^N} d\mathbf{t}\int_{\mathbb{R}^d} dx\int_{\mathbb{R}^d} dy\ f(x)f(y) \\
&\qquad\times |\mathbf{s}-\mathbf{t}|^{-d/2}\exp\Big(-\frac{|y-x|^2}{2^{N+1}N^{1/2}|\mathbf{s}-\mathbf{t}|}\Big) \\
&\quad\leqslant K_5(M;N,d)\int_{\mathbf{s}\in[0,1]^N}\int_{\mathbb{R}^d} dx\int_{\mathbb{R}^d} dy\ f(x)f(y) \\
&\qquad\times |\mathbf{s}|^{-d/2}\exp\Big(-\frac{|y-x|^2}{2^{N+1}N^{1/2}|\mathbf{s}|}\Big) \\
&\quad= K_5(M;N,d)\,(2^{N+1}N^{1/2})^{d/2-N} \\
&\qquad\times\int_{\mathbb{R}^d} dx\int_{\mathbb{R}^d} dy\ f(x)f(y)|x-y|^{-d+2N}\varphi\Big(\frac{2^{N+1}N^{1/2}}{|x-y|^2}\Big).
\end{aligned}$$

We obtain the desired result from Lemma 3.5. $\diamondsuit$

4. The Proof of Theorem 1.1

In order to keep the exposition notationally simple, we will prove Theorem 1.1 for $[\mathbf{a},\mathbf{b}] = [1,3/2]^N$. The general case follows by similar arguments.

Fix $\varepsilon \in (0,1)$ and define E^ε to be the closed ε–enlargement of E. That is,

$$E^\varepsilon \triangleq \big\{x \in \mathbb{R}^d : \mathrm{dist}(x,E) \leqslant \varepsilon\big\}.$$

Let

$$S_1(\varepsilon) \triangleq \inf\big\{s_1 \in [1,3/2]\cap\mathbb{Q} :\ B(\mathbf{s}) \in E^\varepsilon,\ \text{for some } s_2,\cdots,s_N \in [1,3/2]\cap\mathbb{Q}\big\}.$$

with the usual convention that $\inf\varnothing = \infty$. By path continuity, if $S_1(\varepsilon) < \infty$, there exist $S_2(\varepsilon),\cdots,S_N(\varepsilon) \in [1,3/2]\cap\mathbb{Q}$ such that $B(\mathbf{S}(\varepsilon)) \in E^\varepsilon$. Moreover, since E^ε has a non–void interior and B is Gaussian, $\mathbb{P}\big(\mathbf{S}(\varepsilon) \in [1,3/2]^N \cap \mathbb{Q}^N\big) > 0$. This means that we can (classically) condition on the (measurable) event $\{\mathbf{S}(\varepsilon) \in [1,3/2]^N \cap \mathbb{Q}^N\}$. For all Borel sets $A \subset \mathbb{R}^d$, define

$$\mu_\varepsilon(A) \triangleq \mathbb{P}\big(B(\mathbf{S}(\varepsilon)) \in A \;\big|\; \mathbf{S}(\varepsilon) \in [1,3/2]^N \cap \mathbb{Q}^N\big).$$

The previous discussion shows that $\mu_\varepsilon \in \mathcal{P}(E^\varepsilon)$. Let $\mathcal{B}_d(x,r)$ denote the closed ball of radius $r > 0$ about $x \in \mathbb{R}^d$. Define V_d to be the volume of $\mathcal{B}_d(0,1)$.

With the definition of μ_ε in mind, define for all $x \in \mathbb{R}^d$,

$$f_\varepsilon(x) \triangleq \frac{\mu_\varepsilon\big(\mathcal{B}_d(x,\varepsilon)\big)}{V_d\, \varepsilon^d}.$$

It is easy to see that μ_ε is atomless. Therefore, f_ε is a probability density on $\mathbb{R}^d$. This is a consequence of the fact that the volume functional on $\mathbb{R}^d$ is translation invariant. Note that

$$\mathbb{1}\big\{|B(\mathbf{S}(\varepsilon))| \leqslant M+1\big\} \geqslant \mathbb{1}\big\{\mathbf{S}(\varepsilon) \in [1,3/2]^N \cap \mathbb{Q}^N\big\}.$$

For this choice of f_ε, the above observations, together with Proposition 3.7 imply the following:

$$\begin{aligned}&\sup_{\mathbf{s} \in [1,3/2]^N \cap \mathbb{Q}^N} \mathbb{E}\Big(\int_{[1,3/2]^N} f_\varepsilon\big(B(\mathbf{u})\big) d\mathbf{u} \;\Big|\; \mathcal{F}(\mathbf{s})\Big) \\ &\qquad \geqslant K_3(2M+2; N, d) \int_{\mathbb{R}^d} \kappa(b) f_\varepsilon\big(b + B(\mathbf{S}(\varepsilon))\big) db \; \mathbb{1}\big\{\mathbf{S}(\varepsilon) \in [1,3/2]^N \cap \mathbb{Q}^N\big\}.\end{aligned}$$

We wish to square both sides and take expectations. By Lemmas 2.2 and 3.8,

$$\begin{aligned}\mathbb{E}\Big[\Big[\sup_{\mathbf{s} \in [1,3/2]^N \cap \mathbb{Q}^N} \mathbb{E}\Big(\int_{[1,3/2]^N} f_\varepsilon\big(B(\mathbf{u})\big) d\mathbf{u} \;\Big|\; \mathcal{F}(\mathbf{s})\Big)\Big]^2\Big] \\ \leqslant 4^N K_4(M+1; N, d)\, \mathcal{E}_{d-2N}(f_\varepsilon).\end{aligned}$$

Therefore, by the Cauchy–Schwarz inequality,

$$\begin{aligned}&4^N K_4(M+1; N, d)\, \mathcal{E}_{d-2N}(f_\varepsilon) \\ &\quad \geqslant K_3^2(2M+2; N, d) \mathbb{P}\big(\mathbf{S}(\varepsilon) \in [1,3/2]^N \cap \mathbb{Q}^N\big) \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} \kappa(b) f_\varepsilon(a+b) db\Big)^2 \mu_\varepsilon(da) \\ &\quad \geqslant K_3^2(2M+2; N, d) \mathbb{P}\big(\mathbf{S}(\varepsilon) \in [1,3/2]^N \cap \mathbb{Q}^N\big) \Big(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \kappa(b-a) f_\varepsilon(b) db \mu_\varepsilon(da)\Big)^2.\end{aligned}$$

Now, we need to let $\varepsilon \to 0^+$. Since for any $\varepsilon \in (0,1)$, μ_ε is supported in $\mathcal{B}_d(0, M+1)$ and since the latter is compact, $\big(\mu_\varepsilon; 0 < \varepsilon < 1\big)$ is a tight family of probability measures on $\mathbb{R}^d$. By Prohorov's theorem, μ_ε has a subsequential weak limit μ_0. Note that f_ε is the convolution of μ_ε with the step function $\mathbb{1}\{|x| \leqslant \varepsilon\}/V_d\, \varepsilon^d$. Therefore, by going along a further subsequence, we see that $f_\varepsilon \otimes \mu_\varepsilon$ has $\mu_0 \otimes \mu_0$ as a subsequential weak limit. By standard arguments (cf. Theorem 11.11 of [9]), we can let $\varepsilon \to 0^+$ along an appropriate subsequence to see that when $\mathcal{E}_{d-2N}(\mu_0) > 0$,

$$\liminf_{\varepsilon\to 0^+} \mathbb{P}\big(\mathbf{S}(\varepsilon) \in [1,3/2]^N \cap \mathbb{Q}^N\big) \leqslant \frac{4^N K_4(M+1;N,d)}{K_3^2(2M+2;N,d)\mathcal{E}_{d-2N}(\mu_0)} \triangleq \frac{K_0(M;N,d)}{\mathcal{E}_{d-2N}(\mu_0)}.$$

By path continuity and by compactness, the left hand side is exactly $\mathbb{P}\big(B([1,3/2]^N)\cap E \neq \varnothing\big)$. Since $\mu_0 \in \mathcal{P}(E)$, the right hand side is less than $K_0(M;N,d)\mathrm{Cap}_{d-2N}(E)$. Therefore, when $\mathcal{E}_{d-2N}(\mu_0) > 0$, we have the upper bound in Theorem 1.1. When $\mathcal{E}_{d-2N}(\mu_0) = 0$, consider E^ε for $\varepsilon \in (0,1)$. It is easy to see that E^ε has positive $(d-2N)$ capacity. (Indeed, normalized Lebesgue measure will work as an equilibrium measure.) By what we have proven thusfar, for all $\varepsilon \in (0,1)$,

$$\mathbb{P}\big(B([1,3/2]^N)\cap E^\varepsilon \neq \varnothing\big) \leqslant K_0(M+1;N,d)\mathrm{Cap}_{d-2N}(E^\varepsilon).$$

Letting $\varepsilon \to 0^+$, we obtain the general upper bound.

To prove the lower bound, fix $\varepsilon \in (0,1)$ and take probability density f on $\mathbb{R}^d$ whose support is E^ε. Define,

$$\mathbf{I} \triangleq \int_{[1,3/2]^N} f\big(B(\mathbf{u})\big)d\mathbf{u}.$$

We shall only consider the case where $\mathcal{E}_{d-2N}(f) > 0$. The other case is handled by taking limits as in the preceding proof of the upper bound. Of course,

$$\begin{aligned}
\mathbb{E}[\mathbf{I}] &= (2\pi)^{-d/2} \int_{[1,3/2]^N} \int f(a)\Big(\prod_{j=1}^N u_j\Big)^{-d/2} \exp\Big(-\frac{|a|^2}{2\prod_{j=1}^N u_j}\Big) da\; d\mathbf{u} \\
&\geqslant (3\pi)^{-d/2} 2^{-N} \int f(a) e^{-|a|^2/2} da \\
&\geqslant (3\pi)^{-d/2} 2^{-N} \exp\big(-(M+1)^2/2\big) \\
&\triangleq K_6(M,N,d). \qquad (4.1)
\end{aligned}$$

On the other hand, by Lemma 3.8,

$$\mathbb{E}\big[\mathbf{I}^2\big] \leqslant K_4(M+1;N,d)\mathcal{E}_{d-2N}(f). \qquad (4.2)$$

By the Cauchy–Schwarz inequality,

$$\mathbb{E}[\mathbf{I}] = \mathbb{E}[\mathbf{I}; \mathbf{I} > 0] \leqslant \big\{\mathbb{E}\big[\mathbf{I}^2\big]\ \mathbb{P}\big(\mathbf{I} > 0\big)\big\}^{1/2}.$$

Eqs. (4.1) and (4.2) imply the following:

$$\mathbb{P}\big(B([1,3/2]^N)\cap E^\varepsilon \neq \varnothing\big) \geqslant \mathbb{P}\big(\mathbf{I} > 0\big) \geqslant \frac{K_6^2(M;N,d)}{K_4(M+1;N,d)\mathcal{E}_{d-2N}(f)}.$$

By a density argument, we can take the infimum over all $f(x)dx \in \mathcal{P}(E^\varepsilon)$ and let $\varepsilon \to 0^+$ to obtain the capacity of E. $\diamondsuit$

5. Escape Rates

Let $\big(B(\mathbf{u}); \mathbf{u} \in \mathbb{R}_+^N\big)$ be an N–parameter Brownian sheet taking values in $\mathbb{R}^d$. According to [24], B is transient if and only if $d > 2N$. As the process B is zero on the axes, one needs to be careful about the notion of transience here. Following [17], we say that B is **transient**, if for any $R > 1$,

$$\mathbb{P}\Big(\liminf_{\substack{|\mathbf{u}|\to\infty \\ \mathbf{u}\in\mathcal{C}(R)}} |B(\mathbf{u})| = \infty \Big) > 0.$$

Here, $\mathcal{C}(R)$ denotes the R–**cone** defined by

$$\mathcal{C}(R) \triangleq \Big\{ \mathbf{u} \in (1,\infty)^N : \max_{1 \leqslant i,j \leqslant N} \frac{u_i}{u_j} \leqslant R \Big\}.$$

From Kolmogorov's 0–1 law, one can deduce that $\mathbb{P}\big(\text{transience}\big) \in \{0,1\}$. In this section, we address the rate of transience. When $N = 1$ and $d \geqslant 3$, B is d–dimensional Brownian motion ($d \geqslant 3$) and the rate of transience is determined by [5]. The more subtle neighborhood recurrent case, that is when $N = 1$ and $d = 2$, can be found in [34]. When $d > 2N$, in [17] we used rather general Gaussian techniques to determine the rate of transience of B. The end result is the following:

Theorem 5.1. (cf. [17, Theorem 5.1]) *If $\psi : \mathbb{R}_+^1 \mapsto \mathbb{R}_+^1$ is decreasing and $d > 2N$, then for any $R > 1$,*

$$\liminf_{\substack{|\mathbf{u}|\to\infty \\ \mathbf{u}\in\mathcal{C}(R)}} \frac{|B(\mathbf{u})|}{|\mathbf{u}|^{N/2}\psi(|\mathbf{u}|)} = \begin{cases} \infty, & \text{if } \int_1^\infty s^{-1}\psi^{d-2N}(s)ds < \infty \\ 0, & \text{otherwise} \end{cases}, \quad \text{a.s.}$$

The goal of this section is to describe Spitzer's test for the critical case, i.e., when $d = 2N$. Indeed, we offer the following:

Theorem 5.2. *Let B denote an $(N, 2N)$ Brownian sheet. If $\psi : \mathbb{R}_+^1 \mapsto \mathbb{R}_+^1$ is decreasing, then for all $R > 1$,*

$$\liminf_{\substack{|\mathbf{u}|\to\infty \\ \mathbf{u}\in\mathcal{C}(R)}} \frac{|B(\mathbf{u})|}{|\mathbf{u}|^{N/2}\psi(|\mathbf{u}|)} = \begin{cases} \infty, & \text{if } \int_1^\infty \frac{ds}{s|\ln\psi(s)|} < \infty \\ 0, & \text{otherwise} \end{cases}, \quad \text{a.s.}$$

Proof. Without loss of much generality, we can assume that $\lim_{s\to\infty}\psi(s) = 0$. Define,

$$\mathcal{C}_0(R) \triangleq \big\{ \mathbf{x} \in \mathcal{C}(R) : \ 1 \leqslant |\mathbf{x}| \leqslant R \big\}.$$

Then for

$$\mathcal{C}_n(R) \triangleq R^n \mathcal{C}_0(R),$$

consider the (measurable) event,

$$E_n(R) \triangleq \big\{\omega : \inf_{\mathbf{u}\in\mathcal{C}_n(R)} |B(\mathbf{u})| \leqslant R^{nN/2}\psi(R^n)\big\}.$$

By the scaling property of B,

$$\mathbb{P}\big(E_n(R)\big) = \mathbb{P}\big(\inf_{\mathbf{u}\in\mathcal{C}_0(R)} |B(\mathbf{u})| \leqslant \psi(R^n)\big).$$

Theorem 1.1 shows that

$$K_0^{-1}\mathrm{Cap}_0\big(\mathcal{B}_{2N}(0,\psi(R^n))\big) \leqslant \mathbb{P}\big(E_n(R)\big) \leqslant K_0\mathrm{Cap}_0\big(\mathcal{B}_{2N}(0,\psi(R^n))\big),$$

where

$$\mathcal{B}_{2N}(0,r) \triangleq \big\{\mathbf{x}\in\mathbb{R}^{2N} : |\mathbf{x}| \leqslant r\big\}.$$

The above 0–capacity is of order $1/|\ln\psi(R^n)|$; cf. [27, Proposition 3.4.11]. For a probabilistic alternative, in the proof of Theorem 1.1, replace μ by the Lebesgue measure everywhere and directly calculate. The upshot is the existence of some $K_7(R;N,d) \in (0,\infty)$, such that for all $n \geqslant 1$,

$$\frac{1}{K_7(R;N,d)\big|\ln\psi(R^n)\big|} \leqslant \mathbb{P}\big(E_n(R)\big) \leqslant \frac{K_7(R;N,d)}{\big|\ln\psi(R^n)\big|}.$$

In particular,

$$\sum_n \mathbb{P}\big(E_n(R)\big) < \infty \ \Leftrightarrow \int_1^\infty \frac{ds}{s\big|\ln\psi(s)\big|} < \infty.$$

With this estimate established, the rest of the proof follows that of [17, Theorem 5.1] nearly exactly. $\diamondsuit$

6. Additive Brownian Motion

Suppose — on a possibly different probability space — we have N independent standard $\mathbb{R}^d$–valued Brownian motions $W_1, \cdots, W_N$. The (N,d) **additive Brownian motion** is defined as the following multi–parameter process:

$$W(\mathbf{t}) \triangleq \sum_{j=1}^N W_j(t_j), \qquad \mathbf{t}\in\mathbb{R}_+^N.$$

The goal of this section is to record some facts about the process W. First, we mention the following consequence of the proof of Theorem 1.1.

Theorem 6.1. *Suppose $M > 0$ and $0 < a_k < b_k < \infty$ $(k = 1, \cdots, N)$ are fixed. Then there exists a finite positive constant K_8 which only depends on the parameters M, N, d, $\min_{1 \leqslant j \leqslant N} a_j$ and $\max_{1 \leqslant j \leqslant N} b_j$, such that for all compact sets $E \subset \{x \in \mathbb{R}^d : |x| \leqslant M\}$,*

$$K_8^{-1} \mathrm{Cap}_{d-2N}(E) \leqslant \mathbb{P}\Big(W([\mathbf{a}, \mathbf{b}]) \cap E \neq \varnothing \Big) \leqslant K_8 \mathrm{Cap}_{d-2N}(E).$$

The proof of Theorem 6.1 involves a simplification of the arguments of Sections 3 and 4. Thus, we will merely give an outline. For all $\mathbf{s}, \mathbf{t} \in \mathbb{R}_+^N$, define $\widetilde{Z}_{\mathbf{s},\mathbf{t}} \triangleq W(\mathbf{s} + \mathbf{t}) - W(\mathbf{s})$ and let $\widetilde{\mathcal{F}}_{\mathbf{t}}$ denote the σ–field generated by $\big(W(\mathbf{s});\ \mathbf{0} \leqslant \mathbf{s} \leqslant \mathbf{t}\big)$. The process $\widetilde{Z}_{\mathbf{s},\mathbf{t}}$ is clearly the additive Brownian motion analogue of the process $Z_{\mathbf{s},\mathbf{t}}$ of Lemma 3.1. The following analogues of Lemmas 3.1 and 3.8 as well as Proposition 3.4 are much simpler to prove.

(i) for all $\mathbf{s}, \mathbf{t} \in \mathbb{R}_+^N$, the random vector $\widetilde{Z}_{\mathbf{s},\mathbf{t}}$ is independent of $\widetilde{\mathcal{F}}(\mathbf{s})$ and has the same distribution as $W_1\big(\sum_{j=1}^N t_j\big)$;

(ii) let f be as in Proposition 3.7. Then a.s., for all $\mathbf{s} \in [1, 3/2]^N \cap \mathbb{Q}^N$,

$$\begin{aligned}
&\mathbb{E}\Big[\int_{[1,2]^N} f\big(W(\mathbf{u})\big) d\mathbf{u} \ \Big| \ \widetilde{\mathcal{F}}(\mathbf{s}) \Big] \\
&\qquad \geqslant \mathbb{E}\Big[\int_{\substack{[1,2]^N \\ \mathbf{t} \geqslant \mathbf{s}}} f\big(\widetilde{Z}_{\mathbf{s},\mathbf{t}} + W(\mathbf{s})\big) d\mathbf{t} \ \Big| \ \widetilde{\mathcal{F}}(\mathbf{s}) \Big] \\
&\qquad \geqslant K_3'(M; N, d) \int_{\mathbb{R}^d} \kappa(b) f\big(b + W(\mathbf{s})\big) db \ \mathbb{1}\big\{|W(\mathbf{s})| \leqslant M/2\big\},
\end{aligned}$$

for some constant $K_3'(M; N, d)$;

(iii) in the notation of Lemma 3.8,

$$\mathbb{E}\Big[\Big(\int_{[1,2]^N} f\big(W(\mathbf{u})\big) d\mathbf{u} \Big)^2 \Big] \leqslant K_4'(M; N, d) \mathcal{E}_{d-2N}(f),$$

for some constant $K_4'(M; N, d)$.

Theorem 6.1 can now be proved using exactly the same argument as that presented in Section 4. However, all applications of Lemmas 3.1 and 3.8 and those of Proposition 3.4 are to be replaced by applications of (i), (iii) and (ii), respectively.

As a consequence of Theorems 1.1 and 6.1, we have the following curious result.

Corollary 6.2. *The (N, d) additive Brownian motion W and the (N, d) Brownian sheet B are* **locally intersection equivalent** *in the following sense: for all $M > 0$ and all $0 < a_k < b_k < \infty$ $(k = 1, \cdots, N)$, there exists a constant K_9 depending only on the parameters $\big(M, N, d, \min_{1 \leqslant j \leqslant N} a_j, \max_{1 \leqslant j \leqslant N} b_j\big)$ such that for all compact sets $E \subset \{x \in \mathbb{R}^d : |x| \leqslant M\}$,*

$$K_9^{-1} \leqslant \frac{\mathbb{P}\big(W([\mathbf{a}, \mathbf{b}]) \cap E \neq \varnothing\big)}{\mathbb{P}\big(B([\mathbf{a}, \mathbf{b}]) \cap E \neq \varnothing\big)} \leqslant K_9.$$

Remarks 6.2.1.
(i) The notion of local intersection equivalence is essentially due to [26].
(ii) Brownian sheet and additive Brownian motion have other connections than potential theoretic ones as well. For a sampler, see [4].
(iii) The above is interpreted with the convention that $0/0 \triangleq 1$. That is, when one of the two probabilities is 0, so is the other one. Otherwise, they are both of the same rough order.

Roughly speaking, Corollary 6.2 says that Brownian sheet B has the same potential theory as additive Brownian motion W. The latter process turns out to have some very nice analytical properties. The remainder of this section is devoted to a brief discussion of some of them.

For any Borel measurable function $f : \mathbb{R}^d \mapsto \mathbb{R}^1_+$, every $x \in \mathbb{R}^d$ and all $\mathbf{t} \in \mathbb{R}^N_+$, define

$$\mathcal{Q}_{\mathbf{t}} f(x) \triangleq \mathbb{E}\big[f\big(W(\mathbf{t}) + x\big)\big].$$

We can extend the domain of the definition of $\mathcal{Q}_{\mathbf{t}}$ to all measurable $f : \mathbb{R}^d \mapsto \mathbb{R}^1_+$ if, for example, $\mathcal{Q}_{\mathbf{t}}|f|(x) < \infty$ for all $x \in \mathbb{R}^d$. Elementary properties of $\mathcal{Q}_{\mathbf{t}}$ are listed below. Recall that for $\mathbf{s}, \mathbf{t} \in \mathbb{R}^N_+$, $\mathbf{s} \leqslant \mathbf{t}$ if and only if $s_j \leqslant t_j$ for all $1 \leqslant j \leqslant N$.

Proposition 6.3. *For all $\mathbf{t}, \mathbf{s} \in \mathbb{R}^N_+$, $\mathcal{Q}_{\mathbf{t}+\mathbf{s}} = \mathcal{Q}_{\mathbf{t}} + \mathcal{Q}_{\mathbf{s}}$. That is, $\big(\mathcal{Q}_{\mathbf{t}}; \mathbf{t} \in \mathbb{R}^N_+\big)$ is a semi–group indexed by $\big(\mathbb{R}^N_+; \leqslant\big)$. Furthermore, for any $\mathbf{t} > \mathbf{0}$, $\mathcal{Q}_{\mathbf{t}} : C_b(\mathbb{R}^d_+, \mathbb{R}^1_+) \mapsto C_b(\mathbb{R}^d_+, \mathbb{R}^1_+)$.*

Remarks 6.3.1.
(i) As is customary, $C_b(\mathbb{X}, \mathbb{Y})$ denotes the collection of all bounded continuous functions $f : \mathbb{X} \mapsto \mathbb{Y}$.
(ii) Proposition 6.3 says that $\big(\mathcal{Q}_{\mathbf{t}}; \mathbf{t} \in \mathbb{R}^N_+\big)$ is a multi–parameter "Feller semi–group".

Proof of Proposition 6.3. A simple consequence of Theorem 6.1 is the following: fix some $\mathbf{s} \in \mathbb{R}^N_+$, then as processes indexed by $\mathbf{t} \in \mathbb{R}^N_+$,

$$W(\mathbf{s} + \mathbf{t}) = W(\mathbf{s}) + \widetilde{W}(\mathbf{t}),$$

where $\widetilde{W}$ is independent of $\mathcal{G}(\mathbf{s})$. Now pick $\mathbf{s}, \mathbf{t} \in \mathbb{R}^N_+$ and pick a bounded measurable $f : \mathbb{R}^d \mapsto \mathbb{R}^1_+$. The above decomposition shows that for all $x \in \mathbb{R}^d$,

$$\begin{aligned}
\mathcal{Q}_{\mathbf{s}+\mathbf{t}} f(x) &= \mathbb{E}\big[f\big(W(\mathbf{s}) + \widetilde{W}(\mathbf{t}) + x\big)\big] \\
&= \mathbb{E}\big[\mathcal{Q}_{\mathbf{s}} f\big(\widetilde{W}(\mathbf{t}) + x\big)\big] \\
&= \mathcal{Q}_{\mathbf{s}} \mathbb{E}\big[f\big(\widetilde{W}(\mathbf{t}) + x\big)\big] \\
&= \mathcal{Q}_{\mathbf{s}} \mathbb{E}\big[f\big(W(\mathbf{t}) + x\big)\big] \\
&= \mathcal{Q}_{\mathbf{s}} \mathcal{Q}_{\mathbf{t}} f(x).
\end{aligned}$$

Since $f \mapsto \mathcal{Q}_{\mathbf{t}} f$ is linear, for all $\mathbf{t} \in \mathbb{R}^N_+$, a monotone class argument shows the desired semi–group property. To prove the Feller property, let β define a standard $\mathbb{R}^d$–valued

Brownian motion. Corresponding to β, let $\big(\mathcal{H}_t; t \geqslant 0\big)$ be the heat semi–group. That is, for all bounded measurable $f : \mathbb{R}^d \mapsto \mathbb{R}^1_+$ and all $x \in \mathbb{R}^d$, define

$$\mathcal{H}_t f(x) \triangleq \mathbb{E}\big[f\big(\beta(t) + x\big)\big].$$

It is well–known that for all $t > 0$, $\mathcal{H}_t : C_b(\mathbb{R}^d, \mathbb{R}^1_+) \mapsto C_b(\mathbb{R}^d, \mathbb{R}^1_+)$. Fix some $\mathbf{t} > \mathbf{0}$ and some $f \in C_b(\mathbb{R}^d, \mathbb{R}^1_+)$. Define, $c \triangleq \big\{\prod_{j=2}^N t_j\big\}^{1/2}$ and $g(x) \triangleq f(cx)$. Since the random vector $W(\mathbf{t})$ has the same distribution as $c\beta(t_1)$, it follows that

$$\mathcal{Q}_{\mathbf{t}} f(x) = \mathcal{H}_{t_1} g\Big(\frac{x}{c}\Big).$$

The desired Feller property follows immediately. $\diamondsuit$

A somewhat surprising fact is that one merely needs a **one–parameter** family of "resolvents".

For any $\lambda > 0$ and $\mathbf{t} \in \mathbb{R}^N$, define $\lambda \cdot \mathbf{t} \triangleq \lambda \sum_{j=1}^N t_j$. Define,

$$\mathcal{U}^\lambda \triangleq \int_{\mathbb{R}^N_+} e^{-\lambda \cdot \mathbf{t}} \mathcal{Q}_{\mathbf{t}} d\mathbf{t}, \qquad \lambda > 0.$$

The above satisfies a multi–parameter "resolvent equation". More precisely,

Lemma 6.4. *For any $\lambda > 0$, $\mathcal{U}^\lambda = \big(\mathcal{V}^\lambda\big)^N$, where $\mathcal{V}^\lambda$ is the heat resolvent:*

$$\mathcal{V}^\lambda \triangleq \int_0^\infty e^{-\lambda s} \mathcal{H}_s ds.$$

Remarks 6.4.1.

(i) Combining the above with the resolvent equation for $\mathcal{V}^\lambda$, we see that if $f : \mathbb{R}^d \mapsto \mathbb{R}^1_+$ is Borel measurable and $\mathcal{U}^\lambda f = 0$, then $f = 0$. Thus, $\big(\mathcal{U}^\lambda; \lambda > 0\big)$ seperates points. It is this desriable property which justifies the use of the term resolvent for $\big(\mathcal{V}^\lambda; \lambda > 0\big)$.

(ii) Using induction on N, we arrive at the following heat kernel representation of $\mathcal{U}^\lambda$:

$$\mathcal{U}^\lambda = \frac{1}{(N-1)!} \int_0^\infty s^{N-1} e^{-\lambda s} \mathcal{H}_s ds.$$

(iii) It is possible to show that $\mathcal{Q}_{\mathbf{t}}$ solves the following weak operator equation:

$$\begin{aligned} \frac{\partial^N}{\partial t_1 \cdots \partial t_N} \mathcal{Q}_{\mathbf{t}} \Big|_{\mathbf{t}=\mathbf{0}} &= 2^{-N} \Delta^N, \\ \frac{\partial}{\partial t_i} \mathcal{Q}_{\mathbf{t}} \Big|_{\mathbf{t}=\mathbf{0}} &= \frac{1}{2} \Delta, \end{aligned}$$

where Δ is the (spatial) Laplacian on $\mathbb{R}^N$. As such, it follows that the N–Laplacian $2^{-N}\Delta^N$ can be called the "generator" of W.

Proof of Lemma 6.4. Throughout this proof, whenever $\mathbf{t} \in \mathbb{R}^N$, we shall write $[\mathbf{t}]$ for the N–vector $(0, t_2, \cdots, t_N)$. Since $\mathbf{t} = (t_1, 0, \cdots, 0) + [\mathbf{t}]$, Proposition 6.3 implies that $\mathcal{Q}_{\mathbf{t}} = \mathcal{Q}_{(t_1,0,\cdots,0)}\mathcal{Q}_{[\mathbf{t}]}$. However, it is simple to check that $\mathcal{H}_{t_1} = \mathcal{Q}_{(t_1,0,\cdots,0)}$. Therefore,

$$\begin{aligned}
\mathcal{U}^\lambda &= \int_{\mathbb{R}^N_+} e^{-\lambda\cdot\mathbf{t}} \mathcal{H}_{t_1}\mathcal{Q}_{[\mathbf{t}]} d\mathbf{t} \\
&= \int_{\mathbb{R}^{N-1}_+} dt_N \cdots dt_2 \Big(\int_0^\infty e^{-\lambda t_1} \mathcal{H}_{t_1} dt_1 \Big) \mathcal{Q}_{[\mathbf{t}]} \exp\big(-\lambda \sum_{j=2}^N t_j \big) \\
&= \int_{\mathbb{R}^{N-1}_+} dt_N \cdots dt_2 \mathcal{V}^\lambda \mathcal{Q}_{[\mathbf{t}]} \exp\big(-\lambda \sum_{j=2}^N t_j \big) \\
&= \mathcal{V}^\lambda \int_{\mathbb{R}^{N-1}_+} dt_N \cdots dt_2 \mathcal{Q}_{[\mathbf{t}]} \exp\big(-\lambda \sum_{j=2}^N t_j \big).
\end{aligned}$$

Since $\mathcal{Q}_{[\mathbf{t}]}$ is the semi–group for $(N-1, d)$ additive Brownian motion, we obtain the result by induction on N. $\diamondsuit$

7. Applications to Analysis on Wiener Space

Recall from [21, 23] that the d–dimensional Ornstein–Uhlenbeck process $O(\cdot)$ on Wiener space is a stationary ergodic diffusion on $C(\mathbb{R}^1_+, \mathbb{R}^d)$ whose stationary measure is d–dimensional Wiener measure, $\mathbf{w}$. As observed in [36], this process can be realized as follows: take a $(2, d)$ Brownian sheet B. Then

$$O(s) \triangleq e^{-s/2} B(e^s, \cdot), \qquad 0 \leqslant s \leqslant 1. \tag{7.1}$$

For each fixed $s \geqslant 0$, $O(s)$ is a Brownian motion on $\mathbb{R}^d$. Corresponding to O, there is a notion of capacity; cf. [7, 8, 32]. Indeed, the following is a Choquet capacity, defined on analytic subsets of $C(\mathbb{R}^1_+, \mathbb{R}^d)$:

$$\mathrm{Cap}_\infty(F) \triangleq \int_0^\infty e^{-r} \mathbb{P}^{\mathbf{w}}\big(O(s) \in F, \text{ for some } 0 \leqslant s \leqslant r \big) dr, \tag{7.2}$$

where

$$\mathbb{P}^{\mathbf{w}}(\cdots) \triangleq \int_{C(\mathbb{R}^1_+, \mathbb{R}^d)} \mathbb{P}\big(\cdots \bigm| O(0) = f \big) \mathbf{w}(df).$$

When $\mathrm{Cap}_\infty(F) > 0$, we say that F happens **quasi–surely**. From the properties of Laplace transforms, it is not hard to see the following (cf. Lemma 2.2.1(ii) of [8], for example)

$$\mathrm{Cap}_\infty(F) > 0 \iff \mathrm{Cap}^{(t)}_\infty(F) > 0,$$

where $\mathrm{Cap}^{(t)}_\infty$ is the **incomplete capacity** on Wiener space defined as follows:

$$\mathrm{Cap}^{(t)}_\infty(F) \triangleq \mathbb{P}\big(O(s) \in F, \text{ for some } 0 \leqslant s \leqslant t \big).$$

Since $t \mapsto \mathrm{Cap}_\infty^{(t)}(F)$ is increasing, we see that

$$\mathrm{Cap}_\infty(F) > 0 \;\Leftrightarrow\; \mathrm{Cap}_\infty^{(t)}(F) > 0, \text{ for some } t \geqslant 0. \tag{7.3}$$

We say that a Borel set $F \subset C(\mathbb{R}^1_+, \mathbb{R}^d)$ is **exceptional**, if $\mathrm{Cap}_\infty(F) > 0$ while $\mathbf{w}(F) = 0$. It is an intereseting problem — due to David Williams (cf. [36]) — to find non–trivial exceptional sets. Various classes of such exceptional sets F have been found in the literature; cf. [7, 20, 22, 25, 31]. In particular, [7, Theorem 7] implies that

$$d > 4 \Rightarrow \mathrm{Cap}_\infty\Big\{ f \in C(\mathbb{R}^1_+, \mathbb{R}^d) \;:\; \lim_{t\to\infty} |f(t)| = \infty \Big\} > 0.$$

In other words, paths in $C(\mathbb{R}^1_+, \mathbb{R}^d)$ are transient quasi–surely, if $d > 4$. Conversely, by [18], if $d \leqslant 4$, paths in $C(\mathbb{R}^1_+, \mathbb{R}^d)$ are not transient; for another proof of this fact, use (7.7) below and standard capacity estimates. In other words,

$$d > 4 \Leftrightarrow \mathrm{Cap}_\infty\Big\{ f \in C(\mathbb{R}^1_+, \mathbb{R}^d) \;:\; \lim_{t\to\infty} |f(t)| = \infty \Big\} > 0. \tag{7.4}$$

On the other hand, classical results of [5] imply that, paths in $C(\mathbb{R}^1_+, \mathbb{R}^d)$ are transient $\mathbf{w}$–almost surely if and only if $d > 2$. That is,

$$d > 2 \Leftrightarrow \mathbf{w}\Big\{ f \in C(\mathbb{R}^1_+, \mathbb{R}^d) \;:\; \lim_{t\to\infty} |f(t)| = \infty \Big\} > 0. \tag{7.5}$$

A comparison of Eqs. (7.4) and (7.5) shows that the collection of transient paths in $C(\mathbb{R}^1_+, \mathbb{R}^d)$ is exceptional when $2 < d \leqslant 4$. In this section, we present a quantitative extension of this result in terms of a precise integral test for this rate of transience in the case $d > 4$. First note that upon combining (7.1)–(7.4), together with a compactness argument, we obtain the following:

$$\lim_{t\to\infty} |f(t)| = \infty, \qquad \text{quasi–surely } [f],$$

if and only if with probability one,

$$\lim_{t\to\infty} \inf_{1 \leqslant s \leqslant 2} |B(s,t)| = \infty. \tag{7.6}$$

The arguments of [17] which lead to our Theorems 5.1 and 5.2 can be used closely to prove the following quantitative version of (7.6):

Theorem 7.1. *Suppose $\psi : \mathbb{R}^1_+ \mapsto \mathbb{R}^1_+$ is decreasing. With probability one,*

$$\liminf_{t\to\infty} \inf_{1 \leqslant s \leqslant 2} \frac{|B(s,t)|}{t^{1/2}\psi(t)} = \begin{cases} \infty, & \text{if } \int_1^\infty (\kappa \circ \psi(x))\, x^{-1}\, dx < \infty \\ 0, & \text{if } \int_1^\infty (\kappa \circ \psi(x))\, x^{-1}\, dx = \infty \end{cases}.$$

As another application, consider a compact set $F \subset \mathbb{R}^d$ and define

$$\mathtt{Hit}(F) \triangleq \Big\{ f \in C(\mathbb{R}^1_+, \mathbb{R}^d) \ : \ \inf_{0 \leqslant t \leqslant 1} \operatorname{dist}\big(f(t), F\big) = 0 \Big\}.$$

Then, Theorem 1.1, (7.1)–(7.3) and some calculus jointly imply the following: for all $t > 0$, there exists a non–trivial constant K_{10} which depends only on d, t and $\sup F$, such that

$$K_{10}^{-1} \mathrm{Cap}_{d-4}(F) \leqslant \mathrm{Cap}_\infty^{(t)}\big(\mathtt{Hit}(F)\big) \leqslant K_{10} \mathrm{Cap}_{d-4}(F). \tag{7.7}$$

Actually, the calculus involved is technical but the ideas can all be found in the proof of [17, Lemma 6.3]. Moreover, one can almost as easily show that for some K_{11} which depends only on d and $\sup F$,

$$K_{11}^{-1} \mathrm{Cap}_{d-4}(F) \leqslant \mathrm{Cap}_\infty(F) \leqslant K_{11} \mathrm{Cap}_{d-4}(F).$$

We omit the details.

A consequence of the above and (7.3) is that for any Borel set $F \subset C(\mathbb{R}^1_+, \mathbb{R}^d)$,

$$\mathrm{Cap}_\infty\big(\mathtt{Hit}(F)\big) > 0 \Leftrightarrow \mathrm{Cap}_{d-4}(F) > 0. \tag{7.8}$$

This should be compared to the classical result of [13]:

$$\mathbf{w}\big(\mathtt{Hit}(F)\big) > 0 \Leftrightarrow \mathrm{Cap}_{d-2}(F) > 0. \tag{7.9}$$

Used in conjunction, Eqs. (7.8) and (7.9) show that $\mathtt{Hit}(F)$ is exceptional whenever we have $\mathrm{Cap}_{d-2}(F) = 0$ but $\mathrm{Cap}_{d-4}(F) > 0$. As an example, consider any Borel set $F \subset \mathbb{R}^d$ such that $d - 4 < \dim_H(F) < d - 2$. Here, $\dim_H$ denotes Hausdorff dimension. By Frostman's lemma of classical potential theory, $\mathrm{Cap}_{d-4}(F) > 0$ while $\mathrm{Cap}_{d-2}(F) = 0$. In such a case, $\mathtt{Hit}(F)$ is exceptional.

As yet another class of applications, let us note that together with standard capacity estimates, (7.3) and (7.7) extend the results of [20, Section 3].

Corollary 7.2. *For any $x \in \mathbb{R}^d$ and $r > 0$,*

$$\mathrm{Cap}_\infty\Big(\mathtt{Hit}\big(\{x\}\big)\Big) > 0 \Leftrightarrow d \leqslant 3,$$

while

$$\mathrm{Cap}_\infty\Big(\mathtt{Hit}\big(\mathcal{B}_d(x,r)\big)\Big) > 0 \Leftrightarrow d \leqslant 4.$$

Remark 7.2.1. The curious relationship between $d-2$ and $d-4$ in (7.8) and (7.9) seems to belong to a class of so–called **Ciesielski–Taylor** results. For earlier occurrences of such phenomena (in several other contexts), see [3, 12, 16, 37].

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