STOCHASTIC PROCESSES AND THEIR APPLICATIONS **74** 89–121 (1998)

A LAW OF THE ITERATED LOGARITHM FOR STABLE PROCESSES IN RANDOM SCENERY

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Abstract. We prove a law of the iterated logarithm for stable processes in a random scenery. The proof relies on the analysis of a new class of stochastic processes which exhibit long-range dependence.

Keywords. Brownian motion in stable scenery; law of the iterated logarithm; quasi–association

1991 Mathematics Subject Classification.
Primary. 60G18; Secondary. 60E15, 60F15.

* Research partially supported by grants from the National Science Foundation and National Security Agency

§1. Introduction

In this paper we study the sample paths of a family of stochastic processes called stable processes in random scenery. To place our results in context, first we will describe a result of Kesten and Spitzer (1979) which shows that a stable process in random scenery can be realized as the limit in distribution of a random walk in random scenery.

Let $\mathcal{Y} = \{y(i) : i \in \mathbb{Z}\}$ denote a collection of independent, identically–distributed, real–valued random variables and let $\mathcal{X} = \{x_i : i \geqslant 1\}$ denote a collection of independent, identically–distributed, integer–valued random variables. We will assume that the collections $\mathcal{Y}$ and $\mathcal{X}$ are defined on a common probability space and that they generate independent σ–fields. Let $s_0 = 0$ and, for each $n \geqslant 1$, let

$$s_n = \sum_{i=1}^{n} x_i.$$

In this context, $\mathcal{Y}$ is called the *random scenery* and $\mathcal{S} = \{s_n : n \geqslant 0\}$ is called the *random walk.* For each $n \geqslant 0$, let

$$g_n = \sum_{j=0}^{n} y(s_j). \tag{1.1}$$

The process $\mathcal{G} = \{g_n : n \geqslant 0\}$ is called *random walk in random scenery.* Stated simply, a random walk in random scenery is a cumulative sum process whose summands are drawn from the scenery; the order in which the summands are drawn is determined by the path of the random walk.

For purposes of comparison, it is useful to have an alternative representation of $\mathcal{G}$. For each $n \geqslant 0$ and each $a \in \mathbb{Z}$, let

$$\ell_n^a = \sum_{j=0}^{n} \mathbf{1}\{s_j = a\}.$$

$\mathcal{L} = \{\ell_n^a : n \geqslant 0,\, a \in \mathbb{Z}\}$ is the *local–time process* of $\mathcal{S}$. In this notation, it follows that, for each $n \geqslant 0$,

$$g_n = \sum_{a \in \mathbb{Z}} \ell_n^a \, y(a). \tag{1.2}$$

To develop the result of Kesten and Spitzer, we will need to impose some mild conditions on the random scenery and the random walk. Concerning the scenery, we will assume that $\mathbb{E}\big(y(0)\big) = 0$ and $\mathbb{E}\big(y^2(0)\big) = 1$. Concerning the walk, we will assume that $\mathbb{E}(x_1) = 0$ and that x_1 is in the domain of attraction of a strictly stable random variable of index α $(1 < \alpha \leqslant 2)$. Thus, we assume that there exists a strictly stable random variable R_α of index α such that $n^{-\frac{1}{\alpha}} s_n$ converges in distribution to R_α as $n \to \infty$. Since R_α is strictly stable, its characteristic function must assume the following form (see, for example, Theorem 9.32 of Breiman (1968)): there exist real numbers $\chi > 0$ and $\nu \in [-1, 1]$ such that, for all $\xi \in \mathbb{R}$,

$$\mathbb{E}\exp(i\xi R_\alpha) = \exp\Big(-|\xi|^\alpha \frac{1 + i\nu \operatorname{sgn}(\xi)\tan(\alpha\pi/2)}{\chi}\Big).$$

Criteria for a random variable being in the domain of attraction of a stable law can be found, for example, in Theorem 9.34 of Breiman (1968).

Let $Y_\pm = \{Y_\pm(t) : t \geqslant 0\}$ denote two standard Brownian motions and let $X = \{X_t : t \geqslant 0\}$ be a strictly stable Lévy process with index α $(1 < \alpha \leqslant 2)$. We will assume that Y_+, Y_- and X are defined on a common probability space and that they generate independent σ–fields. In addition, we will assume that X_1 has the same distribution as R_α. As such, the characteristic function of X_t is given by

$$\mathbb{E}\exp(i\xi X(t)) = \exp\Big(-t|\xi|^\alpha \frac{1 + i\nu \operatorname{sgn}(\xi)\tan(\alpha\pi/2))}{\chi}\Big). \tag{1.3}$$

We will define a *two–sided Brownian motion* $Y = \{Y(t) : t \in \mathbb{R}\}$ according to the rule

$$Y(t) = \begin{cases} Y_+(t), & \text{if } t \geqslant 0 \\ Y_-(-t), & \text{if } t < 0 \end{cases}$$

Given a function $f : \mathbb{R} \to \mathbb{R}$, we will let

$$\int_{\mathbb{R}} f(x) dY(x) \triangleq \int_0^\infty f(x) dY_+(x) + \int_0^\infty f(-x) dY_-(x)$$

provided that both of the Itô integrals on the right–hand side are defined.

Let $L = \{L_t^x : t \geqslant 0,\ x \in \mathbb{R}\}$ denote the process of local times of X; thus, L satisfies the *occupation density formula*: for each measurable $f : \mathbb{R} \mapsto \mathbb{R}$ and for each $t \geqslant 0$,

$$\int_0^t f\big(X(s)\big) ds = \int_{\mathbb{R}} f(a) L_t^a da. \tag{1.4}$$

Using the result of Boylan (1964), we can assume, without loss of generality, that L has continuous trajectories. With this in mind, the following process is well defined: for each $t \geqslant 0$, let

$$G(t) \triangleq \int L_t^x dY(x). \tag{1.5}$$

Due to the resemblance between (1.2) and (1.5), the stochastic process $G = \{G_t : t \geqslant 0\}$ is called a *stable process in random scenery.*

Given a sequence of càdlàg processes $\{U_n : n \geqslant 1\}$ defined on $[0,1]$ and a càdlàg process V defined on $[0,1]$, we will write $U_n \Rightarrow V$ provided that U_n converges in distribution to V in the space $D_{\mathbb{R}}([0,1])$ (see, for example, Billingsley (1979) regarding convergence in distribution). Let

$$\delta \triangleq 1 - \frac{1}{2\alpha}. \tag{1.6}$$

Then the result of Kesten and Spitzer is

$$\left\{n^{-\delta} g_{[nt]} : 0 \leqslant t \leqslant 1\right\} \Rightarrow \left\{G(t) : 0 \leqslant t \leqslant 1\right\}. \tag{1.7}$$

Thus, normalized random walk in random scenery converges in distribution to a stable process in random scenery. For additional information on random walks in random scenery and stable processes in random scenery, see Bolthausen (1989), Kesten and Spitzer (1979), Lang and Nguyen (1983), Lewis (1992), Lewis (1993), Lou (1985), and Rémillard and Dawson (1991).

Viewing (1.7) as the central limit theorem for random walk in random scenery, it is natural to investigate the law of the iterated logarithm, which would describe the asymptotic behavior of g_n as $n \to \infty$. To give one such result, for each $n \geqslant 0$ let

$$v_n = \sum_{a \in \mathbb{Z}} \left(\ell_n^a\right)^2.$$

The process $\mathcal{V} = \{v_n : n \geqslant 0\}$ is called the *self–intersection local time* of the random walk. Throughout this paper, we will write $\log_e$ to denote the natural logarithm. For $x \in \mathbb{R}$, define $\ln(x) = \log_e(x \vee e)$. In Lewis (1992), it has been shown that if $\mathbb{E}|y(0)|^3 < \infty$, then

$$\limsup_{n\to\infty} \frac{g_n}{\sqrt{2v_n \ln\ln(n)}} = 1, \qquad \text{a.s.}$$

This is called a *self–normalized* law of the iterated logarithm in that the rate of growth of g_n as $n \to \infty$ is described by a random function of the process itself. The goal of this article is to present deterministically normalized laws of the iterated logarithm for stable processes in random scenery and random walk in random scenery.

From (1.3), you will recall that the distribution of X_1 is determined by three parameters: α (the index), χ and ν. Here is our main theorem.

Theorem 1.1. *There exists a real number $\gamma = \gamma(\alpha, \chi, \nu) \in (0, \infty)$ such that*

$$\limsup_{t\to\infty} \Big(\frac{\ln\ln t}{t}\Big)^{\delta} \frac{G(t)}{\big(\ln\ln t\big)^{3/2}} = \gamma \qquad a.s.$$

When $\alpha = \chi = 2$, X is a standard Brownian motion and, in this case, G is called *Brownian motion in random scenery.* For each $t \geqslant 0$, define $Z(t) = Y(X(t))$. The process $Z = \{Z_t : t \geqslant 0\}$ is called *iterated Brownian motion.* Our interest in investigating the path properties of stable processes in random scenery was motivated, in part, by some newly found connections between this process and iterated Brownian motion. In Khoshnevisan and Lewis (1996), we have related the quadratic and quartic variations of iterated Brownian motion to Brownian motion in random scenery. These connections suggest that there is a duality between these processes; Theorem 1.1 may be useful in precisely defining the meaning of "duality" in this context.

Another source of interest in stable processes in random scenery is that they are processes which exhibit long–range dependence. Indeed, by our Theorem 5.2, for each $t \geqslant 0$, as $s \to \infty$,

$$\mathrm{Cov}\big(G(t), G(t+s)\big) \sim \frac{\alpha t}{\alpha - 1} s^{(\alpha-1)/\alpha}.$$

This long–range dependence presents certain difficulties in the proof of the lower bound of Theorem 1.1. To overcome these difficulties, we introduce and study *quasi–associated* collections of random variables, which may be of independent interest and worthy of further examination.

In our next result, we present a law of the iterated logarithm for random walk in random scenery. The proof of this result relies on strong approximations and Theorem

1.1. We will call $\mathcal{G}$ a *simple symmetric random walk in Gaussian scenery* provided that $y(0)$ has a standard normal distribution and

$$\mathbb{P}(x_1 = +1) = \mathbb{P}(x_1 = -1) = \frac{1}{2}.$$

In the statement of our next theorem, we will use $\gamma(2,2,0)$ to denote the constant from Theorem 1.1 for the parameters $\alpha = 2$, $\chi = 2$ and $\nu = 0$.

Theorem 1.2. *There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which supports a Brownian motion in random scenery G and a simple symmetric random walk in Gaussian scenery $\mathcal{G}$ such that, for each $q > 1/2$,*

$$\lim_{n\to\infty} \frac{\sup_{0 \leqslant t \leqslant 1} |G(nt) - g([nt])|}{n^q} = 0 \qquad \text{a.s.}$$

Thus,

$$\limsup_{n\to\infty} \frac{g_n}{\big(n \ln\ln(n)\big)^{\frac{3}{4}}} = \gamma(2,2,0) \qquad \textit{a.s.}$$

A brief outline of the paper is in order. In §2 we prove a maximal inequality for a class of Gaussian processes, and we apply this result to stable processes in random scenery. In §3 we introduce the class of quasi–associated random variables; we show that disjoint increments of $\mathcal{G}$ (hence G) are quasi–associated. §4 contains a correlation inequality which is reminiscent of a result of Hoeffding (see Lehmann (1966) and Newman and Wright (1981)); we use this correlation inequality to prove a simple Borel–Cantelli Lemma for certain sequences of dependent random variables, which is an important step in the proof of the lower bound in Theorem 1.1. §5 contains the main probability calculations, significantly a large deviation estimate for $\mathbb{P}(G_1 > x)$ as $x \to \infty$. In §6 we marshal the results of the previous sections and give a proof of Theorem 1.1. Finally, the proof the Theorem 1.2 is presented in §7.

Remark 1.2. As is customary, we will say that stochastic processes U and V are *equivalent*, denoted by $U \overset{d}{=} V$, provided that they have the same finite–dimensional distributions. We will say that the stochastic process U is *self–similar with index p* ($p > 0$) provided that, for each $c > 0$,

$$\{U_{ct} : t \geqslant 0\} \overset{d}{=} \{c^p U_t : t \geqslant 0\}.$$

Since X is a strictly stable Lévy process of index α, it is self–similar with index α^{-1}. The process of local times L inherits a scaling law from X : for each $c > 0$,

$$\{L_{ct}^x : t \geqslant 0, x \in \mathbb{R}\} \stackrel{d}{=} \{c^{1-\frac{1}{\alpha}} L_t^{xc^{-1/\alpha}} : t \geqslant 0, x \in \mathbb{R}\}.$$

Since a standard Brownian motion is self–similar with index 1/2, it follows that G is self–similar with index $\delta = 1 - (2\alpha)^{-1}$.

§2. A maximal inequality for subadditive Gaussian processes

The main result of this section is a maximal inequality for stable processes in random scenery, which we state presently.

Theorem 2.1. *Let G be a stable process in random scenery and let $t, \lambda \geqslant 0$. Then*

$$\mathbb{P}\Big(\sup_{0 \leqslant s \leqslant t} G_s \geqslant \lambda\Big) \leqslant 2\mathbb{P}(G_t \geqslant \lambda).$$

The proof of this theorem rests on two observations. First we will establish a maximal inequality for a certain class of Gaussian processes. Then we will show that G is a member of this class conditional on the σ–field generated by the underlying stable process X.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space which supports a centered, real–valued Gaussian process $Z = \{Z_t : t \geqslant 0\}$. We will assume that Z has a continuous version. For each $s, t \geqslant 0$, let

$$d(s,t) \triangleq \big(\mathbb{E}(Z_s - Z_t)^2\big)^{1/2},$$

which defines a psuedo–metric on $\mathbb{R}^+$, and let

$$\sigma(t) \triangleq d(0,t).$$

We will say that Z is $\mathbb{P}$–*subadditive* provided that

$$\sigma^2(t) - \sigma^2(s) \geqslant d^2(s,t) \tag{2.1}$$

for all $t \geqslant s \geqslant 0$.

Remark. If, in addition, Z has stationary increments, then $d^2(s,t) = \sigma^2(|t-s|)$. In this case, the subadditivity of Z can be stated as follows: for all $s, t \geqslant 0$,

$$\sigma^2(t) + \sigma^2(s) \leqslant \sigma^2(s+t).$$

In other words, σ^2 is subadditive in the classical sense. Moreover, in this case, Z becomes *sub–diffusive*, that is,

$$\lim_{t\to\infty} \frac{\sigma(t)}{t^{1/2}} = \sup_{s>0} \frac{\sigma(s)}{s^{1/2}}.$$

It is significant that subadditive Gaussian processes satisfy the following maximal inequality:

Proposition 2.2. *Let Z be a centered, $\mathbb{P}$–subadditive, $\mathbb{P}$–Gaussian process on $\mathbb{R}$ and let $t, \lambda \geqslant 0$. Then*

$$\mathbb{P}\big(\sup_{0\leqslant s\leqslant t} Z_s \geqslant \lambda\big) \leqslant 2\mathbb{P}\big(Z_t \geqslant \lambda\big).$$

Proof. Let B be a linear Brownian motion under the probability measure $\mathbb{P}$, and, for each $t \geqslant 0$, let

$$T_t \triangleq B\big(\sigma^2(t)\big).$$

Since T is a centered, $\mathbb{P}$–Gaussian process on $\mathbb{R}$ with independent increments, it follows that, for each $t \geqslant s \geqslant 0$,

$$\begin{aligned} \mathbb{E}(T_t^2) &= \sigma^2(t), \\ \mathbb{E}\big(T_s(T_t - T_s)\big) &= 0. \end{aligned} \tag{2.2}$$

Since T_u and Z_u have the same law for each $u \geqslant 0$, by (2.1) and (2.2) we may conclude that

$$\begin{aligned} \mathbb{E}(Z_s Z_t) - \mathbb{E}(T_s T_t) &= \mathbb{E}(Z_s^2) + \mathbb{E}\big(Z_s(Z_t - Z_s)\big) - \mathbb{E}\big(T_s(T_t - T_s)\big) - \mathbb{E}(T_s^2) \\ &= \mathbb{E}\big(Z_s(Z_t - Z_s)\big) \\ &= \frac{1}{2}\big(\sigma^2(t) - \sigma^2(s) - d^2(s,t)\big) \\ &\geqslant 0. \end{aligned}$$

These calculations demonstrate that $\mathbb{E}(Z_t^2) = \mathbb{E}(T_t^2)$ and $\mathbb{E}\big(Z_t - Z_s\big)^2 \leqslant \mathbb{E}\big(T_t - T_s\big)^2$ for all $t \geqslant s \geqslant 0$. By Slepian's lemma (see p. 48 of Adler (1990)),

$$\mathbb{P}\big(\sup_{0\leqslant s \leq t} Z_s \geqslant \lambda\big) \leqslant \mathbb{P}\big(\sup_{0\leqslant s\leqslant t} T_s \geqslant \lambda\big). \tag{2.3}$$

By (2.1), the map $t \mapsto \sigma(t)$ is nondecreasing. Thus, by the definition of T, (2.3), the reflection principle, and the fact that T_t and Z_t have the same distribution for each $t \geqslant 0$, we may conclude that

$$\begin{aligned}\mathbb{P}\big(\sup_{0 \leqslant s \leqslant t} Z_s \geqslant \lambda\big) &\leqslant \mathbb{P}\big(\sup_{0 \leqslant s \leqslant t} T_s \geqslant \lambda\big) \\ &\leqslant \mathbb{P}\big(\sup_{0 \leqslant s \leqslant \sigma^2(t)} B_s \geqslant \lambda\big) \\ &= 2\mathbb{P}\big(B(\sigma^2(t)) \geqslant \lambda\big) \\ &= 2\mathbb{P}\big(Z_t \geqslant \lambda\big),\end{aligned}$$

which proves the result in question. ◇

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a Markov process $M = \{M_t : t \geqslant 0\}$ and an independent, two–sided Brownian motion $Y = \{Y_t : t \in \mathbb{R}\}$. We will assume that M has a jointly measurable local–time process $L = \{L_t^x : t \geqslant 0, x \in \mathbb{R}\}$. For each $t \geqslant 0$, let

$$\mathcal{G}_t \triangleq \int L_t^x dY(x).$$

The process $\mathcal{G} = \{\mathcal{G}_t : t \geqslant 0\}$ is called a *Markov process in random scenery.* For $t \in [0, \infty]$, let $\mathfrak{M}_t$ denote the $\mathbb{P}$–complete, right–continuous extension of the σ–field generated by the process $\{M_s : 0 \leqslant s < t\}$. Let $\mathfrak{M} \triangleq \mathfrak{M}_\infty$ and let $\mathbb{P}_{\mathfrak{M}}$ be the measure $\mathbb{P}$ conditional on $\mathfrak{M}$. Fix $u \geqslant 0$ and, for each $s \geqslant 0$, define

$$g_s \triangleq \mathcal{G}_{s+u} - \mathcal{G}_u.$$

Let $g \triangleq \{g_s : s \geqslant 0\}$.

Proposition 2.3. *g is a centered, $\mathbb{P}_{\mathfrak{M}}$–subadditive, $\mathbb{P}_{\mathfrak{M}}$–Gaussian process on $\mathbb{R}$, almost surely* $[\mathbb{P}]$.

Proof. The fact that g is a centered $\mathbb{P}_{\mathfrak{M}}$–Gaussian process on $\mathbb{R}$ almost surely $[\mathbb{P}]$ is a direct consequence of the additivity property of Gaussian processes. (This statement only holds almost surely $\mathbb{P}$, since local times are defined only on a set of full $\mathbb{P}$ measure.) Let $t \geqslant s \geqslant 0$, and note that

$$g_t - g_s = \int_{\mathbb{R}} \big(L_{t+u}^x - L_{s+u}^x\big) dY(x).$$

Since Y is independent of $\mathfrak{M}$, we have, by Itô isometry,

$$\begin{aligned} d^2(s,t) &= \mathbb{E}_{\mathfrak{M}}\left(g_t - g_s\right)^2 \\ &= \int_{\mathbb{R}} \left(L_{t+u}^x - L_{s+u}^x\right)^2 dx. \end{aligned}$$

Since the local time at x is an increasing process, for all$t \geqslant s \geqslant 0$,

$$\sigma^2(t) - \sigma^2(s) - d^2(s,t) = 2\int_{\mathbb{R}} \left(L_{u+t}^x - L_{u+s}^x\right)\left(L_{s+u}^x - L_u^x\right) dx \geqslant 0,$$

almost surely $[\mathbb{P}]$. ◇

Proof of Theorem 2.1. By Proposition 2.2 and Proposition 2.3, it follows that

$$\mathbb{P}_{\mathfrak{M}}\left(\sup_{0 \leqslant s \leqslant t} G_s \geqslant \lambda\right) \leqslant 2\mathbb{P}_{\mathfrak{M}}(G_t \geqslant \lambda)$$

almost surely $[\mathbb{P}]$. The result follows upon taking expectations. ◇

§3. Quasi–association

Let $\mathcal{Z} = \{Z_1, Z_2, \cdots, Z_n\}$ be a collection of random variables defined on a common probability space. We will say that $\mathcal{Z}$ is *quasi–associated* provided that

$$\mathrm{Cov}\Big(f(Z_1, \cdots, Z_i), g(Z_{i+1}, \cdots, Z_n)\Big) \geqslant 0, \tag{3.1}$$

for all $1 \leqslant i \leqslant n-1$ and all coordinatewise–nondecreasing, measurable functions $f : \mathbb{R}^i \mapsto \mathbb{R}$ and $g : \mathbb{R}^{n-i} \mapsto \mathbb{R}$. The property of quasi–association is closely related to the property of association. Following Esary, Proschan, and Walkup (1967), we will say that $\mathcal{Z}$ is *associated* provided that

$$\mathrm{Cov}\Big(f(Z_1, \cdots, Z_n),\ g(Z_1, \cdots, Z_n)\Big) \geqslant 0 \tag{3.2}$$

for all coordinatewise–nondecreasing, measurable functions $f, g : \mathbb{R}^n \to \mathbb{R}$. Clearly a collection is quasi–associated whenever it is associated. In verifying either (3.1) or (3.2), we can, without loss of generality, further restrict the set of test functions by assuming that they are bounded and continuous as well.

A generalization of association to collections of random vectors (called *weak association*) was initiated by Burton, Dabrowski, and Dehling (1986) and further investigated by Dabrowski and Dehling (1988). For random variables, weak association is a stronger condition than quasi–association.

As with association, quasi–association is preserved under certain actions on the collection. One such action can be described as follows: Suppose that $\mathcal{Z}$ is quasi–associated, and let $A_1, A_2, \cdots, A_k$ be disjoint subsets of $\{1, 2, \cdots, n\}$ with the property that for each integer j, each element of A_j dominates every element of A_{j-1} and is dominated, in turn, by each element of A_{j+1}. For each integer $1 \leqslant j \leqslant n$, let U_j be a nondecreasing function of the random variables $\{Z_i : i \in A_j\}$. Then it can be shown that the collection $\{U_1, U_2, \cdots, U_k\}$ is quasi–associated as well. We will call the action of forming the collection $\{U_1, \cdots, U_k\}$ *ordered blocking*; thus, quasi–association is preserved under the action of ordered blocking.

Another natural action which preserves quasi–association could be called *passage to the limit.* To describe this action, suppose that, for each $k \geqslant 1$, the collection

$$\mathcal{Z}^{(k)} = \{Z_1^{(k)}, Z_2^{(k)}, \cdots, Z_n^{(k)}\}$$

is quasi–associated, and let $\mathcal{Z} = \{Z_1, Z_2, \cdots, Z_n\}$. If $(Z_1^{(k)}, \cdots, Z_n^{(k)})$ converges in distribution to $(Z_1, \cdots, Z_n)$, then it follows that the collection $\mathcal{Z}$ is quasi–associated. In other words, quasi–association is preserved under the action of passage to the limit.

Our next result states that certain collections of non–overlapping increments of a stable process in random scenery are quasi–associated.

Proposition 3.1. *Let G be a stable process in random scenery, and let $0 \leqslant s_1 < t_1 \leqslant s_2 < t_2 \leqslant \cdots \leqslant s_m < t_m$. Then the collection*

$$\{G(t_1) - G(s_1), G(t_2) - G(s_2), \cdots, G(t_m) - G(s_m)\}$$

is quasi–associated.

Remark 3.2. At present, it is not known whether the collection

$$\{G(t_1) - G(s_1), G(t_2) - G(s_2), \cdots, G(t_m) - G(s_m)\}$$

is associated.

Proof. We will prove a provisional form of this result for random walk in random scenery. Let $n, m \geqslant 1$ be integers and consider the collection of random variables

$$\{y(s_0), \cdots, y(s_{n-1}), y(s_n), \cdots, y(s_{n+m-1})\}.$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^m \to \mathbb{R}$ be measurable and coordinatewise nondecreasing. Since the random scenery is independent of the random walk,

$$\begin{aligned} &\mathbb{E}\big[f\big(y(s_0), \cdots, y(s_{n-1})\big)g\big(y(s_n), \cdots, y(s_{n+m-1})\big)\big] \\ &\qquad = \sum \mathbb{E}\big[f\big(y(0), \cdots, y(\alpha_{n-1})\big)g\big(y(\alpha_n), \cdots, y(\alpha_{n+m-1})\big)\big] \\ &\qquad\qquad \times \mathbb{P}(s_0 = 0, s_1 = \alpha_1, \cdots, s_{n+m-1} = \alpha_{n+m-1}), \end{aligned} \tag{3.3}$$

where the sum extends over all choices of $\alpha_i \in \mathbb{Z}$, $1 \leqslant i \leqslant n+m-1$. By Esary, Proschan, and Walkup (1967), the collection of random variables

$$\{y(0), y(\alpha_1), \cdots, y(\alpha_{n+m-1})\}$$

is associated; thus, by (3.2), we obtain

$$\begin{aligned} &\mathbb{E}\big[f\big(y(0), \cdots, y(\alpha_{n-1})\big)g\big(y(\alpha_n), \cdots, y(\alpha_{n+m-1})\big)\big] \\ &\qquad \geqslant \mathbb{E}\big[f\big(y(0), \cdots, y(\alpha_{n-1})\big)\big]\mathbb{E}\big[g\big(y(\alpha_n), \cdots, y(\alpha_{n+m-1})\big)\big]. \end{aligned} \tag{3.4}$$

Since the scenery is stationary,

$$\mathbb{E}\big(g\big(y(\alpha_n), \cdots, y(\alpha_{n+m-1})\big)\big) = \mathbb{E}\big(g\big(y(0), \cdots, y(\alpha_{n+m-1} - \alpha_n)\big)\big). \tag{3.5}$$

On the other hand, since s is a random walk,

$$\begin{aligned} &\mathbb{P}(s_0 = 0, \cdots, s_{n+m-1} = \alpha_{n+m-1}) \\ &\qquad = \mathbb{P}(s_0 = 0, \cdots, s_{n-1} = \alpha_{n-1})\mathbb{P}(s_1 = \alpha_n - \alpha_{n-1}) \\ &\qquad\qquad \times \mathbb{P}(s_0 = 0, s_1 = \alpha_{n+1} - \alpha_n, \cdots, s_{m-1} = \alpha_{n+m-1} - \alpha_n). \end{aligned} \tag{3.6}$$

Insert (3.4), (3.5), and (3.6) into (3.3). If we sum first on $\alpha_{n+1}, \cdots, \alpha_{n+m-1}$, and then on the remaining indices, we obtain

$$\begin{aligned} &\mathbb{E}\big[f\big(y(s_0), \cdots, y(s_{n-1})\big)g\big(y(s_n), \cdots, y(s_{n+m-1})\big)\big] \\ &\qquad \geqslant \mathbb{E}\big[f\big(y(s_0), \cdots, y(s_{n-1})\big)\big]\mathbb{E}\big[g\big(y(s_0), \cdots, y(s_{m-1})\big)\big] \end{aligned} \tag{3.7}$$

Finally, since s has stationary increments and y and s are independent,

$$\mathbb{E}\big[g\big(y(s_0),\cdots,y(s_{m-1})\big)\big]=\mathbb{E}\big[g\big(y(s_n),\cdots,y(s_{n+m-1})\big)\big].$$

Which, when inserted into (3.7), yields

$$\mathbb{E}\big[f\big(y(s_0),\cdots,y(s_{n-1})\big)g\big(y(s_n),\cdots,y(s_{n+m-1})\big)\big] \geqslant \mathbb{E}\big[f\big(y(s_0),\cdots,y(s_{n-1})\big)\big]\mathbb{E}\big[g\big(y(s_n),\cdots,y(s_{n+m-1})\big)\big].$$

This argument demonstrates that, for any integer N, the collection $\{y(s_0),\cdots,y(s_N)\}$ is quasi–associated. Since association is preserved under ordered blocking, the collection

$$\Big\{n^{-\delta}\big(g_{[nt_1]}-g_{[ns_1]}\big),n^{-\delta}\big(g_{[nt_2]}-g_{[ns_2]}\big),\cdots,n^{-\delta}\big(g_{[nt_m]}-g_{[ns_m]}\big)\Big\}$$

is also quasi–associated. By the result of Kesten and Spitzer, the random vector

$$\Big(n^{-\delta}\big(g_{[nt_1]}-g_{[ns_1]}\big),n^{-\delta}\big(g_{[nt_2]}-g_{[ns_2]}\big),\cdots,n^{-\delta}\big(g_{[nt_m]}-g_{[ns_m]}\big)\Big)$$

converges in distribution to

$$\big(G(t_1)-G(s_1),G(t_2)-G(s_2),\cdots,G(t_m)-G(s_m)\big),$$

which finishes the proof, since quasi–association is preserved under passage to the limit. $\diamond$

§4. A correlation inequality

Given random variables U and V defined on a common probability space and real numbers a and b, let

$$\mathcal{Q}_{U,V}(a,b)\triangleq\mathbb{P}\big(U>a,V>b\big)-\mathbb{P}\big(U>a\big)\,\mathbb{P}\big(V>b\big).$$

Following Lehmann (1966), we will say that U and V are *positively quadrant dependent* provided that $\mathcal{Q}_{U,V}(a,b)\geqslant 0$ for all $a,b\in\mathbb{R}$. In Esary, Proschan, and Walkup (1967), it is shown that U and V are positively quadrant dependent if and only if

$$\mathrm{Cov}\big(f(U),g(V)\big)\geqslant 0$$

for all nondecreasing measurable functions $f, g : \mathbb{R} \to \mathbb{R}$. Thus U and V are positively quadrant dependent if and only if the collection $\{U, V\}$ is quasi–associated.

The main result of this section is a form of the Kochen–Stone Lemma (see Kochen and Stone (1964)) for pairwise positively quadrant dependent random variables.

Proposition 4.1. *Let $\{Z_k : k \geqslant 1\}$ be a sequence of pairwise positively quadrant dependent random variables with bounded second moments. If*

(a) $\displaystyle\sum_{k=1}^{\infty} \mathbb{P}(Z_k \geqslant 0) = \infty$

and

(b) $\displaystyle\liminf_{n\to\infty} \frac{\sum_{1\leqslant j<k\leqslant n} \mathrm{Cov}(Z_j, Z_k)}{\Big(\sum_{k=1}^{n} \mathbb{P}(Z_k \geqslant 0)\Big)^2} = 0,$

then $\limsup_{n\to\infty} Z_n \geqslant 0$ *almost surely.*

Before proving this result, we will develop some notation and prove a technical lemma. Let $C_b^2(\mathbb{R}^2)$ denote the set of all functions from $\mathbb{R}^2$ to $\mathbb{R}$ with bounded and continuous mixed second–order partial derivatives. For $f \in C_b^2(\mathbb{R}^2, \mathbb{R})$, let

$$M(f) \triangleq \sup_{(s,t)\in\mathbb{R}^2} |f_{xy}(s,t)|$$

The above is not a norm, as it cannot distinguish between affine transformations of f.

Lemma 4.2. *Let X, Y, $\tilde{X}$, and $\tilde{Y}$ be random variables with bounded second moments, defined on a common probability space. Let $X \stackrel{d}{=} \tilde{X}$, let $Y \stackrel{d}{=} \tilde{Y}$, and let $\tilde{X}$ and $\tilde{Y}$ be independent. Then, for each $f \in C_b^2(\mathbb{R}^2, \mathbb{R})$,*

(a) $\displaystyle\mathbb{E}(f(X,Y)) - \mathbb{E}(f(\tilde{X},\tilde{Y})) = \int_{\mathbb{R}^2} f_{xy}(s,t)\mathcal{Q}_{X,Y}(s,t)\, ds\, dt.$

(b) *If, in addition, X and Y are positively quadrant dependent, then*

$$|\mathbb{E}(f(X,Y)) - \mathbb{E}(f(\tilde{X},\tilde{Y}))| \leqslant M(f)\mathrm{Cov}(X,Y).$$

Remark. This lemma is a simple generalization of a result attributed to Hoeffding (see Lemma 2 of Lehmann (1966)), which states that

$$\mathrm{Cov}(X,Y) = \int_{\mathbb{R}^2} \mathcal{Q}_{X,Y}(s,t)dsdt, \tag{4.1}$$

whenever the covariance in question is defined.

Proof. Without loss of generality, we may assume that (X,Y) and $(\tilde{X},\tilde{Y})$ are independent. Let

$$I(u,x)=\begin{cases}1 & \text{if } u<x;\\ 0 & \text{if } u\geqslant x.\end{cases}$$

Then

$$\mathbb{E}\big(|X-\tilde{X}||Y-\tilde{Y}|\big)=\mathbb{E}\int_{\mathbb{R}^2}|I(u,X)-I(u,\tilde{X})||I(v,Y)-I(v,\tilde{Y})|dudv \tag{4.2}$$

Observe that

$$\begin{aligned}&\mathbb{E}\big(f(X,Y)-f(\tilde{X},Y)+f(\tilde{X},\tilde{Y})-f(X,\tilde{Y})\big)\\ &\qquad=\mathbb{E}\int_{\mathbb{R}^2}f_{xy}(u,v)\big(I(u,X)-I(u,\tilde{X})\big)\big(I(v,Y)-I(v,\tilde{Y})\big)dudv\end{aligned}$$

The integrand on the right is bounded by

$$M(f)|I(u,X)-I(u,\tilde{X})||I(v,Y)-I(v,\tilde{Y})|,$$

and by (4.2) we may interchange the order of integration, which yields

$$\mathbb{E}\big(f(X,Y)\big)-\mathbb{E}\big(f(\tilde{X},\tilde{Y})\big)=\int_{\mathbb{R}^2}f_{xy}(u,v)\mathcal{Q}_{X,Y}(u,v)dudv,$$

demonstrating item (a).

If X and Y are positively quadrant dependent, then $\mathcal{Q}_{X,Y}$ is nonnegative, and item (b) follows from item (a), an elementary bound, and (4.1).

Proof of Proposition 4.1. Given $\varepsilon>0$, let $\varphi:\mathbb{R}\to\mathbb{R}$ be an infinitely differentiable, nondecreasing function satisfying: $\varphi(x)=0$ if $x\leqslant-\varepsilon$, $\varphi(x)=1$ if $x\geqslant 0$, and $\varphi'(x)>0$ if $x\in(-\varepsilon,0)$. Given integers $n\geqslant m\geqslant 1$, let

$$B_{m,n}=\sum_{k=m}^{n}\varphi(Z_k).$$

Since $\mathbf{1}(x\geqslant 0)\leqslant\varphi(x)$, it follows that

$$\sum_{k=1}^{n}\mathbb{P}(Z_k\geqslant 0)\leqslant\sum_{k=1}^{n}\mathbb{E}(\varphi(Z_k))=\mathbb{E}(B_{1,n}) \tag{4.3}$$

In particular, by item (a) of this proposition, we may conclude that $\mathbb{E}(B_{1,n}) \to \infty$ as $n \to \infty$.

The main observation is that

$$\{B_{m,n} > 0\} = \cup_{k=m}^{n} \{Z_k > -\varepsilon\}.$$

Hence, by the Cauchy–Schwarz inequality,

$$\mathbb{P}\big(\cup_{k=m}^{n} \{Z_k > -\varepsilon\}\big) \geqslant \frac{\big(\mathbb{E}(B_{m,n})\big)^2}{\mathbb{E}(B_{m,n}^2)}.$$

Since $\mathbb{E}(B_{1,n}) \to \infty$ as $n \to \infty$, it is evident that

$$\lim_{n\to\infty} \frac{\big(E(B_{m,n})\big)^2}{\big(E(B_{1,n})\big)^2} = 1. \tag{4.4}$$

In addition, we will show that

$$\liminf_{n\to\infty} \frac{\mathbb{E}(B_{m,n}^2)}{\big(\mathbb{E}(B_{1,n})\big)^2} \leqslant 1. \tag{4.5}$$

From (4.4) and (4.5) we may conclude that $\mathbb{P}(\cup_{k=m}^{\infty}\{Z_k > -\varepsilon\}) = 1$, and, since this is true for each $m \geqslant 1$, it follows that $\mathbb{P}(Z_k > -\varepsilon \text{ i.o.}) = 1$; hence,

$$\limsup_{n\to\infty} Z_n > -\varepsilon \qquad \text{a.s.}$$

Since $\varepsilon > 0$ is arbitrary, this gives the desired conclusion.

We are left to prove (4.4). To this end, observe that

$$\begin{aligned}\mathbb{E}(B_{m,n}^2) &= \sum_{k=1}^{n} \mathbb{E}(\varphi^2(Z_k)) + 2 \sum_{m \leqslant j<k \leqslant n} \mathbb{E}(\varphi(Z_j)\varphi(Z_k)) \\ &\leqslant \mathbb{E}(B_{1,n}) + 2 \sum_{1 \leqslant j<k \leqslant n} \mathbb{E}(\varphi(Z_j)\varphi(Z_k))\end{aligned}$$

Thus, by Lemma 4.2, there exists a positive constant $C = C(\varepsilon)$ such that

$$\begin{aligned}\mathbb{E}(B_{m,n}^2) &\leqslant \mathbb{E}(B_{1,n}) + 2 \sum_{1 \leqslant j<k \leqslant n} \mathbb{E}(\varphi(Z_j))\mathbb{E}(\varphi(Z_k)) + C \sum_{1 \leqslant j<k \leqslant n} \mathrm{Cov}(Z_j, Z_k) \\ &\leqslant \mathbb{E}(B_{1,n}) + \big(\mathbb{E}(B_{1,n})\big)^2 + C \sum_{1 \leqslant j<k \leqslant n} \mathrm{Cov}(Z_j, Z_k).\end{aligned}$$

Upon dividing both sides of this inequality by $\big(\mathbb{E}(B_{1,n})\big)^2$ and using (4.3), we obtain

$$\frac{\mathbb{E}(B_{m,n}^2)}{\big(\mathbb{E}(B_{1,n})\big)^2} \leqslant o(1) + 1 + C\frac{\sum_{1\leqslant j<k\leqslant n}\mathrm{Cov}(Z_j, Z_k)}{\Big(\sum_{k=1}^n \mathbb{P}(Z_k \geqslant 0)\Big)^2}$$

which, by condition (b) of this proposition, yields

$$\liminf_{n\to\infty} \frac{\mathbb{E}(B_{m,n}^2)}{\big(\mathbb{E}(B_{1,n})\big)^2} \leqslant 1,$$

which is (4.5). ◇

§5. Probability calculations

In this section we will prove an assortment of probability estimates for Brownian motion in random scenery and related stochastic processes. This section contains two main results, the first of which is a large deviation estimate for $\mathbb{P}(G_1 > t)$. You will recall that α $(1 < \alpha \leqslant 2)$ is the index of the Lévy process X and that $\delta = 1 - (2\alpha)^{-1}$.

Theorem 5.1. *There exists a positive real number $\gamma = \gamma(\alpha)$ such that*

$$\lim_{\lambda\to\infty} \lambda^{-\frac{2\alpha}{1+\alpha}} \ln \mathbb{P}(G_1 \geqslant \lambda) = -\gamma.$$

As the proof of this theorem will show, we can shed some light on the dependence of γ on α (see Remark 5.7).

The second main result of this section is an estimate for the covariance of certain non–overlapping increments of G.

Theorem 5.2. *Fix $\lambda \in (0, 1)$. Let s, t, u, and v be nonnegative real numbers satisfying*

$$s \leqslant \lambda t < t \leqslant u \leqslant \lambda v < v.$$

Then

$$\mathrm{Cov}\Big(\frac{G(t) - G(s)}{(t-s)^\delta}\ ,\ \frac{G(v) - G(u)}{(v-u)^\delta}\Big) \leqslant \frac{\chi^{1/\alpha}\Gamma(1/\alpha)}{(1-\lambda)^{1/(2\alpha)}(\alpha-1)\pi}\Big(\frac{t}{v}\Big)^{1/(2\alpha)}.$$

First we will attend to the proof of Theorem 5.1, which will require some prefatory definitions and lemmas. For each $t \geqslant 0$, let

$$V_t = \int_{\mathbb{R}} (L_t^x)^2 dx$$
$$S_t = \sqrt{V_t}$$

For each $t \geqslant 0$, V_t is the conditional variance of G_t given $\mathfrak{X}_t$, that is,

$$V_t = \mathbb{E}(G_t^2 \mid \mathfrak{X}_t).$$

V and S inherit scaling laws from G. For future reference let us note that

$$\{S_{ct} : t \geqslant 0\} \stackrel{d}{=} \{c^\delta S_t : t \geqslant 0\}. \tag{5.1}$$

A significant part of our work will be an asymptotic analysis of the moment generating function of S_1. For each $\xi \geqslant 0$, let

$$\mu(\xi) = \mathbb{E}\big(\exp(\xi S_1)\big).$$

The next few lemmas are directed towards demonstrating that there is a positive real number κ such that

$$\lim_{t\to\infty} t^{-\frac{1}{\delta}} \ln \mu(t) = \kappa. \tag{5.2}$$

To this end, our first lemma concerns the asymptotic behavior of certain integrals. Fix $p > 1$ and $c > 0$ and, for each $t \geqslant 0$, let

$$g(t) = t - ct^p.$$

Let t_0 denote the unique stationary point of g on $[0, \infty)$ and, for $\xi \geqslant 0$, let

$$I(\xi) = \int_0^\infty \exp\left(\xi\lambda - c\lambda^p\right) d\lambda.$$

Lemma 5.3. *As* $\xi \to \infty$,

$$I(\xi) \sim \sqrt{\frac{2\pi}{|g''(t_0)|}}\, \xi^{\frac{2-p}{2(p-1)}} \, \exp\left(\xi^{\frac{p}{p-1}} g(t_0)\right).$$

Proof. Consider the change of variables

$$t = \xi^{-\frac{1}{p-1}} \lambda.$$

Under this assignment, and by the definition of g, we obtain

$$\xi\lambda - c\lambda^p = \xi^{\frac{p}{p-1}} g(t).$$

Thus

$$I(\xi) = \xi^{\frac{1}{p-1}} \int_0^\infty \exp\Big(\xi^{\frac{p}{p-1}} g(t)\Big)\, dt.$$

The asymptotic relation follows by the method of Laplace (see, for example, pp. 36–37 of Erdelyi (1956) for a discussion of this method). ◇

Our next lemma contains a provisional form of (5.2).

Lemma 5.4. *There exist positive real numbers* $c_1 = c(\alpha)$ *and* $c_2 = c_2(\alpha)$ *such that, for each* $t \geqslant 0$,

$$c_1 t^{1/\delta} \leqslant \ln \mu(t) \leqslant c_2 t^{1/\delta}.$$

Proof. For simplicity, let

$$L_1^* = \sup_{x\in\mathbb{R}} L_1^x \qquad \text{and} \qquad X_1^* = \sup_{0\leqslant s\leqslant 1} |X_s|.$$

First we will prove the following comparison result: with probability one,

$$\frac{1}{2X_1^*} \leqslant V_1 \leqslant L_1^*. \tag{5.3}$$

Both bounds are a consequence of the occupation density formula (1.4). Since $\int_{\mathbb{R}} L_1^x dx = 1$,

$$\begin{aligned} V_1 &= \int_{\mathbb{R}} (L_1^x)^2 dx \\ &\leqslant L_1^* \int_{\mathbb{R}} L_1^x dx \\ &= L_1^*, \end{aligned}$$

which is the upper bound. To obtain the lower bound, we use the Cauchy–Schwartz inequality. Let $m(\cdot)$ denote Lebesgue measure on $\mathbb{R}$ and observe that

$$\begin{aligned} 1 &= \int_{\mathbb{R}} L_1^x \mathbf{1}\{x : |x| \leqslant X_1^*\} dx \\ &\leqslant \big(V_1\, m\big(\{x : |x| \leqslant X_1^*\}\big)\big)^{1/2} \\ &\leqslant (V_1\, 2X_1^*)^{1/2}, \end{aligned}$$

which is the lower bound. As a consequence of (5.3), for each $\lambda > 0$,

$$\mathbb{P}(X_1^* \leqslant (2\lambda)^{-1}) \leqslant \mathbb{P}(V_1 \geqslant \lambda) \leqslant \mathbb{P}(L_1^* \geqslant \lambda)$$

Combining this with Proposition 10.3 of Fristedt (1974) and Theorem 1.4 of Lacey (1990), we see that there are two positive real numbers $c_3 = c_3(\alpha)$ and $c_4 = c_4(\alpha)$ such that, for each $\lambda \geqslant 0$,

$$e^{-c_3\lambda^\alpha} \leqslant \mathbb{P}(V_1 > \lambda) \leqslant e^{-c_4\lambda^\alpha}.$$

Equivalently, for each $\lambda \geqslant 0$,

$$e^{-c_3\lambda^{2\alpha}} \leqslant \mathbb{P}(S_1 > \lambda) \leqslant e^{-c_4\lambda^{2\alpha}}.$$

Since, after an integration by parts,

$$\mu(\xi) = \xi \int_0^\infty e^{\xi\lambda} \mathbb{P}(S_1 > \lambda) d\lambda,$$

it follows that

$$\xi \int_0^\infty \exp\left(\xi\lambda - c_3\lambda^{2\alpha}\right) d\lambda \leqslant \mu(\xi) \leqslant \xi \int_0^\infty \exp\left(\xi\lambda - c_4\lambda^{2\alpha}\right) d\lambda.$$

We obtain the desired bounds on $\mu(\xi)$ by an appeal to Lemma 5.3 and some algebra. ◇

Lemma 5.5. *There exists a positive real number κ such that*

$$\lim_{t\to\infty} \frac{\ln \mu(t)}{t^{1/\delta}} = \kappa.$$

Proof. Let

$$\kappa = \inf_{t \geqslant 1} \frac{\ln \mu(t)}{t^{1/\delta}}.$$

By Lemma 5.4, $\kappa \in (0, \infty)$.

We will finish the proof with a subadditivity argument. For each $u \geqslant 0$ and $t \geqslant 0$, let

$$X^u(t) \triangleq X(t+u) - X(u)$$

From the elementary properties of Lévy processes, $X^u = \{X^u(t) : t \geqslant 0\}$ is equivalent to X and is independent of $\mathfrak{X}_u$. Let $L(X^u)$ denote the process of local times of X^u. Then, for each $t \geqslant 0$ and $x \in \mathbb{R}$,

$$L_t^x(X^u) = L_{t+u}^{x+X(u)} - L_u^{x+X(u)}. \tag{5.4}$$

Let

$$\tilde{S}_t = \int_{\mathbb{R}} \big(L_t^x(X^u)\big)^2 dx.$$

Since $L(X^u)$ is equivalent to L and is independent of $\mathfrak{X}_u$, $\tilde{S}$ is equivalent to S and independent of $\mathfrak{X}_u$. Moreover, by a change of variables, with probability one

$$\int_{\mathbb{R}} \big(L_{t+u}^x - L_u^x\big)^2 dx = \int_{\mathbb{R}} \big(L_t^x(X^u)\big)^2 dx.$$

Consequently, by Minkowski's inequality, with probability one

$$\begin{aligned} S_{t+u} &\leqslant \sqrt{\int_{\mathbb{R}} (L_u^x)^2 dx} + \sqrt{\int_{\mathbb{R}} (L_{t+u}^x - L_u^x)^2 dx} \\ &= S_u + \tilde{S}_t. \end{aligned}$$

By the scaling law for S (see (5.1)) and the independence of $\tilde{S}_t$ and S_u,

$$\begin{aligned} \mu\big((t+u)^\delta\big) &= \mathbb{E}\big(\exp((t+u)^\delta S_1)\big) \\ &= \mathbb{E}\big(\exp(S_{t+u}\big) \\ &\leqslant \mathbb{E}\big(\exp(\tilde{S}_t + S_u)\big) \\ &\leqslant \mathbb{E}\big(\exp(S_t)\big)\mathbb{E}\big(\exp(S_u)\big) \\ &= \mu(t^\delta)\mu(u^\delta) \end{aligned}$$

This demonstrates that the function $t \mapsto \ln \mu(t^\delta)$ is subadditive. By a classical result,

$$\lim_{t \to \infty} \frac{\ln \mu(t^\delta)}{t} = \kappa,$$

which, up to a minor modification in form, proves the lemma in question. ◇

Corollary 5.6. *There exists a real number* $\zeta \in (0, \infty)$ *such that*

$$\lim_{\lambda \to \infty} \lambda^{-\alpha} \ln \mathbb{P}(V_1 \geqslant \lambda) = -\zeta.$$

Proof. By the result of Davies (1976) and Lemma 5.5, it follows that there exists a positive real number ζ such that

$$\lim_{\lambda \to \infty} \lambda^{-2\alpha} \ln \mathbb{P}(S_1 \geqslant \lambda) = -\zeta.$$

Since $V_1 = S_1^2$, the result follows. ◇

Finally, we give the proof of Theorem 5.1.

Proof of Theorem 5.1. Let

$$\Phi(s) = \int_s^\infty e^{-u^2/2} \frac{du}{\sqrt{2\pi}}$$

For each $t \geqslant 0$, $\mathbb{P}(G_1 \geqslant \lambda \,|\, V_1 = z) = \Phi(\lambda z^{-1/2})$; thus,

$$\begin{aligned} \mathbb{P}(G_1 \geqslant \lambda) &= \int_0^\infty \mathbb{P}(G_1 \geqslant \lambda \,|\, V_1 = z) \mathbb{P}(V_1 \in dz) \\ &= \int_0^\infty \Phi(\lambda z^{-1/2}) \mathbb{P}(V_1 \in dz) \end{aligned} \tag{5.5}$$

For each $u > 0$, let

$$f(u) = \frac{1}{2u} + \zeta u^\alpha.$$

Let

$$u^* = (2\alpha\zeta)^{-\frac{1}{1+\alpha}}$$

and note that u^* is the unique stationary point of f on the set $(0, \infty)$ and that $f(u^*) \leqslant f(u)$ for all $u > 0$. For future reference, we observe that

$$f(u^*) = \frac{\alpha + 1}{2\alpha} (2\alpha\zeta)^{\frac{1}{1+\alpha}}. \tag{5.6}$$

Let $0 < A < u^* < B < \infty$ be chosen so that

$$\frac{1}{2A} \wedge \zeta B^\alpha > f(u^*)$$

and let δ be chosen so that

$$0 < \delta < \frac{1}{2A} \wedge \zeta B^\alpha - f(u^*). \tag{5.7}$$

Let $A = x_0 < x_1 < \cdots < x_n = B$ be a partition of $[A, B]$ which is fine enough so that

$$\zeta\big(x_k^\alpha - x_{k-1}^\alpha\big) < \delta. \tag{5.8}$$

Moreover, we require that $x_i = u^*$ for some index $0 < i < n$.

For each $\lambda > 0$ and each $1 \leqslant k \leqslant n$, let

$$s_k = s_k(\lambda) = x_k \lambda^{\frac{2}{1+\alpha}}.$$

We have

$$\mathbb{P}(G_1 \geqslant \lambda) = \int_0^{s_0} \Phi(\lambda z^{-1/2}) \mathbb{P}(V_1 \in dz) + \int_{s_n}^\infty \Phi(\lambda z^{-1/2}) \mathbb{P}(V_1 \in dz) + \sum_{k=1}^n \int_{s_{k-1}}^{s_k} \Phi(\lambda z^{-1/2}) \mathbb{P}(V_1 \in dz).$$

Since $z \mapsto \Phi(\lambda z^{-1/2})$ is increasing, it follows that

$$\int_0^{s_0} \Phi(\lambda z^{-1/2}) \mathbb{P}(V_1 \in dz) \leqslant \Phi(\lambda s_0^{-1/2}) = \Phi\big(A^{-1/2} \lambda^{\frac{\alpha}{1+\alpha}}\big).$$

By elementary properties, we have

$$\lim_{\lambda \to \infty} \lambda^{-\frac{2\alpha}{1+\alpha}} \ln \Phi\big(A^{-1/2} \lambda^{\frac{\alpha}{1+\alpha}}\big) = -\frac{1}{2A}. \tag{5.9}$$

Similar considerations lead us to conclude that

$$\int_{s_n}^\infty \Phi(\lambda z^{-1/2}) \mathbb{P}(V_1 \in dz) \leqslant \mathbb{P}(V_1 \geqslant s_n) = \mathbb{P}\big(V_1 \geqslant B \lambda^{\frac{2}{1+\alpha}}\big).$$

By Corollary 5.6, we conclude

$$\lim_{\lambda \to \infty} \lambda^{-\frac{2\alpha}{1+\alpha}} \ln \mathbb{P}\big(V_1 \geqslant B \lambda^{\frac{2}{1+\alpha}}\big) = -\zeta B^\alpha. \tag{5.10}$$

Finally, for $1 \leqslant k \leqslant n$,

$$\int_{s_{k-1}}^{s_k} \Phi(\lambda z^{-1/2}) \mathbb{P}(V_1 \in dz) \geqslant \Phi(\lambda s_k^{-1/2}) \mathbb{P}(V_1 \geqslant s_{k-1}) \\ = \Phi\big(x_k^{-1/2} \lambda^{\frac{\alpha}{1+\alpha}}\big) \mathbb{P}\big(V_1 \geqslant x_{k-1} \lambda^{\frac{2\alpha}{1+\alpha}}\big).$$

Thus, by Corollary 5.6 and (5.8),

$$\lim_{\lambda\to\infty} \lambda^{-\frac{2\alpha}{1+\alpha}} \ln \Phi\big(x_k^{-1/2} \lambda^{\frac{\alpha}{1+\alpha}}\big) \mathbb{P}\big(V_1 \geqslant x_{k-1} \lambda^{\frac{\alpha}{1+\alpha}}\big) = -\frac{1}{2x_k} - \zeta x_{k-1}^{\alpha} \\ \leqslant -f(x_k) + \delta. \tag{5.11}$$

By (5.9), (5.10) and (5.11), we obtain

$$\limsup_{\lambda\to\infty} \lambda^{-\frac{2\alpha}{1+\alpha}} \ln \mathbb{P}(G_1 \geqslant \lambda) \leqslant -\min\left\{\frac{1}{2A}, \zeta B^{\alpha}, \min_{1\leqslant k\leqslant n}\{f(x_k) - \delta\}\right\} \\ = -f(u^*) + \delta,$$

where we have used (5.7) and the definition of u^* to obtain this last equality. Letting $\delta \to 0$, we obtain

$$\limsup_{\lambda\to\infty} \lambda^{-\frac{2\alpha}{1+\alpha}} \ln \mathbb{P}(G_1 \geqslant \lambda) = -f(u^*). \tag{5.12}$$

To obtain a lower bound, let

$$a = a(\lambda) = u^* \lambda^{\frac{2}{1+\alpha}}$$

and note that

$$\mathbb{P}(G_1 \geqslant \lambda) \geqslant \int_a^\infty \Phi(\lambda z^{-1/2}) \mathbb{P}(V_1 \in dz) \\ \geqslant \Phi(\lambda a^{-1/2}) \mathbb{P}(V_1 \geqslant a) \\ = \Phi\big((u^*)^{-1/2} \lambda^{\frac{\alpha}{1+\alpha}}\big) \mathbb{P}\big(V_1 \geqslant u^* \lambda^{\frac{2}{1+\lambda}}\big)$$

Consequently, by Corollary 5.6,

$$\lim_{\lambda\to\infty} \lambda^{-\frac{2\alpha}{1+\alpha}} \ln \Phi\big((u^*)^{-1/2} \lambda^{\frac{\alpha}{1+\alpha}}\big) \mathbb{P}\big(V_1 \geqslant u^* \lambda^{\frac{2}{1+\lambda}}\big) = -\frac{1}{2u^*} - \zeta {u^*}^{\alpha} \\ = -f(u^*)$$

It follows that

$$\liminf_{\lambda\to\infty} \lambda^{-\frac{2\alpha}{1+\alpha}} \ln \mathbb{P}(G_1 \geqslant \lambda) = -f(u^*). \tag{5.13}$$

Combining (5.12) and (5.13) and recalling (5.6), we obtain the desired result. ◇

Remark 5.7. As the proof of Theorem 5.1 demonstrates, we have actually shown that

$$\gamma = f(u^*) = \frac{\alpha+1}{2\alpha}(2\alpha\zeta)^{\frac{1}{1+\alpha}}. \tag{5.14}$$

At present, we have only shown that ζ is a positive real number. However, in certain cases (for example, Brownian motion) it might be possible to determine the precise value of ζ, in which case the value of γ will be given by (5.14).

The remainder of this section is directed towards a proof of Theorem 5.2. First we will make a connection between Brownian motion in random scenery and classical *β–energy.* Suppose μ is a probability measure on $\mathbb{R}^1$ endowed with its Borel sets. Then, for any $\beta > 0$, we define the β–energy of μ as

$$\mathcal{E}_\beta(d\mu) = \int_{\mathbb{R}^2} |x-y|^{-\beta} d\mu(x)\ d\mu(y).$$

Lemma 5.8. *For any $s, r > 0$,*

$$\mathbb{E}G(r)G(s) = \frac{\chi^{1/\alpha}\Gamma(1/\alpha)}{\alpha\pi}\int_0^r\int_0^s |x-y|^{-1/\alpha}dx\ dy.$$

In particular,

$$\mathbb{E}G^2(r) = \frac{r^{2\delta}\chi^{1/\alpha}\Gamma(1/\alpha)}{\alpha\pi}\mathcal{E}_\alpha(dx|_{[0,1]}).$$

Remark. Let us mention the following calculation as an aside:

$$\mathcal{E}_\alpha(dx|_{[0,1]}) = \frac{2\alpha^2}{(\alpha-1)(2\alpha-1)}.$$

Proof. The proof of Lemma 5.8 involves some Fourier analysis. By (1.3) and properties of Lévy processes, for all $\xi \in \mathbb{R}$ and all $r, s > 0$,

$$\mathbb{E}e^{i\xi\{X(r)-X(s)\}} = e^{-|\xi|^\alpha|r-s|/\chi}. \tag{5.15}$$

Let $\psi_r(x) = L_r^x$ and note that $\mathbb{E}G(r)G(s) = \mathbb{E}\int \psi_r(x)\psi_s(x)dx$. By Parseval's identity,

$$\mathbb{E}G(r)G(s) = \frac{1}{2\pi}\mathbb{E}\int \widehat{\psi}_r(\xi)\overline{\widehat{\psi}_s}d\xi. \tag{5.16}$$

However, by the occupation density formula (1.4),

$$\widehat{\psi}_r(\xi) = \int e^{i\xi x} L_r^x dx = \int_0^r e^{i\xi X(u)} du.$$

Therefore, by (5.15)

$$\begin{aligned} \mathbb{E}\widehat{\psi}_r(\xi)\overline{\widehat{\psi}_s(\xi)} &= \mathbb{E}\int_0^r \int_0^s e^{i\xi\{X(u)-X(v)\}} du\ dv \\ &= \int_0^r \int_0^s e^{-|\xi|^\alpha |u-v|/\chi} du\ dv. \end{aligned}$$

By (5.16), Fubini's theorem, and symmetry,

$$\begin{aligned} \mathbb{E}(G(r)G(s)) &= \frac{1}{\pi}\int_0^r \int_0^s \int_0^\infty e^{-|\xi|^\alpha |u-v|/\chi} d\xi\ du\ dv \\ &= \frac{\chi^{1/\alpha}\Gamma(1/\alpha)}{\alpha\pi} \int_0^r \int_0^s |u-v|^{-1/\alpha} du\ dv, \end{aligned}$$

which proves the lemma. ◇

Proof of Theorem 5.2. Since G is a centered process,

$$\begin{aligned} \mathcal{C} &\triangleq \mathrm{Cov}\big(G(t)-G(s)\ ,\ G(v)-G(u)\big) \\ &= \mathbb{E}\big(G(t)-G(s)\big)\big(G(v)-G(u)\big) \\ &= \mathbb{E}\big(G(t)G(v)\big) - \mathbb{E}\big(G(t)G(u)\big) - \mathbb{E}\big(G(s)G(v)\big) + \mathbb{E}\big(G(s)G(u)\big). \end{aligned}$$

By Lemma 5.8 and some algebra, this covariance may be expressed compactly as

$$\mathcal{C} = \frac{\chi^{1/a}\Gamma(1/\alpha)}{\alpha\pi} \int_s^t \int_u^v |x-y|^{-1/\alpha} dx\ dy. \tag{5.17}$$

Define $f(b) = \int_u^v (a-b)^{-1/\alpha} da$ and note that, for $b \leqslant u$, $f(b) \leqslant f(u)$. In other words,

$$\int_u^v (a-b)^{-1/\alpha} da \leqslant \int_u^v (a-u)^{-1/\alpha} da = \Big(\frac{\alpha}{\alpha-1}\Big)(v-u)^{(\alpha-1)/\alpha}.$$

Therefore, by (5.17),

$$\mathcal{C} \leqslant \frac{\chi^{1/\alpha}\Gamma\big(1/\alpha\big)}{(\alpha-1)\pi}(v-u)^{(\alpha-1)/\alpha}(t-s).$$

A symmetric analysis shows that

$$\mathcal{C} \leqslant \frac{\chi^{1/\alpha}\Gamma\big(1/\alpha\big)}{(\alpha-1)\pi}(t-s)^{(\alpha-1)/\alpha}(v-u).$$

Together with (5.17), we have

$$\begin{aligned}\mathcal{C} &\leqslant \frac{\chi^{1/\alpha}\Gamma\big(1/\alpha\big)}{(\alpha-1)\pi}\Big\{(t-s)^{(\alpha-1)/\alpha}(v-u)\wedge(v-u)^{(\alpha-1)/\alpha}(t-s)\Big\}\\ &= \frac{\chi^{1/\alpha}\Gamma\big(1/\alpha\big)}{(\alpha-1)\pi}(v-u)^{\delta}(t-s)^{\delta}\Big\{\Big(\frac{v-u}{t-s}\Big)^{1/(2\alpha)}\wedge\Big(\frac{t-s}{v-u}\Big)^{1/(2\alpha)}\Big\}\\ &\leqslant \frac{\chi^{1/\alpha}\Gamma\big(1/\alpha\big)}{(\alpha-1)\pi}(v-u)^{\delta}(t-s)^{\delta}\Big\{\Big(\frac{v}{t-s}\Big)^{1/(2\alpha)}\wedge\Big(\frac{t}{v-u}\Big)^{1/(2\alpha)}\Big\}.\end{aligned}$$

Recall that $0<s<t\leqslant u<v$. Since $s\leqslant \lambda t$ and $u\leqslant\lambda v$,

$$\Big(\frac{v}{t-s}\Big)\wedge\Big(\frac{t}{v-u}\Big)\leqslant(1-\lambda)^{-1}(t/v).$$

The result follows from the above and some arithmetic. ◇

§6. The proof of Theorem 1.1

For $x\in\mathbb{R}$, let

$$U(x)\triangleq\big(\ln\ln(x)\big)^{\frac{1+\alpha}{2\alpha}}$$

and recall the number γ from Theorem 5.1. In this section we will prove a stronger version of Theorem 1.1. We will demonstrate that

$$\limsup_{t\to\infty}\frac{G(t)}{t^{\delta}U(t)}=\gamma^{-\frac{1+\alpha}{2\alpha}}\qquad\text{a.s.}\tag{6.1}$$

As is customary, the proof of (6.1) will be divided into two parts: an upper–bound argument, in which we show that the limit superior is bounded above, and a lower–bound argument, in which we show that the limit superior is bounded below.

The upper–bound argument. Let $\varepsilon>0$ and define

$$\eta\triangleq\left(\frac{1+\varepsilon}{\gamma}\right)^{\frac{1+\alpha}{2\alpha}}.\tag{6.2}$$

For future reference, let us observe that

$$\gamma\eta^{\frac{2\alpha}{1+\alpha}}=1+\varepsilon.\tag{6.3}$$

Let $\rho > 1$ and, for each $k \geqslant 1$, let $n_k \triangleq \rho^k$ and

$$A_k \triangleq \Big\{\omega : \sup_{0 \leqslant s \leqslant n_k} G_s > \eta\, n_k^{\delta}\, U(n_k)\Big\}.$$

First we will show that $\mathbb{P}(A_k, \text{ i.o.}) = 0$.

By Theorem 2.1 and the fact that G is self–similar with index δ, we have

$$\mathbb{P}(A_k) \leqslant 2\mathbb{P}\big(G_1 > \eta U(n_k)\big).$$

Since $\ln\ln(n_k) \sim \ln(k)$ as $k \to \infty$, by Theorem 5.1 and (6.3), it follows that

$$\begin{aligned}\lim_{k\to\infty} \frac{\ln P(A_k)}{\ln(k)} &\leqslant -\gamma\eta^{\frac{2\alpha}{1+\alpha}} \\ &= -(1+\varepsilon)\end{aligned}$$

Let $1 < p < (1+\varepsilon)$. Then there exists an integer $N \geqslant 1$ such that, for each $k \geqslant N$, $\mathbb{P}(A_k) \leqslant k^{-p}$. Hence,

$$\sum_{k=1}^{\infty} \mathbb{P}(A_k) < \infty.$$

By the Borel–Cantelli lemma, $\mathbb{P}(A_k, \text{ i.o.}) = 0$, from which we conclude that

$$\limsup_{k\to\infty} \frac{\sup_{0 \leqslant s \leqslant n_k} G(s)}{n_k^{\delta} U(n_k)} \leqslant \eta \qquad \text{a.s.} \tag{6.4}$$

Let $t \in [n_k, n_{k+1}]$. Since $n_{k+1}/n_k = \rho$,

$$\frac{\sup_{0 \leqslant s \leqslant t} G(s)}{t^{\delta} U(t)} \leqslant \rho^{\delta}\, \frac{\sup_{0 \leqslant s \leqslant n_{k+1}} G(s)}{n_{k+1}^{\delta} U(n_{k+1})}\, \frac{U(n_{k+1})}{U(n_k)}.$$

Thus, by (6.2) and (6.4),

$$\limsup_{t\to\infty} \frac{\sup_{0 \leqslant s \leqslant t} G(s)}{t^{\delta} U(t)} \leqslant \rho^{\delta} \left(\frac{1+\varepsilon}{\gamma}\right)^{\frac{1+\alpha}{2\alpha}}, \qquad \text{a.s.}$$

The left–hand side is independent of ρ and ε. We achieve the upper bound in the law of the iterated logarithm by letting ρ and ε decrease to 1 and 0, respectively.

The lower–bound argument. For each $1 < p < 2$ and each integer $k \geqslant 0$, let

$$n_k = \exp\big(k^p\big).$$

In the course of our work, we will need one technical fact regarding the sequence $\{n_k : k \geqslant 0\}$. Let $0 \leqslant j \leqslant k$. Since, by the mean value theorem, $j^p - k^p \leqslant -pj^{p-1}(k-j)$, it follows that

$$\frac{n_j}{n_k} \leqslant \exp\big(-pj^{p-1}(k-j)\big) \tag{6.5}$$

Let $0 < \varepsilon < 1$ and define

$$\eta \triangleq \left(\frac{1-\varepsilon}{\gamma p}\right)^{\frac{1+\alpha}{2\alpha}}. \tag{6.6}$$

For future reference, let us observe that

$$\gamma\, p\, \eta^{\frac{2\alpha}{1+\alpha}} = 1 - \varepsilon. \tag{6.7}$$

We claim that the proof of the lower bound can be reduced to the following proposition: for each $1 < p < 2$ and each $0 < \varepsilon < 1$,

$$\limsup_{j\to\infty} \frac{G(n_j) - G(n_{j-1})}{(n_j - n_{j-1})^\delta U(n_j)} \geqslant \eta \qquad \text{a.s.} \tag{6.8}$$

Let us accept this proposition for the moment and see how the proof of the lower bound rests upon it.

By our estimate (6.5), $\lim_{j\to\infty}(n_j - n_{j-1})/n_j = 1$; thus, by (6.8), it follows that

$$\limsup_{j\to\infty} \frac{G(n_j) - G(n_{j-1})}{n_j^\delta\, U(n_j)} \geqslant \eta \qquad \text{a.s.} \tag{6.9}$$

Since, by (6.5), $\lim_{j\to\infty} n_{j-1}/n_j = 0$, and, by the upper bound for the law of the iterated logarithm, the sequence

$$\left\{ \frac{|G(n_{j-1})|}{n_{j-1}^\delta U(n_j)},\ j \geqslant 1 \right\}$$

is bounded, it follows that

$$\lim_{j\to\infty} \frac{|G(n_{j-1})|}{n_j^\delta U(n_j)} = 0 \qquad \text{a.s.} \tag{6.10}$$

Since $G(n_j) \geqslant \big(G(n_j) - G(n_{j-1})\big) - |G(n_{j-1})|$, by combining (6.9) and (6.10), we obtain

$$\limsup_{j\to\infty} \frac{G(n_j)}{n_j^\delta U(n_j)} \geqslant \eta \qquad \text{a.s.}$$

However, by (6.6) and the definition of the limit superior, this implies that

$$\limsup_{t\to\infty} \frac{G(t)}{t^\delta\, U(t)} \geqslant \left(\frac{1-\varepsilon}{\gamma p}\right)^{\frac{1+\alpha}{2\alpha}} \qquad \text{a.s.}$$

The left–hand side is independent of p and ε. We achieve the lower bound in the law of the iterated logarithm by letting p and ε decrease to 1 and 0, respectively.

We are left to verify the proposition (6.8). For $j \geqslant 1$, let

$$Z_j = \frac{G(n_j) - G(n_{j-1})}{(n_j - n_{j-1})^\delta} - \eta U(n_j).$$

Clearly it is enough to show that

$$\limsup_{j\to\infty} Z_j \geqslant 0 \qquad \text{a.s.} \tag{6.11}$$

By Proposition 3.1, the collection of random variables $\{Z_j : j \geqslant 1\}$ is pairwise positively quadrant dependent. Thus to demonstrate (6.11), it would suffice to establish items (a) and (b) of Proposition 4.1.

Since G has stationary increments and is self–similar with index δ,

$$\mathbb{P}(Z_j > 0) = \mathbb{P}\big(G_1 > \eta U(n_j)\big).$$

Since $\ln\ln(n_j) \sim p\ln(j)$, by Theorem 5.1 and (6.7), we can conclude that

$$\begin{aligned}\lim_{j\to\infty} \frac{\ln P(Z_j > 0)}{\ln(j)} &= -\gamma\, p\, \eta^{\frac{2\alpha}{1+\alpha}} \\ &= -(1-\varepsilon).\end{aligned}$$

Let $1-\varepsilon < q < 1$. Then there exists an integer $N \geqslant 1$ such that, for each $j \geqslant N$, we have $\mathbb{P}(Z_j > 0) \geqslant j^{-q}$, which verifies Proposition 4.1(a).

Let $1 \leqslant j \leqslant k$, and recall that $\delta = 1 - 1/(2\alpha)$. Then, by Theorem 5.2 and (6.5), there exists a positive constant $C = C(\alpha)$ such that

$$\begin{aligned}\mathrm{Cov}(Z_j, Z_k) &= \mathrm{Cov}\left(\frac{G(n_j) - G(n_{j-1})}{(n_j - n_{j-1})^\delta}, \frac{G(n_k) - G(n_{k-1})}{(n_k - n_{k-1})^\delta}\right) \\ &\leqslant C\left(\frac{n_j}{n_k}\right)^{1/(2\alpha)} \\ &\leqslant C \exp\left(\frac{-p}{2\alpha} j^{p-1}(k-j)\right)\end{aligned}$$

For $j \geqslant 1$, let

$$b_j = \exp\left(\frac{-p}{2\alpha} j^{p-1}\right),$$

and observe that $\{b_j : j \geqslant 1\}$ is monotone decreasing. Thus

$$\begin{aligned}\sum_{1 \leqslant j<k<\infty} \mathrm{Cov}(Z_j, Z_k) &\leqslant C \sum_{j=1}^{\infty} \sum_{k=j+1}^{\infty} b_j^{(k-j)} \\ &\leqslant C \frac{1}{1-b_1} \sum_{j=1}^{\infty} b_j \\ &< \infty,\end{aligned}$$

which verifies Proposition 4.1(b) hence (6.11), as was to be shown. $\diamond$

§7. The LIL for simple random walk in Gaussian scenery

In this section, we will prove Theorem 1.2, the discrete–time analogue of Theorem 1.1. As indicated in §1, we will restrict our attention to the case where $\mathcal{Y}$ is a collection of independent, standard normal random variables and $\mathcal{S}$ is a simple, symmetric random walk on the integers.

The proof of Theorem 1.2 relies on relies on ideas of Révész (see, for example, Chapter 10 of Révész (1990)), some of which can be traced to Knight (see Knight (1981)). Let X be a standard Brownian motion and let Y be a standard two–sided Brownian motion. We will assume that these processes are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and generate independent σ–fields. We will define a Gaussian scenery $\mathcal{Y}$ and a simple symmetric random walk on $(\Omega, \mathcal{F}, \mathbb{P})$ as follows: for each $a \in \mathbb{Z}$, let

$$y(a) = Y(a+1) - Y(a),$$

which defines the scenery. Let $\tau(0) = 0$ and, for each $k \geqslant 1$, let

$$\tau(k) \triangleq \inf\big(s > \tau(k-1) : \ |X(s) - X(\tau(k-1))| = 1\big).$$

For each $k \geqslant 0$, let $s_k \triangleq X(\tau(k))$. By the strong Markov property, $\mathcal{S} = \{s_k : k \geqslant 0\}$ is a simple symmetric random walk on $\mathbb{Z}$.

As described in §1, let $\mathcal{L}$ denote the local–time process of $\mathcal{S}$. For each $x \in \mathbb{R}$ and each $n \geqslant 0$, let

$$\ell_n^x = \begin{cases} \ell_n^{[x]}, & \text{if } x \geqslant 0, \\ \ell_n^{-[-x]}, & \text{if } x < 0. \end{cases}$$

In this notation, we have

$$g_n = \int_{\mathbb{R}} \ell_n^x dY(x).$$

Consequently,

$$G_n - g_n = \int_{\mathbb{R}} \big(L_n^x - \ell_n^x\big) dY(x).$$

Our first result is the main lemma of this section. Here and throughout the remainder of the section, we will use the following notation: given nonnegative sequences $\{a_n\}$ and $\{b_n\}$, we will write

$$a_n = O(b_n)$$

provided that there is a constant $C \in (0, \infty)$ such that, for all $n \geqslant 1$,

$$a_n \leqslant C b_n.$$

Lemma 7.1. *For each* $p \geqslant 1$,

$$\mathbb{E}\big(|G_n - g_n|^{2p}\big) = O(n^p).$$

The proof of this crucial lemma will be given in the sequel. At this point, we will give the proof of Theorem 1.2. This proof uses Lemma 7.1 and a standard blocking argument.

Proof of Theorem 1.2. Let $q > 1/2$ and choose $p \geqslant 1$ such that

$$1 + p - 2pq < 0. \tag{7.1}$$

Observe that

$$\sup_{0 \leqslant t \leqslant 1} |G(nt) - g([nt])| \leqslant \max_{1 \leqslant k \leqslant n} \sup_{k-1 \leqslant s \leqslant k} |G(s) - G(k-1)| + \max_{1 \leqslant k \leqslant n} |G_k - g_k|.$$

Let $\varepsilon > 0$ be given. Since G has stationary increments, by a trivial estimate and Theorem 2.1,

$$\begin{aligned} \mathbb{P}\big(\max_{1 \leqslant k \leqslant n} \sup_{k-1 \leqslant s \leqslant k} |G(s) - G(k-1)| > \varepsilon n^q\big) &\leqslant n\mathbb{P}\big(\sup_{0 \leqslant s \leqslant 1} |G(s)| > \varepsilon n^q\big) \\ &\leqslant 4n\mathbb{P}(G_1 > \varepsilon n^q) \end{aligned}$$

By Theorem 5.1, this last term is summable. Since this is true for each $\varepsilon > 0$, by the Borel–Cantelli lemma we can conclude that

$$\lim_{n\to\infty} \max_{1 \leqslant k \leqslant n} \sup_{k-1 \leqslant s \leqslant k} |G(s) - G(k-1)| = 0, \qquad \text{a.s.} \tag{7.2}$$

Let $\varepsilon > 0$ be given. By Markov's inequality and Lemma 7.1, there exists $C > 0$ such that,

$$\begin{aligned} \mathbb{P}(\max_{1 \leqslant k \leqslant 2^j} |G_k - g_k| > \varepsilon 2^{jq}) &\leqslant 2^j \max_{1 \leqslant k \leqslant 2^j} \mathbb{P}(|G_k - g_k| > \varepsilon 2^{jq}) \\ &\leqslant 2^j \max_{1 \leqslant k \leqslant 2^j} \left[\frac{\mathbb{E}(|G_k - g_k|^{2p})}{\varepsilon^{2p} 2^{2jpq}} \right] \\ &\leqslant C \varepsilon^{-2p} 2^{j(1+p-2pq)}. \end{aligned}$$

By (7.1), this last term is summable. Since this is true for each $\varepsilon > 0$, by the Borel–Cantelli lemma we can conclude that

$$\lim_{j\to\infty} \frac{\max_{1 \leqslant k \leqslant 2^j} |G_k - g_k|}{2^{jq}} = 0 \qquad \text{a.s.} \tag{7.3}$$

Finally, for each integer $n \in [2^j, 2^{j+1})$,

$$\frac{\max_{1 \leqslant k \leqslant n} |G_k - g_k|}{n^q} \leqslant 2^q \frac{\max_{1 \leqslant k \leqslant 2^{j+1}} |G_k - g_k|}{2^{(j+1)q}}.$$

This inequality, in conjunction with (7.3), demonstrates that

$$\lim_{n\to\infty} \frac{\max_{1 \leqslant k \leqslant n} |G_k - g_k|}{n^q} = 0, \qquad \text{a.s.}$$

Together with (7.2), this proves Theorem 1.2. $\diamond$

We are left to prove Lemma 7.1. In preparation for this proof, we will develop some terminology and some supporting results.

Let $\sigma(0) \triangleq 0$ and, for $k \geqslant 1$, let

$$\sigma(k) = \inf\{j > \sigma(k-1) : s_j = 0\}$$
$$\Delta_k \triangleq L^0_{\sigma(k)} - L^0_{\sigma(k-1)}$$

In words, $\sigma(k)$ is the time of the kth visit to 0 by the random walk $\mathcal{S}$, while Δ_k is the local time in 0 by X between the $(k-1)$st and kth visits to 0 by $\mathcal{S}$.

Lemma 7.2.

(a) *The random variables* $\{\Delta_j : j \geqslant 1\}$ *are independent and identically distributed.*

(b) $\mathbb{E}(\Delta_1) = 1$.

(c) Δ_1 *has bounded moments of all orders.*

Proof. Item (a) follows from the strong Markov property.

To prove (b) and (c), let us observe that the local time in 0 of X up to time $\sigma(1)$ is only accumulated during the time interval $[0, \tau(1)]$; thus,

$$\Delta_1 = L^0_{\sigma(1)} = L^0_{\tau(1)}.$$

Therefore it suffices to prove (b) and (c) for $L^0_{\tau(1)}$ in place of Δ_1.

By Tanaka's formula (see, for example, Theorem 1.5 of Revuz and Yor (1991)), for each $t \geqslant 0$,

$$|X(t)| = \int_0^t \operatorname{sgn}\big(X(s)\big) dX(s) + L^0_t.$$

Let $n \geqslant 1$. Then, by the optional stopping theorem,

$$\mathbb{E}|X(\tau(1) \wedge n)| = \mathbb{E}(L^0_{\tau(1)\wedge n}).$$

Since $\sup_{n \geqslant 1} |X(\tau(1) \wedge n)| \leqslant 1$, by continuity and Lebesgue's dominated convergence theorem, $\mathbb{E}(L^0_{\tau(1)}) = 1$, which verifies (b).

Finally, let us verify (c). By Tanaka's formula,

$$L^0_{\tau(1)\wedge n} = |X(\tau(1) \wedge n)| - \int_0^{\tau(1)\wedge n} \operatorname{sgn}\big(X(s)\big) dX(s).$$

Let $p \geqslant 1$. Due to the definition of $\tau(1)$, we have the trivial bound $E(|X(\tau(1) \wedge n)|^p) \leqslant 1$ for all $n \geqslant 1$. To bound the pth moment of the integral, first let us note that $\tau(1)$ has bounded moments of all orders. Therefore, by the Burkholder–Davis–Gundy Inequality (see Corollary 4.2 of Revuz and Yor (1991)), there exists a positive constant $C = C(p)$ such that,

$$\begin{aligned}\mathbb{E}\left[\left|\int_0^{\tau(1)\wedge n} \operatorname{sgn}\big(X(s)\big) dX(s)\right|^p\right] &\leqslant C\mathbb{E}\big((\tau(1) \wedge n)^{p/2}\big) \\ &\leqslant C\mathbb{E}\big(\tau(1)^{p/2}\big),\end{aligned}$$

Thus

$$\mathbb{E}(|L^0_{\tau(1)\wedge n}|^p) \leqslant 2^{p-1}\left(1 + C\mathbb{E}\big(\tau(1)^{p/2}\big)\right),$$

which verifies (c). ◇

Our next lemma is the main technical result needed to prove Lemma 7.1.

Lemma 7.3. *For each integer* $p \geqslant 1$,

(a) $\sup_{0\leqslant z\leqslant 1} \mathbb{E}\big(|L^z_n - L^0_n|^p\big) = O(n^{p/4})$.

(b) $\mathbb{E}\big(|L^0_n - L^0_{\tau_n}|^p\big) = O(n^{p/4})$.

(c) $\mathbb{E}\big(|L^0_{\tau_n} - \ell^0_n|^p\big) = O(n^{p/4})$.

Proof. Let $z \in (0,1]$. Define $I = (0,z)$, and

$$f(x) = \begin{cases} 0 & \text{if } x < 0; \\ x & \text{if } 0 \leqslant x \leqslant z; \\ z & \text{if } x > z. \end{cases}$$

By Tanaka's formula,

$$\frac{1}{2}(L^z_n - L^0_n) = f(X_n) - \int_0^n \mathbf{1}(X_t \in I) dX_t.$$

Since $|f|$ is bounded by 1, $\mathbb{E}(|f(X_n)|^p) \leqslant 1$. It remains to show that

$$\mathbb{E}\Big|\int_0^n \mathbf{1}\big(X_t \in I\big) dX_t\Big|^p = O(n^{p/4}).$$

For the moment, let us assume that $p = 2k$ is an even integer, and let

$$J \triangleq \{(t_1, t_2, \cdots, t_k) : 0 \leqslant t_1 < t_2 < \cdots < t_k \leqslant n\}.$$

Then, by the Burkholder–Davis–Gundy inequality (see Corollary 4.2 of Revuz and Yor (1991)), there exists a positive constant $C = C(p)$ such that

$$\begin{aligned} \mathbb{E}\left(\Big|\int_0^n \mathbf{1}(X_t \in I) dX_t\Big|^{2k}\right) &\leqslant C\mathbb{E}\left(\Big|\int_0^n \mathbf{1}(X_t \in I) dt\Big|^{k}\right) \\ &= k!C\int_J \mathbb{P}(X(t_1) \in I, \cdots, X(t_k) \in I) dt_1 \cdots dt_k \end{aligned} \tag{7.4}$$

Observe that the density of $(X(t_1), \cdots, X(t_k))$ is bounded by

$$(2\pi)^{-k/2}\big(t_1(t_2-t_1)\cdots(t_k-t_{k-1})\big)^{-1/2},$$

and the volume of I^k is bounded by 1. Let $u_1 = t_1$ and, for $k \geqslant 2$, let $u_k = t_k - t_{k-1}$. Then

$$\begin{aligned}\int_J \mathbb{P}(X(t_1)\in I, \cdots, X(t_k)\in I)dt_1\cdots dt_k &\leqslant (2\pi)^{-k/2}\int_{[0,n]^k}(u_1\cdots u_k)^{-1/2}du_1\cdots du_k \\ &= 2^k(2\pi)^{-k/2}n^{-k/2}.\end{aligned}$$

In light of (7.4), this gives the desired bound for the moments of even order. Bounds on the moments of odd order can be obtained from these even–order estimates and Jensen's inequality. This proves (a).

For each $t>0$ and $n\geqslant 1$, let

$$\begin{aligned} F &= \{n \leqslant \tau_n \leqslant n + n^{1/2}t\} \\ G &= \{(n-n^{1/2}t)\vee 0 \leqslant \tau_n \leqslant n\} \\ H &= \{|\tau_n - n| \geqslant n^{1/2}t\}\end{aligned}$$

Since $u \mapsto L_u^0$ is increasing,

$$\begin{aligned}\mathbb{P}(|L_n^0 - L_{\tau_n}^0| > n^{1/4}t, F) &\leqslant \mathbb{P}(L^0_{n+n^{1/2}t} - L_n^0 > n^{1/4}t) \\ &\leqslant \mathbb{P}(L^0_{n^{1/2}t} > n^{1/4}t) \\ &= \mathbb{P}\big((L_1^0)^2 > t\big)\end{aligned}$$

If $n - n^{1/2}t \geqslant 0$, then, arguing as above,

$$\mathbb{P}(|L_n^0 - L_{\tau_n}^0| > n^{1/4}t, G) \leqslant \mathbb{P}\big((L_1^0)^2 > t\big).$$

If, however, $n - n^{1/2}t < 0$, then $\sqrt{t} > n^{1/4}$ and

$$\begin{aligned}\mathbb{P}(|L_n^0 - L_{\tau_n}^0| > n^{1/4}t, G) &\leqslant \mathbb{P}(L_n^0 > n^{1/2}t) \\ &= \mathbb{P}(L_1^0 > n^{-1/4}t) \\ &\leqslant \mathbb{P}((L_1^0)^2 > t).\end{aligned}$$

By Markov's inequality and Burkholder's inequality (see, for example, Theorem 2.10 of Hall and Heyde (1980)), there exists a positive constant $C = C(p)$ such that

$$\begin{aligned}\mathbb{P}(H) &\leqslant \frac{\mathbb{E}(|\tau_n - n|^{p+2})}{n^{(p+2)/2}t^{(p+2)}} \\ &\leqslant Ct^{-(p+2)}\end{aligned}$$

In summary,

$$\mathbb{P}(|L_n^0 - L_{\tau_n}^0| > n^{1/4}t) \leqslant 2\mathbb{P}((L_1^0)^2 > t) + (C_q t^{-(p+2)}) \wedge 1,$$

which demonstrates that $|L_n^0 - L_{\tau_n}^0|/n^{1/4}$ has a bounded pth moment. This verifies (b).

Observe that

$$L_{\tau_n}^0 = \begin{cases} \sum_{k=1}^{\ell_n^0} \Delta_k & \text{if } s_n \neq 0; \\ \sum_{k=1}^{\ell_n^0 - 1} \Delta_k & \text{if } s_n = 0. \end{cases}$$

Thus, by a generous bound and Lemma 7.2,

$$|L_{\tau_n}^0 - \ell_n^0| \leqslant \left| \sum_{k=1}^{\ell_n^0} (\Delta_k - \mathbb{E}(\Delta_k)) \right| + \left| \sum_{k=1}^{\ell_n^0 - 1} (\Delta_k - \mathbb{E}(\Delta_k)) \right| \mathbf{1}(\ell_n^0 \geqslant 2) + 1.$$

Since the event $\{\ell_n = j\}$ is independent of the σ–field generated by $\{\Delta_1, \cdots, \Delta_j\}$, it follows that

$$\mathbb{E}\left(\left| \sum_{k=1}^{\ell_n^0} (\Delta_k - \mathbb{E}(\Delta_k)) \right|^p \right) \leqslant \sum_{j=1}^{n} \mathbb{E}\left(\left| \sum_{k=1}^{j} (\Delta_k - \mathbb{E}(\Delta_k)) \right|^p \right) \mathbb{P}(\ell_n^0 = j).$$

By Burkholder's inequality (see, for example, Theorem 2.10 of Hall and Heyde (1980)), there exists a positive constant $C = C(p)$ such that

$$\mathbb{E}\left(\left| \sum_{k=1}^{j} (\Delta_k - \mathbb{E}(\Delta_k)) \right|^p \right) \leqslant C j^{p/2}.$$

Thus

$$\begin{aligned} \mathbb{E}\left(\left| \sum_{k=1}^{\ell_n^0} (\Delta_k - \mathbb{E}(\Delta_k)) \right|^p \right) &\leqslant C\mathbb{E}\big((\ell_n^0)^{p/2} \big) \\ &= O(n^{p/4}) \end{aligned}$$

The other relevant term can be handled similarly. This proves (c) hence the lemma. $\diamond$

Lemma 7.4. *For each $p \geqslant 1$ there exists a constant $C = C(p)$ such that, for all $x \in \mathbb{R}$ and $n \geqslant 1$,*

$$\mathbb{E}\big(|L_n^x - \ell_n^x|^p \big) \leqslant C n^{p/4} \exp\left(-\frac{x^2}{4n} \right).$$

Proof. We will assume, without loss of generality, that $x \geqslant 0$. Let

$$T \triangleq \min\{j \geqslant 0 : s_j = [x]\}.$$

Then, by the strong Markov property,

$$\begin{aligned}\mathbb{E}\big(|L_n^x - \ell_n^x|^p\big) &= \sum_{j=0}^{n} \mathbb{E}\big(|L_{n-j}^{x-[x]} - \ell_{n-j}^0|^p\big)\mathbb{P}(T = j) \\ &\leqslant \max_{0 \leqslant k \leqslant n} \mathbb{E}\big(|L_k^{x-[x]} - \ell_k^0|^p\big)\mathbb{P}\big(\max_{0 \leqslant k \leqslant n} |s_k| \geqslant [x]\big).\end{aligned} \tag{7.5}$$

By the reflection principle, a classical bound, and some algebra, we obtain

$$\begin{aligned}\mathbb{P}\big(\max_{0 \leqslant k \leqslant n} |s_k| \geqslant [x]\big) &\leqslant 4 \exp\left(-\frac{[x]^2}{2n}\right) \\ &\leqslant 4e^{1/2} \exp\left(-\frac{x^2}{4n}\right)\end{aligned} \tag{7.6}$$

By the triangle inequality and Lemma 7.3,

$$\begin{aligned}&\mathbb{E}\big(|L_k^{x-[x]} - \ell_k^0|^p\big) \\ &\quad\leqslant 3^{p-1}\left(\sup_{0 \leqslant z \leqslant 1} \mathbb{E}\big(|L_k^z - L_k^0|^p\big) + \mathbb{E}\big(|L_k^0 - L_{\tau_k}^0|^p\big) + \mathbb{E}\big(|L_{\tau_k}^0 - \ell_k^0|^p\big)\right) \\ &\quad= O(k^{p/4})\end{aligned} \tag{7.7}$$

The proof is completed by combining (7.6) and (7.7) with (7.5). ◇

Proof of Lemma 7.1 Since X and Y are independent, it follows that $G_n - g_n$, conditional on $\mathfrak{X}$, is a centered normal random variable with variance

$$\mathbb{E}_{\mathfrak{X}}\left((G_n - g_n)^2\right) = \int_{\mathbb{R}} (L_n^x - \ell_n^x)^2 dx$$

Thus

$$\begin{aligned}\mathbb{E}\big((G_n - g_n)^{2p}\big) &= \mathbb{E}\left[\mathbb{E}_{\mathfrak{X}}\left((G_n - g_n)^{2p}\right)\right] \\ &= \frac{(2p)!}{2^p p!}\mathbb{E}\left[\left(\int_R (L_n^x - \ell_n^x)^2 dx\right)^p\right]\end{aligned}$$

By Minkowski's inequality, Lemma 7.4, and a standard calculation, there exists a constant $C = C(p)$ such that,

$$\begin{aligned}\sqrt[p]{\mathbb{E}\left[\left(\int_R (L_n^x - \ell_n^x)^2 dx\right)^p\right]} &\leqslant \int_{\mathbb{R}} \|(L_n^x - \ell_n^x)^2\|_p dx \\ &\leqslant C n^{1/2} \int_{\mathbb{R}} \exp\left(-\frac{x^2}{4np}\right) dx \\ &= 2C\sqrt{p\,\pi} n\end{aligned}$$

It follows that $\mathbb{E}\big((G_n - g_n)^{2p}\big) = O(n^p)$, as was to be shown. ⋄

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STOCHASTIC PROCESSES AND THEIR APPLICATIONS **74** 89–121 (1998)

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