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# WEAK UNIMODALITY OF FINITE MEASURES, AND AN APPLICATION TO POTENTIAL THEORY OF ADDITIVE LÉVY PROCESSES

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Abstract. A probability measure $\mu$ on $\mathbb{R}^d$ is called *weakly unimodal* if there exists a constant $\kappa \geq 1$ such that for all $r > 0$,

$$\sup_{a \in \mathbb{R}^d} \mu(B(a,r)) \leq \kappa \mu(B(0,r)). \tag{0.1}$$

Here, $B(a,r)$ denotes the $\ell^\infty$-ball centered at $a \in \mathbb{R}^d$ with radius $r > 0$.

In this note, we derive a sufficient condition for weak unimodality of a measure on the Borel subsets of $\mathbb{R}^d$. In particular, we use this to prove that every symmetric infinitely divisible distribution is weakly unimodal. This result is then applied to improve some recent results of the authors on capacities and level sets of additive Lévy processes.

## 1. Introduction

For any integer $k \geq 1$ and any $x \in \mathbb{R}^k$, let $|x| = \max_{1 \leq \ell \leq k} |x_\ell|$ and $\|x\| = (\sum_{\ell=1}^k x_\ell^2)^{\frac{1}{2}}$ denote the $\ell^\infty$ and $\ell^2$ norms on $\mathbb{R}^k$, respectively. Moreover, $B(x,r) = \{y \in \mathbb{R}^d : |x-y| \leq r\}$ stands for the closed $r$-ball about $x \in \mathbb{R}^d$, while $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ denotes the usual $d$-dimensional Euclidean space, together with its Borel $\sigma$-algebra.

Given a measure $\mu$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, we say that $\mu$ is $\kappa$-*weakly unimodal* if there exists a positive constant $\kappa \geq 1$ such that for all $r > 0$, (0.1) holds, the point being that $\kappa$ can be chosen independently of $r > 0$. In this note, we use Fourier analytical methods to derive a general criterion for the weak unimodality of measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

Our intended application is to the potential theory of additive Lévy processes, as described by Khoshnevisan and Xiao [4]. An *additive Lévy process* $X$ in $\mathbb{R}^d$ is an $N$-parameter stochastic process $X = \{X(t);\, t \in \mathbb{R}_+^N\}$ that has the form $X(t) = \sum_{j=1}^N X_j(t_j)$, where $X_1, \ldots, X_N$ are independent Lévy processes with values in $\mathbb{R}^d$. Using the notation of Khoshnevisan and Xiao [4], we write $X = X_1 \oplus \cdots \oplus X_N$. We refer to Bertoin [2] and Sato [7] for definitions and properties of Lévy processes, infinitely divisible distributions and self-decomposable distributions.

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Following Khoshnevisan and Xiao [4], we say that an additive Lévy process $X$ is *$\kappa$-weakly unimodal* if there exists one $\kappa$, such that the distribution of the random vector $X(t)$ is $\kappa$-weakly unimodal for every $t \in \mathbb{R}^N_+ \backslash \partial\mathbb{R}^N_+$. If and when this is so, we say that the process $X$ is *weakly unimodal*. In Khoshnevisan and Xiao [4], we applied the various results of Anderson [1], Kanter [3], Medgyessy [5], Sato [6], Wolfe [8], [9], [10], and Yamazato [11] to obtain various sufficient conditions for $X$ to be weakly unimodal. For example, we showed that if for every $t \in \mathbb{R}^N_+ \backslash \partial\mathbb{R}^N_+$, the distribution of $X(t)$ is symmetric and self-decomposable, then $X$ is weakly unimodal with $\kappa = 1$. Using the criterion for weak unimodality of the present paper, we are able to prove that *every symmetric infinitely divisible distribution is weakly unimodal*. This can, in turn, be used to improve the recent results of the authors on capacities and level sets of additive Lévy processes.

## 2. Weak unimodality and Fourier analysis

Throughout, $\widehat{\ }$ denotes the Fourier transform on $\mathbb{R}^d$ normalized so that for every $f \in L^1(\mathbb{R}^d)$,

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{i\xi\cdot x} f(x)\, dx, \qquad \forall \xi \in \mathbb{R}^d.$$

Our main result is the following.

**Theorem 2.1.** *Any finite measure $\mu$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is $\kappa$-weakly unimodal with $\kappa = 16^d(1+\delta^2)$, as long as there exists $\delta > 0$, such that*

$$|\operatorname{Im} \widehat{\mu}(\xi)| \le \delta \operatorname{Re} \widehat{\mu}(\xi), \qquad \forall \xi \in \mathbb{R}^d. \tag{2.1}$$

Since $\widehat{\mu}(0) = \mu(\mathbb{R}^d)$, (2.1) is a kind of sector condition. Moreover, it tacitly asserts that $\operatorname{Re}\widehat{\mu} \ge 0$, pointwise. Motivated by this remark, in Section 4 we will exhibit a probability measure that is $\kappa$-weak unimodal but does not satisfy the sector–like property (2.1).

**Corollary 2.2.** *Any finite measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ whose Fourier transform is nonnegative is $\kappa$-weakly unimodal with $\kappa = 16^d$.*

*Proof of Theorem 2.1.* For any $r > 0$, consider the function

$$\varphi_r(x) = \prod_{j=1}^{d} \frac{1-\cos(2rx_j)}{2\pi r x_j^2}, \qquad \forall x \in \mathbb{R}^d.$$

Then $\varphi_r$ is a non-negative function with Fourier transform given by

$$\widehat{\varphi_r}(\xi) = \prod_{j=1}^{d} \Big(1 - \frac{|\xi_i|}{2r}\Big)^+, \qquad \forall \xi \in \mathbb{R}^d, \tag{2.2}$$

where $x^+ = \max(x, 0)$.

Suppose $\mathbf{1}_A$ is the indicator function of the set $A$. Then, $\xi \in B(a,r)$ implies that $1 - (2r)^{-1}|\xi_j - a_j| \ge \frac{1}{2}$ for every $j = 1, \ldots, d$. In light of (2.2), this shows that

$$\mathbf{1}_{B(a,r)}(\xi) \le 2^d \widehat{\varphi_r}(\xi - a), \qquad \forall \xi \in \mathbb{R}^d.$$

Integrating this $[d\mu]$ yields

$$\begin{aligned} \mu(B(a,r)) &\le 2^d \widehat{\varphi_r} \star \mu(a), && \forall a \in \mathbb{R}^d,\ r>0, \\ &\le 2^d \int_{\mathbb{R}^d} \varphi_r(\xi) |\widehat{\mu}(\xi)|\, d\xi, && \forall r > 0, \end{aligned} \tag{2.3}$$

where $\star$ denotes convolution. On the other hand, whenever $1 - \frac{|\xi_j|}{r} > 0$ for all $j = 1, \ldots, d$, we have $\xi \in B(0,r)$. This and (2.2), together, imply

$$\mathbf{1}_{B(0,r)}(\xi) \ge \widehat{\varphi_{r/2}}(\xi), \qquad \forall r > 0,\ \xi \in \mathbb{R}^d.$$

We integrate this with respect to $\mu$ to obtain

$$\begin{aligned} \mu(B(0,r)) &\ge \int_{\mathbb{R}^d} \widehat{\varphi_{r/2}}(\xi)\, \mu(d\xi) \\ &= \int_{\mathbb{R}^d} \varphi_{r/2}(y) \operatorname{Re} \widehat{\mu}(y)\, dy \\ &\ge (1+\delta^2)^{-\frac{1}{2}} \int_{\mathbb{R}^d} \varphi_{r/2}(y) |\widehat{\mu}(y)|\, dy && \text{(by (2.1))} \\ &\ge 2^{-d} (1+\delta^2)^{-\frac{1}{2}} \sup_{a \in \mathbb{R}^d} \mu(B(a, \tfrac{r}{2})), \end{aligned} \tag{2.4}$$

thanks to (2.3). To complete our proof, we use a covering argument. Let $a_1, \ldots, a_{4^d} \in [0, 2r]^d$ be chosen such that

- the interiors of $B(a_\ell, r)$'s are disjoint as $\ell$ varies in $\{1, \ldots, 4^d\}$; and
- $\bigcup_{\ell=1}^{4^d} B(a_\ell; \frac{r}{2}) = B(0, 2r)$.

Applying (2.4) yields

$$\begin{aligned} \sup_{a \in \mathbb{R}^d} \mu(B(a,r)) &\le 2^d (1+\delta^2)^{\frac{1}{2}} \mu(B(0,2r)) \\ &\le 2^d (1+\delta^2)^{\frac{1}{2}} \sum_{\ell=1}^{4^d} \mu(B(a_\ell, \tfrac{r}{2})). \end{aligned}$$

Another application of (2.4) yields the desired result. □

## 3. Infinitely divisible laws and potential theory

Recall that a probability measure $\mu$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is *infinitely divisible* if its Fourier transform has the representation $\widehat{\mu}(\xi) = e^{-\Psi(\xi)}$ $(\forall \xi \in \mathbb{R}^d)$, where $\Psi(x)$ is given by the Lévy-Khintchine formula; see Bertoin [2] or Sato [7] for this and more information. The function, $\Psi$, is called the *Lévy exponent* of $\mu$. Recall that $\mu$ is *symmetric* if for all $A \in \mathcal{B}(\mathbb{R}^d)$, $\mu(A) = \mu(-A)$.

**Corollary 3.1.** *Any symmetric infinitely divisible law $\mu$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is $\kappa$-weakly unimodal with $\kappa = 16^d$.*

*Proof.* Note that $\mu(\bullet) = \frac{1}{2}\mu(\bullet) + \frac{1}{2}\mu(-\bullet)$ has Lévy exponent $\Psi = \overline{\Psi}$; thus, $\Psi$ is real. Moreover, since $|\widehat{\mu}(\xi)| \le 1$, $\Psi(\xi) \ge 0$. Thus, $\widehat{\mu} \ge 0$, pointwise, and Corollary 2.2 completes our proof. □

Thanks to Corollary 3.1, the weak unimodality condition in Khoshnevisan and Xiao [4, Th. 2.9] holds tautologically. This only uses the fact that whenever $X$ is an $\mathbb{R}^d$-valued, $N$-parameter, symmetric additive Lévy process, $X(t)$ has a symmetric

infinitely divisible law on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, for each $t \in \mathbb{R}_+^N \setminus \partial\mathbb{R}_+^N$. In particular, in the notation of Khoshnevisan and Xiao [4], we have

**Theorem 3.2.** *Let $X_1, \ldots, X_N$ be $N$ independent symmetric Lévy processes on $\mathbb{R}^d$ and let $X = X_1 \oplus \cdots \oplus X_N$. Suppose $X$ is absolutely continuous and $\Phi$ denotes the gauge function of $X$. Then $\Phi \in L^1_{loc}(\mathbb{R}^N)$, if and only if any of the following conditions are satisfied:*

a) $\mathbb{P}\{\mathrm{Leb}\{X([c,\infty[^N)\} > 0\} = 1$, *for all* $c > 0$;
b) $\mathbb{P}\{\mathrm{Leb}\{X([c,\infty[^N)\} > 0\} > 0$, *for all* $c > 0$;
c) $\mathbb{P}\{\mathrm{Leb}\{X([c,\infty[^N)\} > 0\} > 0$, *for some* $c > 0$;
d) $\mathbb{P}\{X^{-1}(0) \cap [c,\infty[^N \neq \emptyset\} > 0$, *for all* $c > 0$;
e) $\mathbb{P}\{X^{-1}(0) \cap [c,\infty[^N \neq \emptyset\} > 0$, *for some* $c > 0$,

*where* Leb *denotes Lebesgue's measure on* $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

When $X^{-1}(0) \neq \emptyset$, it is of interest to determine its Hausdorff dimension. Our next theorem provides upper and lower bounds for $\dim_H X^{-1}(0)$ in terms of the following two indices associated to the gauge function $\Phi$:

$$\overline{\gamma} = \inf\big\{\beta > 0 : \liminf_{s\to 0} \|s\|^{N-\beta}\Phi(s) > 0\big\},$$

$$\gamma = \sup\Big\{\beta > 0 : \int_{[0,1]^N} \frac{1}{\|s\|^\beta}\ \Phi(s)\ ds < \infty\Big\}.$$

It is easy to verify that $0 \leq \gamma \leq \overline{\gamma} \leq N$.

Henceforth, $\|\mathbf{s}\|$ designates the $N$-dimensional vector $(\|s\|, \ldots, \|s\|)$.

**Theorem 3.3.** *Given the conditions of Theorem 3.2, for any* $0 < c < C < \infty$,

$$\mathbb{P}\big\{\gamma \ \leq \dim_H(X^{-1}(0) \cap [c,C]^N) \leq \ \overline{\gamma}\big\} > 0. \tag{3.1}$$

*Moreover, if there exists a constant* $K_1 > 0$ *such that*

$$\Phi(s) \leq \Phi(K_1\|\mathbf{s}\|) \qquad \text{for all} \ \ s \in [0,1]^N, \tag{3.2}$$

*then* $\mathbb{P}\{\dim_H(X^{-1}(0) \cap [c,C]^N) = \gamma\} > 0$.

*Remark* 3.4. Clearly, if $X_1, \ldots, X_N$ have the same Lévy exponent, then (3.2) holds. In particular, it follows from Theorems 3.2 and 3.3 that if $X_1, \ldots, X_N$ are independent isotropic stable Lévy processes in $\mathbb{R}^d$ with index $\alpha \in ]0,2]$ and $X = X_1 \oplus \cdots \oplus X_N$, then

(i) $\mathbb{P}\{X^{-1}(0) \neq \emptyset\} > 0$ if and only if $N\alpha > d$; and
(ii) if $N\alpha > d$, then $\mathbb{P}\big\{\dim_H X^{-1}(0) = N - \frac{d}{\alpha}\big\} > 0$.

In this case, it would be interesting to determine a Hausdorff measure function $\psi$ such that $0 < \psi\text{-}m(X^{-1}(0)) < \infty$, where $\psi$-$m$ denote Hausdorff measure.

## 4. A counter-example

Consider a Lévy process $X = \{X(t);\, t \geq 0\}$ on $\mathbb{R}^d$ and let $\mu_t$ denote the distribution of the vector $X(t)$ for each $t \geq 0$. Clearly, $\{\mu_t;\, t \geq 0\}$ is a convolution semigroup whose 1-potential is

$$U(A) = \int_0^\infty e^{-s}\mu_s(A)\, ds, \qquad \forall A \in \mathcal{B}(\mathbb{R}^d).$$

On the other hand, in §3 we already saw that $\widehat{\mu_t}(\xi) = e^{-t\Psi(\xi)}$ for a Lévy exponent $\Psi$. This yields $\widehat{U} = \{1+\Psi\}^{-1}$. We note that $U$ is a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, and (2.1) for $U$ is *equivalent* to

$$|\operatorname{Im}\Psi(\xi)| \leq \delta\{1 + \operatorname{Re}\Psi(\xi)\}, \qquad \forall \xi \in \mathbb{R}^d.$$

This is precisely the classical *sector condition*, and motivates the few allusions to this condition in §1. On the other hand, the sector condition is not needed for $4^d$-weak unimodality of $U$, as the following shows.

**Lemma 4.1.** *$U$ is always $4^d$-weakly unimodal.*

*Proof.* We present a probabilistic proof that is quite well-known in the context of random walks. Throughout, we shall fix $a \in \mathbb{R}^d$ and $r > 0$, and let $\sigma = \inf\{s > 0 : |X(s) - a| < r\}$. Clearly, $\sigma$ is a stopping time, and

$$\begin{aligned} U(B(a,r)) &= \mathbb{E}\Big[\int_0^\infty e^{-s}\mathbf{1}_{\{|X(s)-a|<r\}}\,ds\Big] \\ &= \mathbb{E}\Big[\int_\sigma^\infty e^{-s}\mathbf{1}_{\{|X(s)-a|<r\}}\,ds\Big], \end{aligned}$$

where $\int_\sigma^\infty(\cdots) = 0$ on $\{\sigma = +\infty\}$. Thus, writing $e^{-\infty} = 0$ and $X(\infty) = 0$, we have

$$\begin{aligned} U(B(a,r)) &\leq \mathbb{E}\Big[\int_0^\infty e^{-(s+\sigma)}\mathbf{1}_{\{|X(s+\sigma)-X(\sigma)+X(\sigma)-a|<r\}}\,ds\Big] \\ &\leq \mathbb{E}\Big[\int_0^\infty e^{-(s+\sigma)}\mathbf{1}_{\{|X(s+\sigma)-X(\sigma)|<2r\}}\,ds\Big]. \end{aligned}$$

By the strong Markov property of $X$, the latter is not greater than $U(B(0,2r))$. We now apply the same covering argument used in our proof of Theorem 2.1 to complete this proof. □

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